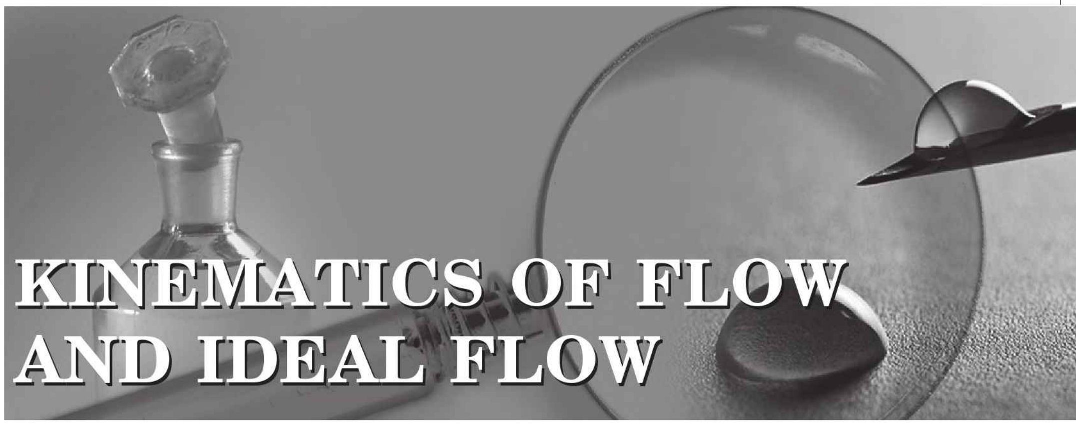
5 CHAPTER



KINEMATICS OF FLOW AND IDEAL FLOW

A. KINEMATICS OF FLOW

▶ 5.1 INTRODUCTION

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics. Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined. In this chapter, the methods of determining velocity and acceleration are discussed.

▶ 5.2 METHODS OF DESCRIBING FLUID MOTION

The fluid motion is described by two methods. They are —(*i*) Lagrangian Method, and (*ii*) Eulerian Method. In the Lagrangian method, a **single fluid particle** is followed during its motion and its velocity, acceleration, density, etc., are described. In case of Eulerian method, the velocity, acceleration, pressure, density etc., are described **at a point** in flow field. The Eulerian method is commonly used in fluid mechanics.

▶ 5.3 TYPES OF FLUID FLOW

The fluid flow is classified as :

(*i*) Steady and unsteady flows ;
(*ii*) Uniform and non-uniform flows ;
(*iii*) Laminar and turbulent flows ;
(*iv*) Compressible and incompressible flows ;
(*v*) Rotational and irrotational flows ; and
(*vi*) One, two and three-dimensional flows.

5.3.1 Steady and Unsteady Flows. Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time. Thus for steady flow, mathematically, we have

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

where (x_0, y_0, z_0) is a fixed point in fluid field.

Unsteady flow is that type of flow, in which the velocity, pressure or density at a point changes with respect to time. Thus, mathematically, for unsteady flow

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} \neq 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0 \text{ etc.}$$

5.3.2 Uniform and Non-uniform Flows. Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (*i.e.*, length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t\,=\,\text{constant}} = 0$$

where ∂V = Change of velocity

∂s = Length of flow in the direction S.

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t\,=\,\text{constant}} \neq 0.$$

5.3.3 Laminar and Turbulent Flows. Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream-line flow or viscous flow.

Turbulent flow is that type of flow in which the fluid particles move in a *zig-zag* way. Due to the movement of fluid particles in a *zig-zag* way, the eddies formation takes place which are responsible for high energy loss. For a pipe flow, the type of flow is determined by a non-dimensional number $\frac{VD}{\nu}$ called the Reynold number,

where D = Diameter of pipe

V = Mean velocity of flow in pipe

and ν = Kinematic viscosity of fluid.

If the Reynold number is less than 2000, the flow is called laminar. If the Reynold number is more than 4000, it is called turbulent flow. If the Reynold number lies between 2000 and 4000, the flow may be laminar or turbulent.

5.3.4 Compressible and Incompressible Flows. Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq \text{Constant}$$

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = \text{Constant.}$$

5.3.5 Rotational and Irrotational Flows. Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis. And if the fluid particles while flowing along stream-lines, do not rotate about their own axis then that type of flow is called irrotational flow.

5.3.6 One-, Two- and Three-Dimensional Flows. **One-dimensional flow** is that type of flow in which the flow parameter such as velocity is a function of time and one space co-ordinate only, say x. For a steady one-dimensional flow, the velocity is a function of one-space-co-ordinate only. The variation of velocities in other two mutually perpendicular directions is assumed negligible. Hence mathematically, for one-dimensional flow

$$u = f(x),\ v = 0 \text{ and } w = 0$$

where u, v and w are velocity components in x, y and z directions respectively.

Two-dimensional flow is that type of flow in which the velocity is a function of time and two rectangular space co-ordinates say x and y. For a steady two-dimensional flow the velocity is a function of two space co-ordinates only. The variation of velocity in the third direction is negligible. Thus, mathematically for two-dimensional flow

$$u = f_1(x, y),\ v = f_2(x, y) \text{ and } w = 0.$$

Three-dimensional flow is that type of flow in which the velocity is a function of time and three mutually perpendicular directions. But for a steady three-dimensional flow the fluid parameters are functions of three space co-ordinates (x, y and z) only. Thus, mathematically, for three-dimensional flow

$$u = f_1(x, y, z),\ v = f_2(x, y, z) \text{ and } w = f_3(x, y, z).$$

▶ 5.4 RATE OF FLOW OR DISCHARGE (Q)

It is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume of fluid flowing across the section per second. For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section. Thus

(*i*) For liquids the units of Q are m^3/s or litres/s

(*ii*) For gases the units of Q is kgf/s or Newton/s

Consider a liquid flowing through a pipe in which

A = Cross-sectional area of pipe

V = Average velocity of fluid across the section

Then discharge $Q = A \times V$. ...(5.1)

▶ 5.5 CONTINUITY EQUATION

The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross-section, the quantity of fluid per second is constant. Consider two cross-sections of a pipe as shown in Fig. 5.1.

Let V_1 = Average velocity at cross-section 1-1

ρ_1 = Density at section 1-1

A_1 = Area of pipe at section 1-1

and V_2, ρ_2, A_2 are corresponding values at section, 2-2.

Then rate of flow at section 1-1 = $\rho_1 A_1 V_1$

Rate of flow at section 2-2 = $\rho_2 A_2 V_2$

According to law of conservation of mass

Rate of flow at section 1-1 = Rate of flow at section 2-2

or $$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad ...(5.2)$$

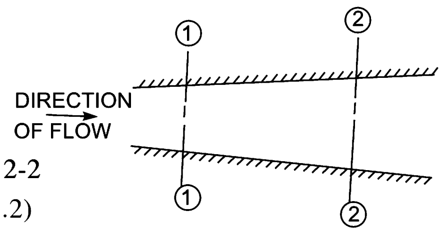


Fig. 5.1 *Fluid flowing through a pipe.*

Equation (5.2) is applicable to the compressible as well as incompressible fluids and is called **Continuity Equation**. If the fluid is incompressible, then $\rho_1 = \rho_2$ and continuity equation (5.2) reduces to

$$A_1 V_1 = A_2 V_2 \qquad ...(5.3)$$

Problem 5.1 *The diameters of a pipe at the sections 1 and 2 are 10 cm and 15 cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 is 5 m/s. Determine also the velocity at section 2.*

Solution. Given :

At section 1, $D_1 = 10 \text{ cm} = 0.1 \text{ m}$

$$A_1 = \frac{\pi}{4}(D_1^2) = \frac{\pi}{4}(.1)^2 = 0.007854 \text{ m}^2$$

$$V_1 = 5 \text{ m/s.}$$

At section 2, $D_2 = 15 \text{ cm} = 0.15 \text{ m}$

$$A_2 = \frac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$$

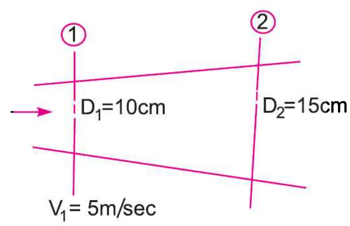


Fig. 5.2

(*i*) Discharge through pipe is given by equation (5.1)

or $$Q = A_1 \times V_1$$

$$= 0.007854 \times 5 = \mathbf{0.03927 \text{ m}^3/\text{s}}. \text{ **Ans.**}$$

Using equation (5.3), we have $A_1 V_1 = A_2 V_2$

(*ii*) $\therefore$ $$V_2 = \frac{A_1 V_1}{A_2} = \frac{0.007854}{0.01767} \times 5.0 = \mathbf{2.22 \text{ m/s.}} \text{ **Ans.**}$$

Problem 5.2 *A 30 cm diameter pipe, conveying water, branches into two pipes of diameters 20 cm and 15 cm respectively. If the average velocity in the 30 cm diameter pipe is 2.5 m/s, find the discharge in this pipe. Also determine the velocity in 15 cm pipe if the average velocity in 20 cm diameter pipe is 2 m/s.*

Solution. Given :

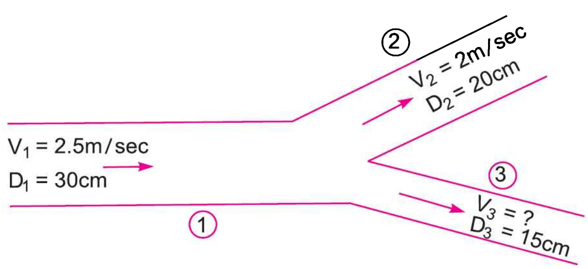


Fig. 5.3

$D_1 = 30 \text{ cm} = 0.30 \text{ m}$

$\therefore \quad A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times .3^2 = 0.07068 \text{ m}^2$

$V_1 = 2.5 \text{ m/s}$

$D_2 = 20 \text{ cm} = 0.20 \text{ m}$

$\therefore \quad A_2 = \frac{\pi}{4} (.2)^2 = \frac{\pi}{4} \times .4 = 0.0314 \text{ m}^2,$

$V_2 = 2 \text{ m/s}$

$D_3 = 15 \text{ cm} = 0.15 \text{ m}$

$\therefore \quad A_3 = \frac{\pi}{4} (.15)^2 = \frac{\pi}{4} \times 0.225 = 0.01767 \text{ m}^2$

Find (*i*) Discharge in pipe 1 or Q_1

(*ii*) Velocity in pipe of dia. 15 cm or V_3

Let Q_1, Q_2 and Q_3 are discharges in pipe 1, 2 and 3 respectively.

Then according to continuity equation

$$Q_1 = Q_2 + Q_3 \qquad \text{...(1)}$$

(*i*) **The discharge Q_1** in pipe 1 is given by

$Q_1 = A_1 V_1 = 0.07068 \times 2.5 \text{ m}^3/\text{s} = \mathbf{0.1767 \text{ m}^3/\text{s}}$. **Ans.**

(*ii*) **Value of V_3**

$Q_2 = A_2 V_2 = 0.0314 \times 2.0 = 0.0628 \text{ m}^3/\text{s}$

Substituting the values of Q_1 and Q_2 in equation (1)

$0.1767 = 0.0628 + Q_3$

$\therefore \quad Q_3 = 0.1767 - 0.0628 = 0.1139 \text{ m}^3/\text{s}$

But $\quad Q_3 = A_3 \times V_3 = 0.01767 \times V_3 \quad \text{or} \quad 0.1139 = 0.01767 \times V_3$

$\therefore \quad V_3 = \frac{0.1139}{0.01767} = \mathbf{6.44 \text{ m/s}}$. **Ans.**

Problem 5.3 *Water flows through a pipe AB 1.2 m diameter at 3 m/s and then passes through a pipe BC 1.5 m diameter. At C, the pipe branches. Branch CD is 0.8 m in diameter and carries one-third of the flow in AB. The flow velocity in branch CE is 2.5 m/s. Find the volume rate of flow in AB, the velocity in BC, the velocity in CD and the diameter of CE.*

Solution. Given :

Diameter of pipe *AB*, $D_{AB} = 1.2 \text{ m}$

Velocity of flow through *AB*, $V_{AB} = 3.0 \text{ m/s}$

Dia. of pipe *BC*, $D_{BC} = 1.5 \text{ m}$

Dia. of branched pipe *CD*, $D_{CD} = 0.8 \text{ m}$

Velocity of flow in pipe *CE*, $V_{CE} = 2.5 \text{ m/s}$

Let the flow rate in pipe $AB = Q \text{ m}^3/\text{s}$

Velocity of flow in pipe $BC = V_{BC} \text{ m/s}$

Velocity of flow in pipe $CD = V_{CD} \text{ m/s}$

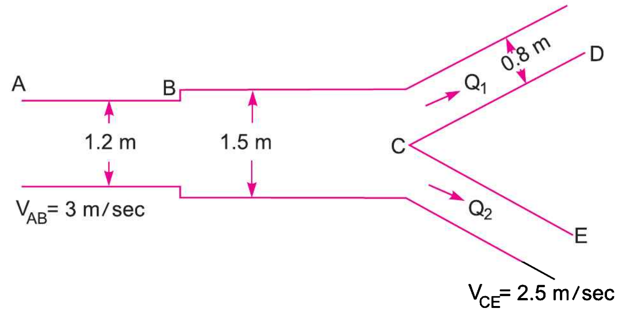


Fig. 5.4

Diameter of pipe $CE = D_{CE}$

Then flow rate through $CD = Q/3$

and flow rate through $CE = Q - Q/3 = \dfrac{2Q}{3}$

(*i*) Now volume flow rate through $AB = Q = V_{AB} \times \text{Area of } AB$

$$= 3.0 \times \frac{\pi}{4}(D_{AB})^2 = 3.0 \times \frac{\pi}{4}(1.2)^2 = \mathbf{3.393\ m^3/s.\ Ans.}$$

(*ii*) Applying continuity equation to pipe *AB* and pipe *BC*,

$$V_{AB} \times \text{Area of pipe } AB = V_{BC} \times \text{Area of pipe } BC$$

or $$3.0 \times \frac{\pi}{4}(D_{AB})^2 = V_{BC} \times \frac{\pi}{4}(D_{BC})^2$$

or $$3.0 \times (1.2)^2 = V_{BC} \times (1.5)^2 \qquad \left[\text{Divide by } \frac{\pi}{4}\right]$$

or $$V_{BC} = \frac{3 \times 1.2^2}{1.5^2} = \mathbf{1.92\ m/s.\ Ans.}$$

(*iii*) The flow rate through pipe

$$CD = Q_1 = \frac{Q}{3} = \frac{3.393}{3} = 1.131\ m^3/s$$

$\therefore$ $$Q_1 = V_{CD} \times \text{Area of pipe } CD \times \frac{\pi}{4}(D_{CD})^2$$

or $$1.131 = V_{CD} \times \frac{\pi}{4} \times 0.8^2 = 0.5026\ V_{CD}$$

$\therefore$ $$V_{CD} = \frac{1.131}{0.5026} = \mathbf{2.25\ m/s.\ Ans.}$$

(*iv*) Flow rate through *CE*,

$$Q_2 = Q - Q_1 = 3.393 - 1.131 = 2.262\ m^3/s$$

$\therefore$ $$Q_2 = V_{CE} \times \text{Area of pipe } CE = V_{CE}\frac{\pi}{4}(D_{CE})^2$$

or $$2.263 = 2.5 \times \frac{\pi}{4} \times (D_{CE})^2$$

or $$D_{CE} = \sqrt{\frac{2.263 \times 4}{2.5 \times \pi}} = \sqrt{1.152} = 1.0735\ m$$

$\therefore$ Diameter of pipe CE = **1.0735 m. Ans.**

Problem 5.4 *A 25 cm diameter pipe carries oil of sp. gr. 0.9 at a velocity of 3 m/s. At another section the diameter is 20 cm. Find the velocity at this section and also mass rate of flow of oil.*

Solution. Given :

at section 1, $D_1 = 25 \text{ cm} = 0.25 \text{ m}$

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.25^2 = 0.049 \text{ m}^3$$

$V_1 = 3$ m/s

at section 2, $D_2 = 20 \text{ cm} = 0.2 \text{ m}$

$$A_2 = \frac{\pi}{4} (0.2)^2 = 0.0314 \text{ m}^2$$

$V_2 = ?$

Mass rate of flow of oil = ?

Applying continuity equation at sections 1 and 2,

$$A_1V_1 = A_2V_2$$

or $0.049 \times 3.0 = 0.0314 \times V_2$

$$\therefore \quad V_2 = \frac{0.049 \times 3.0}{0.0314} = \textbf{4.68 m/s. Ans.}$$

Mass rate of flow of oil $= \text{Mass density} \times Q = \rho \times A_1 \times V_1$

Sp. gr. of oil $= \dfrac{\text{Density of oil}}{\text{Density of water}}$

$\therefore$ Density of oil = Sp. gr. of oil × Density of water

$$= 0.9 \times 1000 \text{ kg/m}^3 = \frac{900 \text{ kg}}{\text{m}^3}$$

$\therefore$ Mass rate of flow $= 900 \times 0.049 \times 3.0$ kg/s = **132.23 kg/s. Ans.**

Problem 5.5 *A jet of water from a 25 mm diameter nozzle is directed vertically upwards. Assuming that the jet remains circular and neglecting any loss of energy, that will be the diameter at a point 4.5 m above the nozzle, if the velocity with which the jet leaves the nozzle is 12 m/s.*

Solution. Given :

Dia. of nozzle, $D_1 = 25 \text{ mm} = 0.025 \text{ m}$

Velocity of jet at nozzle, $V_1 = 12$ m/s

Height of point A, $h = 4.5$ m

Let the velocity of the jet at a height 4.5 m = V_2

Consider the vertical motion of the jet from the outlet of the nozzle to the point A (neglecting any loss of energy).

Initial velocity, $u = V_1 = 12$ m/s

Final velocity, $V = V_2$

Value of $g = -9.81 \text{ m/s}^2$ and $h = 4.5$ m

Using, $V^2 - u^2 = 2gh$, we get

$$V_2^2 - 12^2 = 2 \times (-9.81) \times 4.5$$

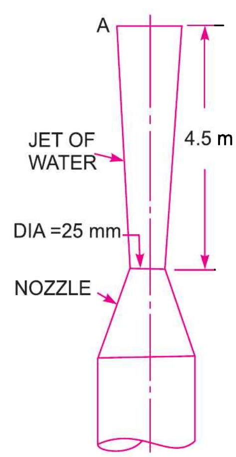


Fig. 5.5

$\therefore \quad V_2 = \sqrt{12^2 - 2 \times 9.81 \times 4.5} = \sqrt{144 - 88.29} = 7.46 \text{ m/s}$

Now applying continuity equation to the outlet of nozzle and at point A, we get

$$A_1 V_1 = A_2 V_2$$

or
$$A_2 = \frac{A_1 V_1}{V_2} = \frac{\frac{\pi}{4} D_1^2 \times V_1}{V_2} = \frac{\pi \times (0.025)^2 \times 12}{4 \times 7.46} = 0.0007896$$

Let D_2 = Diameter of jet at point A.

Then $A_2 = \frac{\pi}{2} D_2^2$ or $0.0007896 = \frac{\pi}{4} \times D_2^2$

$\therefore \quad D_2 = \sqrt{\frac{0.0007896 \times 4}{\pi}} = 0.0317 \text{ m} =$ **31.7 mm. Ans.**

▶ 5.6 CONTINUITY EQUATION IN THREE-DIMENSIONS

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z directions respectively. Mass of fluid entering the face $ABCD$ per second

$$= \rho \times \text{Velocity in } x\text{-direction} \times \text{Area of } ABCD$$
$$= \rho \times u \times (dy \times dz)$$

Then mass of fluid leaving the face $EFGH$ per second $= \rho u \, dydz + \frac{\partial}{\partial x} (\rho u \, dydz) \, dx$

$\therefore$ Gain of mass in x-direction

$$= \text{Mass through } ABCD - \text{Mass through } EFGH \text{ per second}$$
$$= \rho u \, dydz - \rho u \, dydz - \frac{\partial}{\partial x} (\rho u \, dydz) dx$$
$$= -\frac{\partial}{\partial x} (\rho u \, dydz) \, dx$$
$$= -\frac{\partial}{\partial x} (\rho u) \, dx \, dydz \qquad \{\because dydz \text{ is constant}\}$$

Similarly, the net gain of mass in y-direction

$$= -\frac{\partial}{\partial y} (\rho v) \, dxdydz$$

and in z-direction
$$= -\frac{\partial}{\partial z} (\rho w) \, dxdydz$$

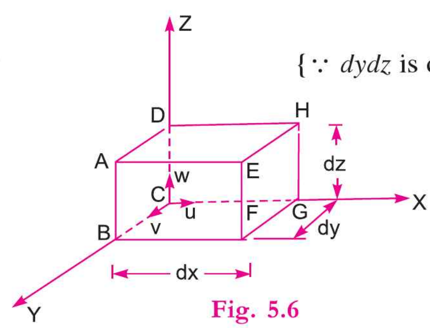


Fig. 5.6

$$\therefore \quad \text{Net gain of masses} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz$$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass of fluid in the element is $\rho.\ dx.\ dy.\ dz$ and its rate of increase with time is $\frac{\partial}{\partial t}(\rho\ dx.\ dy.\ dz)$ or

$\frac{\partial \rho}{\partial t}.\ dx\ dy\ dz.$

Equating the two expressions,

or $$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t}.\ dxdydz$$

or $$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0 \text{ [Cancelling } dx.dy.dz \text{ from both sides]} \quad ...(5.3A)$$

Equation (5.3*A*) is the continuity equation in cartesian co-ordinates in its most general form. This equation is applicable to :

(*i*) Steady and unsteady flow,
(*ii*) Uniform and non-uniform flow, and
(*iii*) Compressible and incompressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.3*A*) becomes as

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0 \quad ...(5.3B)$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \quad ...(5.4)$$

Equation (5.4) is the continuity equation in three-dimensions. For a two-dimensional flow, the component $w = 0$ and hence continuity equation becomes as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \quad ...(5.5)$$

5.6.1 Continuity Equation in Cylindrical Polar Co-ordinates. The continuity equation in cylindrical polar co-ordinates (*i.e.*, r, θ, z co-ordinates) is derived by the procedure given below.

Consider a two-dimensional incompressible flow field. The two-dimensional polar co-ordinates are r and θ. Consider a fluid element *ABCD* between the radii r and $r + dr$ as shown in Fig. 5.7. The angle subtended by the element at the centre is $d\theta$. The components of the velocity V are u_r in the radial direction and u_θ in the tangential direction. The sides of the element are having the lengths as

Side $AB = rd\theta$, $BC = dr$, $DC = (r + dr)\, d\theta$, $AD = dr$.

The thickness of the element perpendicular to the plane of the paper is assumed to be unity.

Consider the flow in radial direction

Mass of fluid entering the face AB per unit time

$= \rho \times$ Velocity in r-direction $\times$ Area

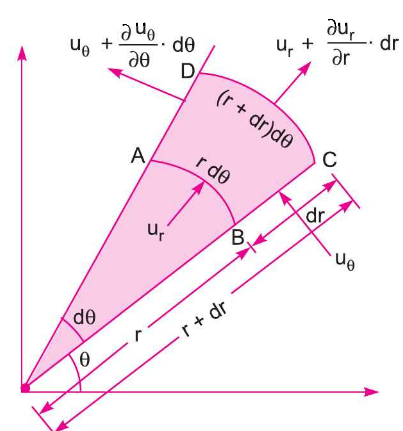


Fig. 5.7

$= \rho \times u_r \times (AB \times 1)$ $\quad$ ($\because$ Area $= AB \times$ Thickness $= rd\theta \times 1$)

$= \rho \times u_r \times (rd\theta \times 1) = \rho \,.\, u_r.\, rd\theta$

Mass of fluid leaving the face CD per unit time

$= \rho \times$ Velocity $\times$ Area

$$= \rho \times \left(u_r + \frac{\partial u_r}{\partial r}.dr\right) \times (CD \times 1) \quad (\because \text{ Area} = CD \times 1)$$

$$= \rho \times \left(u_r + \frac{\partial u_r}{\partial r}\, dr\right) \times (r + dr)d\theta \quad [\because CD = (r + dr)\, d\theta]$$

$$= \rho \times \left[u_r \times r + u_r dr + r\frac{\partial u_r}{\partial r}\, dr + \frac{\partial u_r}{\partial r}(dr)^2\right] d\theta$$

$$= \rho\left[u_r \times r + u_r \times dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

[The term containing $(dr)^2$ is very small and has been neglected]

$\therefore$ Gain of mass in r-direction per unit time

= (Mass through AB – Mass through CD) per unit time

$$= \rho.\, u_r.\, rd\theta - \rho\left[u_r.r + u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

$$= \rho.\, u_r.\, rd\theta - \rho.\, u_r.\, r.\, d\theta - \rho\left[u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

$$= -\,\rho\left[u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right].\, d\theta$$

$$= -\,\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] r.\, dr.\, d\theta$$

[This is written in this form because ($r.\ d\theta.\ dr.\ 1$) is equal to volume of element]

Now consider the flow in θ-*direction*

Gain in mass in θ-direction per unit time

= (Mass through BC – Mass through AD) per unit time

= [$\rho \times$ Velocity through $BC \times$ Area – $\rho \times$ Velocity through $AD \times$ Area]

$$= \left[\rho.u_\theta.dr \times 1 - \rho\left(u_\theta + \frac{\partial u_\theta}{\partial \theta}.d\theta\right) \times dr \times 1\right]$$

$$= -\,\rho\left(\frac{\partial u_\theta}{\partial \theta}.d\theta\right) dr \times 1 \quad (\because \text{ Area} = dr \times 1)$$

$$= -\,\rho\frac{\partial u_\theta}{\partial \theta}.\frac{r.\,d\theta.\,dr}{r} \quad [\text{Multiplying and dividing by } r]$$

$\therefore$ Total gain in fluid mass per unit time

$$= -\,\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right].\, r.\, dr.\, d\theta - \rho\frac{\partial u_\theta}{\partial \theta}.\frac{rd\theta.\,dr}{r} \quad \text{...(5.5}A\text{)}$$

But mass of fluid element $= \rho \times$ Volume of fluid element

$= \rho \times [rd\theta \times dr \times 1]$

$= \rho \times rd\theta \,.\, dr$

Rate of increase of fluid mass in the element with time

$$= \frac{\partial}{\partial t} [\rho \,.\, rd\theta \,.\, dr] = \frac{\partial \rho}{\partial t} \,.\, rd\theta\, dr \qquad ...(5.5B)$$

($\because$ $rd\theta \,.\, dr \,.\, 1$ is the volume of element and is a constant quantity)

Since the mass is neither created nor destroyed in the fluid element, hence net gain of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element.

Hence equating the two expressions given by equations (5.5 *A*) and (5.5 *B*), we get

$$-\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] r \,.\, dr \,.\, d\theta - \rho \frac{\partial u_\theta}{\partial \theta} \frac{rd\theta \,.\, dr}{r} = \frac{\partial \rho}{\partial t}\, rd\theta\, dr$$

or
$$-\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] - \rho \frac{\partial u_\theta}{\partial \theta} \,.\, \frac{1}{r} = \frac{\partial \rho}{\partial t} \qquad \text{[Cancelling } rdr \,.\, d\theta \text{ from both sides]}$$

or
$$\frac{\partial \rho}{\partial t} + \rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] + \rho \frac{\partial u_\theta}{\partial \theta} \,.\, \frac{1}{r} = 0 \qquad ...(5.5C)$$

Equation (5.5 *C*) is the continuity equation in polar co-ordinates for two-dimensional flow.

For steady flow $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.5 *C*) reduces to

$$\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] + \rho \frac{\partial u_\theta}{\partial \theta} \,.\, \frac{1}{r} = 0$$

or
$$\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} \,.\, \frac{1}{r} = 0$$

or
$$u_r + r\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} = 0$$

or
$$\frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial \theta}(u_\theta) = 0 \qquad \left[\because \frac{\partial}{\partial r}(r \,.\, u_r) = r \,.\, \frac{\partial u_r}{\partial r} + u_r\right] \qquad ...(5.5D)$$

Equation (5.5 *D*) represents the continuity equation in polar co-ordinates for two-dimensional steady incompressible flow.

Problem 5.5A *Examine whether the following velocity components represent a physically possible flow ?*

$$u_r = r \sin \theta,\ u_\theta = 2r \cos \theta.$$

Solution. Given : $u_r = r \sin \theta$ and $u_\theta = 2r \cos \theta$

For physically possible flow, the continuity equation,

$$\frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial \theta}(u_\theta) = 0 \text{ should be satisfied.}$$

Now $u_r = r \sin \theta$

Multiplying the above equation by r, we get

$$ru_r = r^2 \sin \theta$$

Differentiating the preceding equation w.r.t. r, we get

$$\frac{\partial}{\partial r}(ru_r) = \frac{\partial}{\partial r}(r^2 \sin\theta)$$

$$= 2r\sin\theta \qquad (\because \ \sin\theta \text{ is constant w.r.t. } r)$$

Now $u_\theta = 2r\cos\theta$

Differentiating the above equation w.r.t. θ, we get

$$\frac{\partial}{\partial \theta}(u_\theta) = \frac{\partial}{\partial \theta}(2r\cos\theta)$$

$$= 2r(-\sin\theta) \qquad (\because \ 2r \text{ is constant w.r.t. } \theta)$$

$$= -2r\sin\theta$$

$$\therefore \quad \frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial \theta}(u_\theta) = 2r\sin\theta - 2r\sin\theta = 0$$

Hence the continuity equation is satisfied. Hence the given velocity components represent a physically possible flow.

▶ 5.7 VELOCITY AND ACCELERATION

Let V is the resultant velocity at any point in a fluid flow. Let u, v and w are its component in x, y and z directions. The velocity components are functions of space-co-ordinates and time. Mathematically, the velocity components are given as

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

and Resultant velocity, $V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x, a_y and a_z are the **total acceleration** in x, y and z directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But $\frac{dx}{dt} = u, \frac{dy}{dt} = v$ and $\frac{dz}{dt} = w$

$$\left.\begin{aligned} \therefore \quad a_x &= \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \\ \text{Similarly,} \quad a_y &= \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\ a_z &= \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \end{aligned}\right] \quad \text{...(5.6)}$$

For steady flow, $\frac{\partial V}{\partial t} = 0$, where V is resultant velocity or $$\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0 \text{ and} \frac{\partial w}{\partial t} = 0$$

Hence acceleration in x, y and z directions becomes

$$\left.\begin{aligned} a_x &= \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right] \quad \text{...(5.7)}$$

Acceleration vector $$\left.\begin{aligned} A &= a_x i + a_y j + a_z k \\ &= \sqrt{a_x^2 + a_y^2 + a_z^2}. \end{aligned}\right] \quad \text{...(5.8)}$$

5.7.1 Local Acceleration and Convective Acceleration. **Local acceleration** is defined as the rate of increase of velocity with respect to time at a given point in a flow field. In the equation given by (5.6), the expression $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ or $\frac{\partial w}{\partial t}$ is known as local acceleration.

Convective acceleration is defined as the rate of change of velocity due to the change of position of fluid particles in a fluid flow. The expressions other than $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ in equation (5.6) are known as convective acceleration.

Problem 5.6 *The velocity vector in a fluid flow is given*

$$V = 4x^3 i - 10x^2 y j + 2tk.$$

Find the velocity and acceleration of a fluid particle at (2, 1, 3) at time t = 1.

Solution. The velocity components u, v and w are $u = 4x^3$, $v = -10x^2 y$, $w = 2t$

For the point (2, 1, 3), we have $x = 2$, $y = 1$ and $z = 3$ at time $t = 1$.

Hence velocity components at (2, 1, 3) are

$$u = 4 \times (2)^3 = 32 \text{ units}$$

$$v = -10(2)^2(1) = -40 \text{ units}$$

$$w = 2 \times 1 = 2 \text{ units}$$

∴ Velocity vector V at (2, 1, 3) = $32i - 40j + 2k$

or Resultant velocity $= \sqrt{u^2 + v^2 + w^2}$

$$= \sqrt{32^2 + (-40)^2 + 2^2} = \sqrt{1024 + 1600 + 4} = \textbf{51.26 units. Ans.}$$

Acceleration is given by equation (5.6)

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Now from velocity components, we have

$$\frac{\partial u}{\partial x} = 12x^2, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = -20xy, \frac{\partial v}{\partial y} = -10x^2, \frac{\partial v}{\partial z} = 0\ \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = 0, \frac{\partial w}{\partial y} = 0, \frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 2.1$$

Substituting the values, the acceleration components at (2, 1, 3) at time $t = 1$ are

$$a_x = 4x^3(12x^2) + (-10x^2y)(0) + 2t \times (0) + 0$$
$$= 48x^5 = 48 \times (2)^5 = 48 \times 32 = 1536 \text{ units}$$
$$a_y = 4x^3(-20xy) + (-10x^2y)(-10x^2) + 2t(0) + 0$$
$$= -80x^4y + 100x^4y$$
$$= -80(2)^4(1) + 100(2)^4 \times 1 = -1280 + 1600 = 320 \text{ units.}$$
$$a_z = 4x^3(0) + (-10x^2y)(0) + (2t)(0) + 2.1 = 2.0 \text{ units}$$

∴ Acceleration is $A = a_x i + a_y j + a_z k =$ **1536i + 320j + 2k. Ans.**

or Resultant $A = \sqrt{(1536)^2 + (320)^2 + (2)^2}$ units

$= \sqrt{2359296 + 102400 + 4}$ = **1568.9 units. Ans.**

Problem 5.7 *The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation :*

(i) $u = x^2 + y^2 + z^2$; $v = xy^2 - yz^2 + xy$

(ii) $v = 2y^2$, $w = 2xyz$.

Solution. The continuity equation for incompressible fluid is given by equation (5.4) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Case I. $u = x^2 + y^2 + z^2$ ∴ $\frac{\partial u}{\partial x} = 2x$

$v = xy^2 - yz^2 + xy$ ∴ $\frac{\partial v}{\partial y} = 2xy - z^2 + x$

Substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ in continuity equation.

$$2x + 2xy - z^2 + x + \frac{\partial w}{\partial z} = 0$$

or $\frac{\partial w}{\partial z} = -3x - 2xy + z^2$ or $\partial w = (-3x - 2xy + z^2)\,\partial z$

Integration of both sides gives $\int dw = \int\left(-3x - 2xy + z^2\right) dz$

or $$w = \left(-3xz - 2xyz + \frac{z^3}{3}\right) + \text{Constant of integration,}$$

where constant of integration cannot be a function of z. But it can be a function of x and y that is $f(x, y)$.

$$\therefore \quad w = \left(\mathbf{-3xz - 2xyz + \frac{z^3}{3}}\right) + \mathbf{f(x, y)}. \textbf{ Ans.}$$

Case II. $$v = 2y^2 \quad \therefore \quad \frac{\partial v}{\partial y} = 4y$$

$$w = 2xyz \quad \therefore \quad \frac{\partial w}{\partial z} = 2xy$$

Substituting the values of $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ in continuity equation, we get

$$\frac{\partial u}{\partial x} + 4y + 2xy = 0$$

or $$\frac{\partial u}{\partial x} = -4y - 2xy \text{ or } du = (-4y - 2xy)\, dx$$

Integrating, we get $$u = -4xy - 2y\frac{x^2}{2} + f(y, z) = \mathbf{-4xy - x^2y + f(y, z)}. \textbf{ Ans.}$$

Problem 5.8 *A fluid flow field is given by*

$$V = x^2yi + y^2zj - (2xyz + yz^2)k$$

Prove that it is a case of possible steady incompressible fluid flow. Calculate the velocity and acceleration at the point (2, 1, 3).

Solution. For the given fluid flow field $u = x^2y \quad \therefore \quad \frac{\partial u}{\partial x} = 2xy$

$$v = y^2z \quad \therefore \quad \frac{\partial v}{\partial y} = 2yz$$

$$w = -2xyz - yz^2 \quad \therefore \quad \frac{\partial w}{\partial z} = -2xy - 2yz.$$

For a case of possible steady incompressible fluid flow, the continuity equation (5.4) should be satisfied.

i.e., $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Substituting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2xy + 2yz - 2xy - 2yz = 0$$

Hence the velocity field $V = x^2yi + y^2zj - (2xyz + yz^2)\ k$ is a possible case of fluid flow. **Ans.**

Velocity at (2, 1, 3)

Substituting the values $x = 2,\ y = 1$ and $z = 3$ in velocity field, we get

$$V = x^2yi + y^2zj - (2xyz + yz^2)\ k$$
$$= 2^2 \times 1i + 1^2 \times 3j - (2 \times 2 \times 1 \times 3 + 1 \times 3^2)\ k$$
$$= \mathbf{4i + 3j - 21k}.\ \textbf{Ans.}$$

and Resultant velocity $= \sqrt{4^2 + 3^2 + (-21)^2} = \sqrt{16 + 9 + 441} = \sqrt{466} =$ **21.587 units. Ans.**

Acceleration at (2, 1, 3)

The acceleration components a_x, a_y and a_z for steady flow are

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$u = x^2y, \frac{\partial u}{\partial x} = 2xy, \frac{\partial u}{\partial y} = x^2 \text{ and } \frac{\partial u}{\partial z} = 0$$

$$v = y^2z, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 2yz, \frac{\partial v}{\partial z} = y^2$$

$$w = -2xyz - yz^2, \frac{\partial w}{\partial x} = -2yz, \frac{\partial w}{\partial y} = -2xz - z^2, \frac{\partial w}{\partial z} = -2xy - 2yz.$$

Substituting these values in acceleration components, we get acceleration at (2, 1, 3)

$$a_x = x^2y\ (2xy) + y^2z\ (x)^2 - (2xyz + yz^2)\ (0)$$
$$= 2x^3y^2 + x^2y^2z$$
$$= 2\ (2)^3 1^2 + 2^2 \times 1^2 \times 3 = 2 \times 8 + 12$$
$$= 16 + 12 = 28 \text{ units}$$

$$a_y = x^2y\ (0) + y^2z\ (2yz) - (2xyz + yz^2)\ (y^2)$$
$$= 2y^3z^2 - 2xy^3z - y^3z^2$$
$$= 2 \times 1^3 \times 3^2 - 2 \times 2 \times 1^3 \times 3 - 1^3 \times 3^2 = 18 - 12 - 9 = -3 \text{ units}$$

$$a_z = x^2y\ (-2yz) + y^2z\ (-2xz - z^2) - (2xyz + yz^2)\ (-2xy - 2yz)$$
$$= -2x^2y^2z - 2xy^2z^2 - y^2z^3 + [4x^2y^2z + 2xy^2z^2 + 4xy^2z^2 + 2y^2z^3]$$
$$= -2 \times 2^2 \times 1^2 \times 3 - 2 \times 2 \times 1^2 \times 3^2 - 1^2 \times 3^3$$
$$+ [4 \times 2^2 \times 1^2 \times 3 + 2 \times 2 \times 1^2 \times 3^2 + 4 \times 2 \times 1^2 \times 3^2 + 2 \times 1^2 \times 3^3]$$
$$= -24 - 36 - 27 + [48 + 36 + 72 + 54]$$
$$= -24 - 36 - 27 + 48 + 36 + 72 + 54 = 123$$

∴ Acceleration $= a_xi + a_yj + a_zk =$ **28i – 3j + 123k. Ans.**

or Resultant acceleration $= \sqrt{28^2 + (-3)^2 + 123^2} = \sqrt{784 + 9 + 15129}$

$= \sqrt{15922}$ = **126.18 units. Ans.**

Problem 5.9 *Find the convective acceleration at the middle of a pipe which converges uniformly from 0.4 m diameter to 0.2 m diameter over 2 m length. The rate of flow is 20 lit/s. If the rate of flow changes uniformly from 20 l/s to 40 l/s in 30 seconds, find the total acceleration at the middle of the pipe at 15th second.*

Solution. Given :

Diameter at section 1, $D_1 = 0.4$ m ; $D_2 = 0.2$ m, $L = 2$ m, $Q = 20$ *l*/s $= 0.02$ m^3/s as one litre $= 0.001\text{m}^3 = 1000\text{ cm}^3$

Find (*i*) Convective acceleration at middle *i.e.*, at *A* when $Q = 20$ *l*/s.

(*ii*) Total acceleration at *A* when *Q* changes from 20 *l*/s to 40 *l*/s in 30 seconds.

Case I. In this case, the rate of flow is constant and equal to 0.02 m^3/s. The velocity of flow is in *x*-direction only. Hence this is one-dimensional flow and velocity components in *y* and *z* directions are zero or $v = 0$, $z = 0$.

$\therefore$ Convective acceleration $= u\dfrac{\partial u}{\partial y}$ only ...(*i*)

Let us find the value of *u* and $\dfrac{\partial u}{\partial x}$ at a distance *x* from inlet

The diameter (D_x) at a distance *x* from inlet or at section *X-X* is given by,

$$D_x = 0.4 - \frac{0.4 - 0.2}{2} \times x$$

$$= (0.4 - 0.1\,x)\text{ m}$$

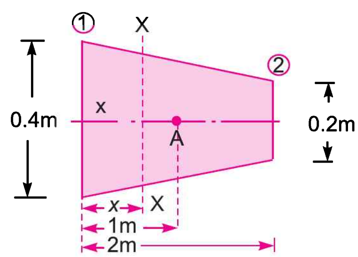


Fig. 5.8

The area of cross-section (A_x) at section *X-X* is given by,

$$A_x = \frac{\pi}{4} D_x^2 = \frac{\pi}{4}(0.4 - 0.1\,x)^2$$

Velocity (*u*) at the section *X-X* in terms of *Q* (*i.e.*, in terms of rate of flow)

$$u = \frac{Q}{\text{Area}} = \frac{Q}{A_x} = \frac{Q}{\frac{\pi}{4}D_x^2} = \frac{4Q}{\pi\,(0.4 - 0.1x)^2}$$

$$= \frac{1.273Q}{(0.4 - 0.1x)^2} = 1.273\,Q\,(0.4 - 0.1\,x)^{-2}\text{ m/s} \qquad ...(ii)$$

To find $\dfrac{\partial u}{\partial x}$, we must differentiate equation (*ii*) with respect to *x*.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[1.273\,Q\,(0.4 - 0.1\,x)^{-2}]$$

$$= 1.273\,Q\,(-2)\,(0.4 - 0.1\,x)^{-1} \times (-0.1) \qquad \text{[Here Q is constant]}$$

$$= 0.2546\,Q\,(0.4 - 0.1\,x)^{-1} \qquad ...(iii)$$

Substituting the value of *u* and $\dfrac{\partial u}{\partial x}$ in equation (*i*), we get

Convective acceleration $= [1.273\,Q\,(0.4 - 0.1\,x)^{-2}] \times [0.2546\,Q\,(0.4 - 0.1\,x)^{-1}]$

$= 1.273 \times 0.2546 \times Q^2 \times (0.4 - 0.1\,x)^{-3}$

$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.4 - 0.1\,x)^{-3}$ $[\because Q = 0.02 \text{ m}^3/\text{s}]$

$\therefore$ Convective acceleration at the middle (where $x = 1$ m)

$$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.4 - 0.1 \times 1)^{-3} \text{ m/s}^2$$
$$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.3)^{-3} \text{ m/s}^2$$
$$= \mathbf{0.0048 \text{ m/s}^2}. \textbf{ Ans.}$$

Case II. When Q changes from 0.02 m^3/s to 0.04 m^3/s in 30 seconds, find the total acceleration at $x = 1$ m and $t = 15$ seconds.

Total acceleration = Convective acceleration + Local acceleration at $t = 15$ seconds.

The rate of flow at $t = 15$ seconds is given by

$$Q = Q_1 + \frac{Q_2 - Q_1}{30} \times 15 \text{ where } Q_2 = 0.04 \text{ m}^3/\text{s and } Q_1 = 0.02 \text{ m}^3/\text{s}$$

$$= 0.02 + \frac{(0.04 - 0.02)}{30} \times 15 = 0.03 \text{ m}^3/\text{s}$$

The velocity (u) and gradient $\left(\frac{\partial u}{\partial x}\right)$ in terms of Q are given by equations (ii) and (iii) respectively

$\therefore$ Convective acceleration $= u.\dfrac{\partial u}{\partial x}$

$$= [1.273\, Q\, (0.4 - 0.1\, x)^{-2}] \times [0.2546\, Q\, (0.4 - 0.1\, x)^{-1}]$$
$$= 1.273 \times 0.2546\, Q^2 \times (0.4 - 0.1 \times 1)^{-3}$$

$\therefore$ Convective acceleration (when $Q = 0.03$ m^3/s and $x = 1$ m)

$$= 1.273 \times 0.2546 \times (0.03)^2 \times (0.4 - 0.1 \times 1)^{-3}$$
$$= 1.273 \times 0.2546 \times (0.03)^2 \times (0.3)^{-3} \text{ m/s}^2$$
$$= 0.0108 \text{ m/s}^2 \qquad \ldots(iv)$$

$$\text{Local acceleration} = \frac{\partial u}{\partial t} = \frac{\partial}{\partial t}[1.273\, Q\, (0.4 - 0.1\, x)^{-2}]$$

$[\because u$ from equation (ii) is $u = 1.273\, Q\, (0.4 - 0.1\, x)^{-2}]$

$$= 1.273 \times (0.4 - 0.1\, x)^{-2} \times \frac{\partial Q}{\partial t}$$

[$\because$ Local acceleration is at a point where x is constant but Q is changing]

Local acceleration (at $x = 1$ m)

$$= 1.273 \times (0.4 - 0.1 \times 1)^{-2} \times \frac{\partial Q}{\partial t}$$

$$= 1.273 \times (0.3)^{-2} \times \frac{0.02}{30} \qquad \left[\because \frac{\partial Q}{\partial t} = \frac{Q_2 - Q_1}{t} = \frac{0.04 - 0.02}{30} = \frac{0.02}{30}\right]$$

$$= 0.00943 \text{ m/s}^2 \qquad \ldots(v)$$

Hence adding equations (iv) and (v), we get total acceleration.

$\therefore$ Total acceleration = Convective acceleration + Local acceleration

$$= 0.0108 + 0.00943 = \mathbf{0.02023 \text{ m/s}^2}. \textbf{ Ans.}$$

▶ 5.8 VELOCITY POTENTIAL FUNCTION AND STREAM FUNCTION

5.8.1 Velocity Potential Function. It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi). Mathematically, the velocity, potential is defined as $\phi = f(x, y, z)$ for steady flow such that

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\} \quad \text{...(5.9)}$$

where u, v and w are the components of velocity in x, y and z directions respectively.

The velocity components in cylindrical polar co-ordinates in terms of velocity potential function are given by

$$\left.\begin{aligned} u_r &= \frac{\partial \phi}{\partial r} \\ u_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} \end{aligned}\right\} \quad \text{...(5.9}A\text{)}$$

where u_r = velocity component in radial direction (*i.e.*, in r direction)

and u_θ = velocity component in tangential direction (*i.e.*, in θ direction)

The continuity equation for an incompressible steady flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$

Substituting the values of u, v and w from equation (5.9), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad \text{...(5.10)}$$

Equation (5.10) is a Laplace equation.

For two-dimension case, equation (5.10) reduces to $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$ ...(5.11)

If any value of ϕ that satisfies the Laplace equation, will correspond to some case of fluid flow.

Properties of the Potential Function. The rotational components* are given by

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

* Please, refer to equation (5.17) on page 192.

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

Substituting the values, of u, v and w from equation (5.9) in the above rotational components, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \phi}{\partial x \partial z}\right]$$

and
$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y}\right]$$

If ϕ is a continuous function, then $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$; $\frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z}$; etc.

$\therefore$
$$\omega_z = \omega_y = \omega_x = 0.$$

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are :

1. If velocity potential (ϕ) exists, the flow should be irrotational.

2. If velocity potential (ϕ) satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

5.8.2 Stream Function. It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (*Psi*) and defined only for two-dimensional flow. Mathematically, for steady flow it is defined as $\psi = f(x, y)$ such that

$$\left.\begin{aligned} \frac{\partial \psi}{\partial x} &= v \\ \text{and} \quad \frac{\partial \psi}{\partial y} &= -u \end{aligned}\right\} \quad \text{...(5.12)}$$

The velocity components in cylindrical polar co-ordinates in terms of stream function are given as

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \text{ and } u_\theta = -\frac{\partial \psi}{\partial r} \quad \text{...(5.12A)}$$

where u_r= radial velocity and u_θ = tangential velocity

The continuity equation for two-dimensional flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

Substituting the values of u and v from equation (5.12), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=0 \text{ or } -\frac{\partial^2 \psi}{\partial x \partial y}+\frac{\partial^2 \psi}{\partial x \partial y}=0.$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component ω_z is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$.

Substituting the values of u and v from equation (5.12) in the above rotational component, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right]=\frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}\right]$$

For irrotational flow, $\omega_z = 0$. Hence above equation becomes as $\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial \psi}{\partial y^2}=0$

which is Laplace equation for ψ.

The **properties** of stream function (ψ) are :

1. If stream function (ψ) exists, it is a possible case of fluid flow which may be rotational or irrotational.
2. If stream function (ψ) satisfies the Laplace equation, it is a possible case of an irrotational flow.

5.8.3 Equipotential Line. A line along which the velocity potential ϕ is constant, is called equipotential line.

For equipotential line $\phi = \text{Constant}$

$\therefore$ $d\psi = 0$

But $\phi = f(x, y)$ for steady flow

$$\therefore \quad d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$$= -\,udx - vdy \qquad \left\{\because \frac{\partial \phi}{\partial x} = -u, \frac{\partial \phi}{\partial x} = -v\right\}$$

$$= -\,(udx + vdy).$$

For equipotential line, $d\phi = 0$

or $-(udx + vdy) = 0$ or $udx + vdy = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{u}{v} \qquad \text{...(5.13)}$$

But $\frac{dy}{dx}$ = Slope of equipotential line.

5.8.4 Line of Constant Stream Function

$\psi = \text{Constant}$

$\therefore$ $d\psi = 0$

$$\text{But} \quad d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = +\,vdx - udy \qquad \left\{\because \frac{\partial \psi}{\partial x} = v\,;\, \frac{\partial \psi}{\partial y} = -u\right\}$$

For a line of constant stream function

$$= d\psi = 0 \text{ or } vdx - udy = 0$$

or
$$\frac{dy}{dx} = \frac{v}{u} \quad ...(5.14)$$

But $\frac{dy}{dx}$ is slope of stream line.

From equations (5.13) and (5.14) it is clear that the product of the slope of the equipotential line and the slope of the stream line at the point of intersection is equal to – 1. Thus the equipotential lines are othogonal to the stream lines at all points of intersection.

5.8.5 Flow Net. A grid obtained by drawing a series of equipotential lines and stream lines is called a flow net. The flow net is an important tool in analysing two-dimensional irrotational flow problems.

5.8.6 Relation between Stream Function and Velocity Potential Function

From equation (5.9),

we have
$$u = -\frac{\partial \phi}{\partial x} \text{ and } v = -\frac{\partial \phi}{\partial y}$$

From equation (5.12), we have $u = -\frac{\partial \psi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$

Thus, we have
$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

Hence
$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \text{and} \quad \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \quad ...(5.15)$$

Problem 5.10 *The velocity potential function (ϕ) is given by an expression*

$$\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3 y}{3} + y^2$$

(*i*) *Find the velocity components in x and y direction.*
(*ii*) *Show that ϕ represents a possible case of flow.*

Solution. Given :
$$\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3 y}{3} + y^2$$

The partial derivatives of ϕ w.r.t. x and y are

$$\frac{\partial \phi}{\partial x} = -\frac{y^3}{3} - 2x + \frac{3x^2 y}{3} \quad ...(1)$$

and
$$\frac{\partial \phi}{\partial y} = -\frac{3xy^2}{3} + \frac{x^3}{3} + 2y \quad ...(2)$$

(*i*) The velocity components u and v are given by equation (5.9)

$$u = -\frac{\partial \phi}{\partial x} = -\left[-\frac{y^3}{3} - 2x + \frac{3x^2 y}{3}\right] = \frac{y^3}{3} + 2x - x^2 y$$

$$\therefore \qquad u = \mathbf{\frac{y^2}{3} + 2x - x^2 y. \text{ Ans.}}$$

$$\therefore \qquad v = -\frac{\partial \phi}{\partial y} = -\left[-\frac{3xy^2}{3} + \frac{x^3}{3} + 2y\right] = \frac{3xy^2}{3} - \frac{x^3}{3} - 2y = \mathbf{xy^2 - \frac{x^3}{3} - 2y.}$$

Ans.

(*ii*) The given value of ϕ, will represent a possible case of flow if it satisfies the Laplace equation, *i.e.*,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

From equations (1) and (2), we have

Now $\qquad \dfrac{\partial \phi}{\partial x} = -y^3/3 - 2x + x^2 y$

$$\therefore \qquad \frac{\partial^2 \phi}{\partial x^2} = -2 + 2xy$$

and $\qquad \dfrac{\partial \phi}{\partial y} = -xy^2 + \dfrac{x^3}{3} + 2y$

$$\therefore \qquad \frac{\partial^2 \phi}{\partial y^2} = -2xy + 2$$

$$\therefore \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (-2 + 2xy) + (-2xy + 2) = 0$$

$\therefore$ Laplace equation is satisfied and hence ϕ represent a possible case of flow. **Ans.**

Problem 5.11 *The velocity potential function is given by $\phi = 5(x^2 - y^2)$. Calculate the velocity components at the point (4, 5).*

Solution. $\qquad \phi = 5(x^2 - y^2)$

$$\therefore \qquad \frac{\partial \phi}{\partial x} = 10x$$

$$\frac{\partial \phi}{\partial y} = -10y.$$

But velocity components u and v are given by equation (5.9) as

$$u = -\frac{\partial \phi}{\partial x} = -10x$$

$$v = -\frac{\partial \phi}{\partial y} = -(-10y) = 10y$$

The velocity components at the point (4, 5), *i.e.*, at $x = 4$, $y = 5$

$u = -10 \times 4 =$ **– 40 units. Ans.**

$v = 10 \times 5 =$ **50 units. Ans.**

Problem 5.12 *A stream function is given by ψ = 5x – 6y.*
Calculate the velocity components and also magnitude and direction of the resultant velocity at any point.

Solution. $\psi = 5x - 6y$

$$\therefore \quad \frac{\partial \psi}{\partial x} = 5 \text{ and } \frac{\partial \psi}{\partial y} = -6.$$

But the velocity components u and v in terms of stream function are given by equation (5.12) as

$$u = -\frac{\partial \psi}{\partial y} = -(-6) = \mathbf{6 \text{ units/sec. Ans.}}$$

$$v = \frac{\partial \psi}{\partial x} = \mathbf{5 \text{ units/sec. Ans.}}$$

Resultant velocity $= \sqrt{u^2 + v^2} = \sqrt{6^2 + 5^2} = \sqrt{36 + 25} = \sqrt{61} = 7.81$ unit/sec

Direction is given by, $\tan \theta = \frac{v}{u} = \frac{5}{6} = 0.833$

$$\therefore \quad \theta = \tan^{-1} .833 = \mathbf{39° \ 48'. \ Ans.}$$

Problem 5.13 *If for a two-dimensional potential flow, the velocity potential is given by*

$$\phi = x(2y - 1)$$

determine the velocity at the point P (4, 5). Determine also the value of stream function ψ at the point P.

Solution. Given : $\phi = x(2y - 1)$

(*i*) The velocity components in the direction of x and y are

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}[x(2y - 1)] = -[2y - 1] = 1 - 2y$$

$$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}[x(2y - 1)] = -[2x] = -2x$$

At the point P (4, 5), *i.e.*, at $x = 4$, $y = 5$

$$u = 1 - 2 \times 5 = -9 \text{ units/sec}$$

$$v = -2 \times 4 = -8 \text{ units/sec}$$

∴ Velocity at P $= -9i - 8j$

or Resultant velocity at P $= \sqrt{9^2 + 8^2} = \sqrt{81 + 64} = 12.04$ units/sec $=$ **12.04 units/sec. Ans.**

(*ii*) **Value of Stream Function at P**

We know that $\frac{\partial \psi}{\partial y} = -u = -(1 - 2y) = 2y - 1$ ...(*i*)

and $\frac{\partial \psi}{\partial x} = v = -2x$ ...(*ii*)

Integrating equation (*i*) w.r.t. '*y*', we get

$$\int d\psi = \int (2y - 1)\, dy \text{ or } \psi = \frac{2y^2}{2} - y + \text{Constant of integration.}$$

The constant of integration is not a function of y but it can be a function of x. Let the value of constant of integration is k. Then

$$\psi = y^2 - y + k. \qquad \text{...}(iii)$$

Differentiating the above equation w.r.t. 'x', we get

$$\frac{\partial \psi}{\partial x} = \frac{\partial k}{\partial x}.$$

But from equation (*ii*), $\dfrac{\partial \psi}{\partial x} = -2x$

Equating the value of $\dfrac{\partial \psi}{\partial x}$, we get $\dfrac{\partial k}{\partial x} = -2x$.

Integrating this equation, we get $k = \int -2x dx = -\dfrac{2x^2}{2} = -x^2$.

Substituting this value of k in equation (*iii*), we get $\psi = \mathbf{y^2 - y - x^2}$. **Ans.**

∴ Stream function ψ at P (4, 5) $= 5^2 - 5 - 4^2 = 25 - 5 - 16 =$ **4 units. Ans.**

Problem 5.14 *The stream function for a two-dimensional flow is given by $\psi = 2xy$, calculate the velocity at the point P (2, 3). Find the velocity potential function ϕ.*

Solution. Given : $\psi = 2xy$

The velocity components u and v in terms of ψ are

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y}(2xy) = -2x$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y.$$

At the point P (2, 3), we get $u = -2 \times 2 = -4$ units/sec

$v = 2 \times 3 = 6$ units/sec

∴ Resultant velocity at $P = \sqrt{u^2 + v^2} = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52} = 7.21$ units/sec.

Velocity Potential Function ϕ

We know

$$\frac{\partial \phi}{\partial x} = -u = -(-2x) = 2x \qquad \text{...}(i)$$

$$\frac{\partial \phi}{\partial y} = -v = -2y \qquad \text{...}(ii)$$

Integrating equation (*i*), we get

$$\int d\phi = \int 2x dx$$

or

$$\phi = \frac{2x^2}{2} + C = x^2 + C \qquad \text{...}(iii)$$

where C is a constant which is independent of x but can be a function of y.

Differentiating equation (*iii*) w.r.t. 'y', we get $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial C}{\partial y}$

But from (*ii*), $$\frac{\partial \phi}{\partial y} = -2y$$

$\therefore$ $$\frac{\partial C}{\partial y} = -2y$$

Integrating this equation, we get $C = \int -2y\, dy = -\frac{2y^2}{2} = -y^2$

Substituting this value of C in equation (*iii*), we get $\phi = \mathbf{x^2 - y^2}$. **Ans.**

Problem 5.15 *Sketch the stream lines represented by $\psi = x^2 + y^2$. Also find out the velocity and its direction at point (1, 2).*

Solution. Given : $\psi = x^2 + y^2$

The velocity components u and v are

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y}(x^2 + y^2) = -2y$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

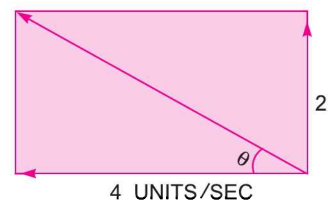


Fig. 5.9

At the point (1, 2), the velocity components are

$$u = -2 \times 2 = -4 \text{ units/sec}$$
$$v = 2 \times 1 = 2 \text{ units/sec}$$

Resultant velocity $$= \sqrt{u^2 + v^2} = \sqrt{(-4)^2 + 2^2}$$

$$= \sqrt{20} = 4.47 \text{ units/sec}$$

and $$\tan\theta = \frac{v}{u} = \frac{2}{4} = \frac{1}{2}$$

$\therefore$ $$\theta = \tan^{-1} .5 = 26° \, 34'$$

$\therefore$ Resultant velocity makes an angle of 26° 34′ with x-axis.

Sketch of Stream Lines

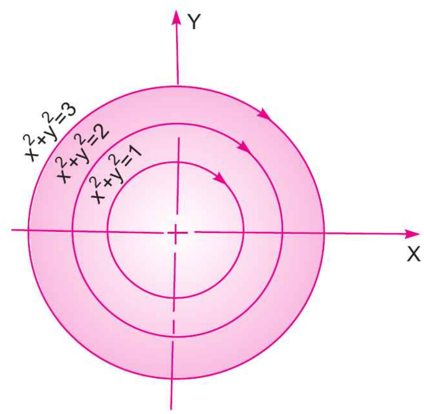


Fig. 5.10

$$\psi = x^2 + y^2$$

Let $\psi = 1, 2, 3$ and so on.

Then we have
$$1 = x^2 + y^2$$
$$2 = x^2 + y^2$$
$$3 = x^2 + y^2$$

and so on.

Each equation is a equation of a circle. Thus we shall get concentric circles of different diameters as shown in Fig. 5.10.

Problem 5.16 *The velocity components in a two-dimensional flow field for an incompressible fluid are as follows :*

$$u = \frac{y^3}{3} + 2x - x^2 y \text{ and } v = xy^2 - 2y - x^3/3$$

obtain an expression for the stream function ψ.

Solution. Given : $u = y^3/3 + 2x - x^2y$

$v = xy^2 - 2y - x^3/3.$

The velocity components in terms of stream function are

$$\frac{\partial \psi}{\partial x} = v = xy^2 - 2y - x^3/3 \quad \text{...(i)}$$

$$\frac{\partial \psi}{\partial y} = -u = -y^3/3 - 2x + x^2y \quad \text{...(ii)}$$

Integrating (*i*) w.r.t. x, we get $\psi = \int (xy^2 - 2y - x^3/3)\, dx$

or
$$\psi = \frac{x^2y^2}{2} - 2xy - \frac{x^4}{4 \times 3} + k, \quad \text{...(iii)}$$

where k is a constant of integration which is independent of x but can be a function of y.

Differentiating equation (*iii*) w.r.t. y, we get

$$\frac{\partial \psi}{\partial y} = \frac{2x^2y}{2} - 2x + \frac{\partial k}{\partial y} = x^2y - 2x + \frac{\partial k}{\partial y}$$

But from (*ii*),
$$\frac{\partial \psi}{\partial y} = -y^3/3 - 2x + x^2y$$

Comparing the value of $\frac{\partial \psi}{\partial y}$, we get $x^2y - 2x + \frac{\partial k}{\partial y} = -y^3/3 - 2x + x^2y$

$\therefore$
$$\frac{\partial k}{\partial y} = -y^3/3$$

Integrating, we get
$$k = \int (-y^3/3)\, dy = \frac{-y^4}{4 \times 3} = \frac{-y^4}{12}$$

Substituting this value in (*iii*), we get

$$\psi = \frac{\mathbf{x^2y^2}}{\mathbf{2}} - \mathbf{2xy} - \frac{\mathbf{x^4}}{\mathbf{12}} - \frac{\mathbf{y^4}}{\mathbf{12}}. \textbf{ Ans.}$$

Problem 5.17 *In a two-dimensional incompressible flow, the fluid velocity components are given by*

$u = x - 4y$ and $v = -y - 4x$.

Show that velocity potential exists and determine its form. Find also the stream function.

Solution. Given : $u = x - 4y$ and $v = -y - 4x$

$\therefore$
$$\frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = -1$$

$\therefore$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$$

Hence flow is continuous and velocity potential exists.

Let ϕ = Velocity potential.

Let velocity components in terms of velocity potential is given by

$$\frac{\partial \phi}{\partial x} = -u = -(x - 4y) = -x + 4y \qquad ...(i)$$

and

$$\frac{\partial \phi}{\partial y} = -v = -(-y - 4x) = y + 4x \qquad ...(ii)$$

Integrating equation (*i*), we get $\phi = -\frac{x^2}{2} + 4xy + C$ ...(*iii*)

where C is a constant of integration, which is independent of x.

This constant can be a function of y.

Differentiating the above equation, *i.e.*, equation (*iii*) with respect to 'y', we get

$$\frac{\partial \phi}{\partial y} = 0 + 4x + \frac{\partial C}{\partial y}$$

But from equation (*iii*), we have $\frac{\partial \phi}{\partial y} = y + 4x$

Equating the two values of $\frac{\partial \phi}{\partial y}$, we get

$$4x + \frac{\partial C}{\partial y} = y + 4x \quad \text{or} \quad \frac{\partial C}{\partial y} = y$$

Integrating the above equation, we get

$$C = \frac{y^2}{2} + C_1$$

where C_1 is a constant of integration, which is independent of x and y.

Taking it equal to zero, we get $C = \frac{y^2}{2}$.

Substituting the value of C in equation (*iii*), we get

$$\phi = -\frac{x^2}{2} + 4xy + \frac{y^2}{2}. \textbf{ Ans.}$$

Value of Stream functions

Let ψ = Stream function

The velocity components in terms of stream function are

$$\frac{\partial \psi}{\partial x} = v = -y - 4x \qquad ...(iv)$$

and

$$\frac{\partial \psi}{\partial y} = -u = -(x - 4y) = -x + 4y \qquad ...(v)$$

Integrating equation (*iv*) w.r.t. x, we get

$$\psi = -yx - \frac{4x^2}{2} + k \qquad ...(vi)$$

where k is a constant of integration which is independent of x but can be a function of y.

Differentiating equation (*vi*) w.r.t. *y*, we get $\dfrac{\partial \psi}{\partial y} = -x - 0 + \dfrac{\partial k}{\partial y}$

But from equation (*v*), we have $\dfrac{\partial \psi}{\partial y} = -x + 4y$

Equating the two values of $\dfrac{\partial \psi}{\partial y}$, we get $-x + \dfrac{\partial k}{\partial y} = -x + 4y$ or $\dfrac{\partial k}{\partial y} = 4y$

Integrating the above equation, we get $k = \dfrac{4y^2}{2} = 2y^2$

Substituting the value of *k* in equation (*vi*), we get

$$\psi = -yx - 2x^2 + 2y^2. \text{ Ans.}$$

▶ 5.9 TYPES OF MOTION

A fluid particle while moving may undergo anyone or combination of following four types of displacements :

(*i*) Linear Translation or Pure Translation,
(*ii*) Linear Deformation,
(*iii*) Angular Deformation, and
(*iv*) Rotation.

5.9.1 Linear Translation. It is defined as the movement of a fluid element in such a way that it moves bodily from one position to another position and the two axes *ab* and *cd* represented in new positions by *a′b′* and *c′d′* are parallel as shown in Fig. 5.11 (*a*).

5.9.2 Linear Deformation. It is defined as the deformation of a fluid element in linear direction when the element moves. The axes of the element in the deformed position and un-deformed position are parallel, but their lengths change as shown in Fig. 5.11 (*b*).

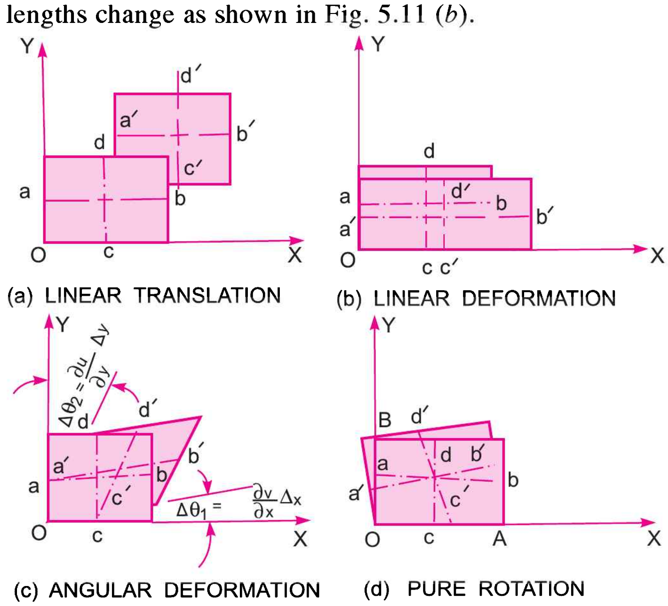


Fig. 5.11. *Displacement of a fluid element.*

5.9.3 Angular Deformation or Shear Deformation. It is defined as the average change in the angle contained by two adjacent sides. Let $\Delta\theta_1$ and $\Delta\theta_2$ is the change in angle between two adjacent sides of a fluid element as shown in Fig. 5.11 (*c*), then angular deformation or shear strain rate

$$= \frac{1}{2}\left[\Delta\theta_1 + \Delta\theta_2\right]$$

Now $$\Delta\theta_1 = \frac{\partial v}{\partial x} \times \frac{\Delta x}{\Delta x} = \frac{\partial v}{\partial x} \text{ and } \Delta\theta_2 = \frac{\partial u}{\partial y} \cdot \frac{\Delta y}{\Delta y} = \frac{\partial u}{\partial y}.$$

∴ $$\textbf{Angular deformation} = \frac{1}{2}\left[\Delta\theta_1 + \Delta\theta_2\right]$$

or $$\textbf{Shear strain rate} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \qquad \text{...(5.16)}$$

5.9.4 Rotation. It is defined as the movement of a fluid element in such a way that both of its axes (horizontal as well as vertical) rotate in the same direction as shown in Fig. 5.11 (*d*). It is equal to $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ for a two-dimensional element in *x-y* plane. The rotational components are

$$\left.\begin{aligned} \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \end{aligned}\right\} \qquad \text{...(5.17)}$$

5.9.5 Vorticity. It is defined as the value twice of the rotation and hence it is given as 2ω.

Problem 5.18 *A fluid flow is given by* $V = 8x^3 i - 10x^2 yj$.
Find the shear strain rate and state whether the flow is rotational or irrotational.

Solution. Given : $$V = 8x^3 i - 10x^2 yj$$

∴ $$u = 8x^3, \frac{\partial u}{\partial x} = 24x^2, \frac{\partial u}{\partial y} = 0$$

and $$v = -10x^2 y, \frac{\partial v}{\partial x} = -20xy, \frac{\partial v}{\partial y} = -10x^2$$

(*i*) Shear strain rate is given by equation (5.16) as

$$= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}(-20xy + 0) = \mathbf{-10xy.\ Ans.}$$

(*ii*) Rotation in x - y plane is given by equation (5.17) or

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\,(-\,20xy - 0) = -\,10xy$$

As rotation $\omega_z \neq 0$. Hence flow is rotational. **Ans.**

Problem 5.19 *The velocity components in a two-dimensional flow are*

$$u = y^3/3 + 2x - x^2y \text{ and } v = xy^2 - 2y - x^3/3.$$

Show that these components represent a possible case of an irrotational flow.

Solution. Given : $u = y^3/3 + 2x - x^2y$

$\therefore$ $$\frac{\partial u}{\partial x} = 2 - 2xy$$

$$\frac{\partial u}{\partial y} = \frac{3y^2}{3} - x^2 = y^2 - x^2$$

Also $v = xy^2 - 2y - x^3/3$

$\therefore$ $$\frac{\partial v}{\partial y} = 2xy - 2$$

$$\frac{\partial v}{\partial x} = y^2 - \frac{3x^2}{3} = y^2 - x^2.$$

(*i*) For a two-dimensional flow, continuity equation is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Substituting the value of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2xy + 2xy - 2 = 0$$

$\therefore$ It is a possible case of fluid flow.

(*ii*) Rotation, ω_z is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\,[(y^2 - x^2) - (y^2 - x^2)] = 0$

$\therefore$ Rotation is zero, which means it is case of irrotational flow. **Ans.**

▶ 5.10 VORTEX FLOW

Vortex flow is defined as the flow of a fluid along a curved path or the flow of a rotating mass of fluid is known a 'Vortex Flow'. The vortex flow is of two types namely :

1. Forced vortex flow, and
2. Free vortex flow.

5.10.1 Forced Vortex Flow. Forced vortex flow is defined as that type of vortex flow, in which some external torque is required to rotate the fluid mass. The fluid mass in this type of flow, rotates at constant angular velocity, ω. The tangential velocity of any fluid particle is given by

$$v = \omega \times r \qquad \text{...(5.18)}$$

where r = Radius of fluid particle from the axis of rotation.

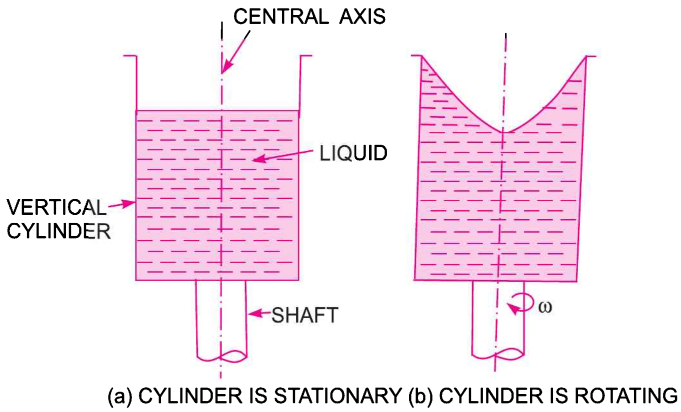


Fig. 5.12 *Forced vortex flow.*

Hence angular velocity ω is given by

$$\omega = \frac{v}{r} = \text{Constant.} \qquad \text{...(5.19)}$$

Examples of forced vortex are :

1. A vertical cylinder containing liquid which is rotated about its central axis with a constant angular velocity ω, as shown in Fig. 5.12.
2. Flow of liquid inside the impeller of a centrifugal pump.
3. Flow of water through the runner of a turbine.

5.10.2 Free Vortex Flow. When no external torque is required to rotate the fluid mass, that type of flow is called free vortex flow. Thus the liquid in case of free vortex is rotating due to the rotation which is imparted to the fluid previously.

Examples of the free vortex flow are :

1. Flow of liquid through a hole provided at the bottom of a container.
2. Flow of liquid around a circular bend in a pipe.
3. A whirlpool in a river.
4. Flow of fluid in a centrifugal pump casing.

The relation between velocity and radius, in free vortex is obtained by putting the value of external torque equal to zero, or, the time rate of change of angular momentum, *i.e.*, moment of momentum must be zero. Consider a fluid particle of mass 'm' at a radial distance r from the axis of rotation, having a tangential velocity v. Then

Angular momentum = Mass × Velocity = $m \times v$

Moment of momentum = Momentum × r = $m \times v \times r$

$\therefore$ Time rate of change of angular momentum = $\dfrac{\partial}{\partial t}(mvr)$

$\therefore$ For free vortex $\dfrac{\partial}{\partial t}(mvr) = 0$

Integrating, we get mvr = Constant or $vr = \dfrac{\text{Constant}}{m}$ = Constant ...(5.20)

5.10.3 Equation of Motion for Vortex Flow. Consider a fluid element *ABCD* (shown shaded) in Fig. 5.13 rotating at a uniform velocity in a horizontal plane about an axis perpendicular to the plane of paper and passing through *O*.

Let r = Radius of the element from *O*.

$\Delta\theta$ = Angle subtended by the element at *O*.

Δr = Radial thickness of the element.

ΔA = Area of cross-section of element.

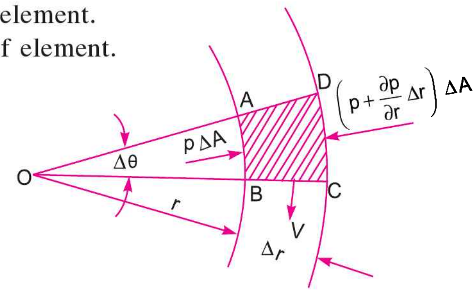


Fig. 5.13

The forces acting on the element are :

(*i*) Pressure force, $p\Delta A$, on the face *AB*.

(*ii*) Pressure force, $\left(p + \frac{\partial p}{\partial r}\Delta r\right)\Delta A$ on the face *CD*.

(*iii*) Centrifugal force, $\frac{mv^2}{r}$ acting in the direction away from the centre, *O*.

Now, the mass of the element = Mass density × Volume

$= \rho \times \Delta A \times \Delta r$

$\therefore$ Centrifugal force $= \rho\Delta A\Delta r\,\frac{v^2}{r}$.

Equating the forces in the radial direction, we get

$$\left(p + \frac{\partial p}{\partial r}\Delta r\right)\Delta A - p\Delta A = \rho\Delta A\Delta r\,\frac{v^2}{r}$$

or
$$\frac{\partial p}{\partial r}\Delta r\,\Delta A = \rho\Delta A\Delta r\,\frac{v^2}{r}.$$

Cancelling $\Delta r \times \Delta A$ from both sides, we get $\frac{\partial p}{\partial r} = \rho\,\frac{v^2}{r}$ ...(5.21)

Equation (5.21) gives the pressure variation along the radial direction for a forced or free vortex flow in a horizontal plane. The expression $\frac{\partial p}{\partial r}$ is called pressure gradient in the radial direction. As $\frac{\partial p}{\partial r}$ is positive, hence pressure increases with the increase of radius 'r'.

The pressure variation in the vertical plane is given by the hydrostatic law, *i.e.*,

$$\frac{\partial p}{\partial z} = -\rho g \qquad ...(5.22)$$

In equation (5.22), z is measured vertically in the upward direction.

The pressure, p varies with respect to r and z or p is a function of r and z and hence total derivative of p is

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz.$$

Substituting the values of $\frac{\partial p}{\partial r}$ from equation (5.21) and $\frac{\partial p}{\partial z}$ from equation (5.22), we get

$$dp = \rho \frac{v^2}{r} dr - \rho g dz \quad ...(5.23)$$

Equation (5.23) gives the variation of pressure of a rotating fluid in any plane.

5.10.4 Equation of Forced Vortex Flow.

For the forced vortex flow, from equation (5.18), we have

$$v = \omega \times r$$

where ω = Angular velocity = Constant.

Substituting the value of v in equation (5.23), we get

$$dp = \rho \times \frac{\omega^2 r^2}{r} dr - \rho g \, dz.$$

Consider two points 1 and 2 in the fluid having forced vortex flow as shown in Fig. 5.14. Integrating the above equation for points 1 and 2, we get

$$\int_1^2 dp = \int_1^2 \rho\omega^2 r dr - \int_1^2 \rho g dz$$

or
$$(p_2 - p_1) = \left[\rho\omega^2 \frac{r^2}{2}\right]_1^2 - \rho g \, [z]_1^2$$

or
$$(p_2 - p_1) = \frac{\rho\omega^2}{2} [r_2^2 - r_1^2] - \rho g \, [z_2 - z_1]$$

$$= \frac{\rho}{2} [\omega^2 r_2^2 - \omega^2 r_1^2] - \rho g \, [z_2 - z_1]$$

$$= \frac{\rho}{2} [v_2^2 - v_1^2] - \rho g \, [z_2 - z_1] \quad \left\{\because \begin{array}{l} v_2 = \omega r_2 \\ v_1 = \omega r_1 \end{array}\right\}$$

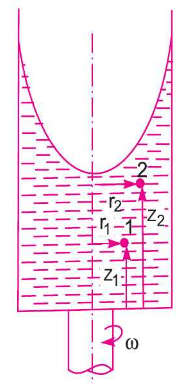


Fig. 5.14

If the points 1 and 2 lie on the free surface of the liquid, then $p_1 = p_2$ and hence above equation becomes

$$0 = \frac{\rho}{2} [v_2^2 - v_1^2] - \rho g \, [z_2 - z_1]$$

or
$$\rho g \, [z_2 - z_1] = \frac{\rho}{2} [v_2^2 - v_1^2]$$

or
$$[z_2 - z_1] = \frac{1}{2g} [v_2^2 - v_1^2].$$

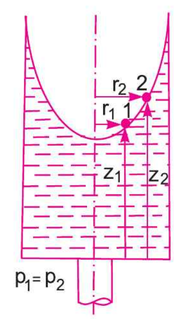


Fig. 5.15

If the point 1 lies on the axis of rotation, then $v_1 = \omega \times r_1 = \omega \times 0 = 0$. The above equation becomes as

$$z_2 - z_1 = \frac{1}{2g} v_2^2 = \frac{v_2^2}{2g}$$

Let $z_2 - z_1 = Z$, then we have $Z = \frac{v_2^2}{2g} = \frac{\omega^2 \times r_2^2}{2g}$ ...(5.24)

Thus Z varies with the square of r. Hence equation (5.24) is an equation of parabola. This means the free surface of the liquid is a paraboloid.

Problem 5.20 *Prove that in case of forced vortex, the rise of liquid level at the ends is equal to the fall of liquid level at the axis of rotation.*

Solution. Let R = radius of the cylinder.

O-O = Initial level of liquid in cylinder when the cylinder is not rotating.

$\therefore$ Initial height of liquid $= (h + x)$

$\therefore$ Volume of liquid in cylinder $= \pi R^2 \times$ Height of liquid

$$= \pi R^2 \times (h + x) \quad \ldots(i)$$

Let the cylinder is rotated at constant angular velocity ω. The liquid will rise at the ends and will fall at the centre.

Let y = Rise of liquid at the ends from O-O

x = Fall of liquid at the centre from O-O.

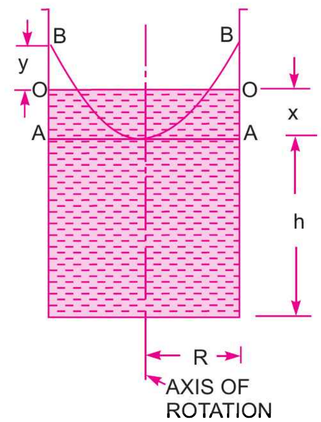


Fig. 5.16

Then volume of liquid

= [Volume of cylinder upto level B-B] – [Volume of paraboloid]

= [$\pi R^2 \times$ Height of liquid upto level B-B] $- \left[\dfrac{\pi R^2}{2} \times \text{Height of paraboloid}\right]$

$$= \pi R^2 \times (h + x + y) - \frac{\pi R^2}{2} \times (x + y)$$

$$= \pi R^2 \times h + \pi R^2 (x + y) - \frac{\pi R^2}{2} \times (x + y)$$

$$= \pi R^2 \times h + \frac{\pi R^2}{2} (x + y) \quad \ldots(ii)$$

Equating (*i*) and (*ii*), we get

$$\pi R^2 (h + x) = \pi R^2 \times h + \frac{\pi R^2}{2} (x + y)$$

or
$$\pi R^2 h + \pi R^2 x = \pi R^2 \times h + \frac{\pi R^2}{2} x + \frac{\pi R^2}{2} y$$

or
$$\pi R^2 x - \frac{\pi R^2}{2} x = \frac{\pi R^2}{2} y \quad \text{or} \quad \frac{\pi R^2}{2} x = \frac{\pi R^2}{2} y \quad \text{or} \quad x = y$$

or Fall of liquid at centre = Rise of liquid at the ends.

Problem 5.21 *An open circular tank of 20 cm diameter and 100 cm long contains water upto a height of 60 cm. The tank is rotated about its vertical axis at 300 r.p.m., find the depth of parabola formed at the free surface of water.*

Solution. Given :

Diameter of cylinder = 20 cm

$\therefore$ Radius, $R = \dfrac{20}{2} = 10$ cm

Height of liquid, $H = 60$ cm

Speed, $N = 300$ r.p.m.

Angular velocity, $\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 300}{60} = 31.41$ rad/sec.

Let the depth of parabola $= Z$

Using equation (5.24), $Z = \frac{\omega^2 r_2^2}{2g}$, where $r_2 = R$

$$= \frac{\omega^2 R^2}{2g} = \frac{(31.41)^2 \times (10)^2}{2 \times 981} = \textbf{50.28 cm. Ans.}$$

Problem 5.22 *An open circular cylinder of 15 cm diameter and 100 cm long contains water upto a height of 80 cm. Find the maximum speed at which the cylinder is to be rotated about its vertical axis so that no water spills.*

Solution. Given :

Diameter of cylinder $= 15$ cm

$\therefore$ Radius, $R = \frac{15}{2} = 7.5$ cm

Length of cylinder, $L = 100$ cm

Initial height of water $= 80$ cm.

Let the cylinder is rotated at an angular speed of ω rad/sec, when the water is about to spill. Then using,

Rise of liquid at ends = Fall of liquid at centre

But rise of liquid at ends = Length – Initial height

$= 100 - 80 = 20$ cm

$\therefore$ Fall of liquid at centre $= 20$ cm

$\therefore$ Height of parabola $= 20 + 20 = 40$ cm

$\therefore$ $Z = 40$ cm

Using the relation, $Z = \frac{\omega^2 R^2}{2g}$, we get $40 = \frac{\omega^2 (7.5)^2}{2 \times 981}$

$$\therefore \quad \omega^2 = \frac{40 \times 2 \times 981}{7.5 \times 7.5} = 1395.2$$

$$\therefore \quad \omega = \sqrt{1395.2} = 37.35 \text{ rad/s}$$

$\therefore$ Speed, N is given by $\omega = \frac{2\pi N}{60}$

or $$N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 37.35}{2 \times \pi} = \textbf{356.66 r.p.m. Ans.}$$

Problem 5.23 *A cylindrical vessel 12 cm in diameter and 30 cm deep is filled with water upto the top. The vessel is open at the top. Find the quantity of liquid left in the vessel, when it is rotated about its vertical axis with a speed of (a) 3000 r.p.m., and (b) 600 r.p.m.*

Solution. Given :

Diameter of cylinder $= 12$ cm

$\therefore$ Radius, $R = 6$ cm

Initial height of water $= 30$ cm

Initial volume of water $= \text{Area} \times \text{Initial height of water}$

$$= \frac{\pi}{4} \times 12^2 \times 30 \text{ cm}^3 = 3392.9 \text{ cm}^3$$

(*a*) Speed, $N = 300$ r.p.m.

$$\therefore \quad \omega = \frac{2\pi N}{60} = \frac{2\pi \times 300}{60} = 31.41 \text{ rad/s}$$

Height of parabola is given by $Z = \dfrac{\omega^2 R^2}{2g} = \dfrac{(31.41)^2 \times 6^2}{2 \times 981} = 18.10$ cm.

As vessel is initially full of water, water will be spilled if it is rotated. Volume of water spilled is equal to the volume of paraboloid.

But volume of paraboloid $= [\text{Area of cross-section} \times \text{Height of parabola}] \div 2$

$$= \frac{\pi}{4} D^2 \times \frac{Z}{2} = \frac{\pi}{4} \times 12^2 \times \frac{18.10}{2} = 1023.53 \text{ cm}^3$$

Volume of water left = Initial volume – Volume of water spilled

$= 3392.9 - 1023.53 =$ **2369.37 cm^3. Ans.**

(*b*) Speed, $N = 600$ r.p.m.

$$\therefore \quad \omega = \frac{2\pi N}{60} = \frac{2\pi \times 600}{60} = 62.82 \text{ rad/s}$$

Height of parabola, $Z = \dfrac{\omega^2 R^2}{2g} = \dfrac{(62.82)^2 \times 6^2}{2 \times 981} = 72.40$ cm.

As the height of parabola is more than the height of cylinder the shape of imaginary parabola will be as shown in Fig. 5.17.

Let r = Radius of the parabola at the bottom of the vessel.

Height of imaginary parabola

$= 72.40 - 30 = 42.40$ cm.

Volume of water left in the vessel

= Volume of water in portions *ABC* and *DEF*

= Initial volume of water

– Volume of paraboloid *AOF*

+ Volume of paraboloid *COD*.

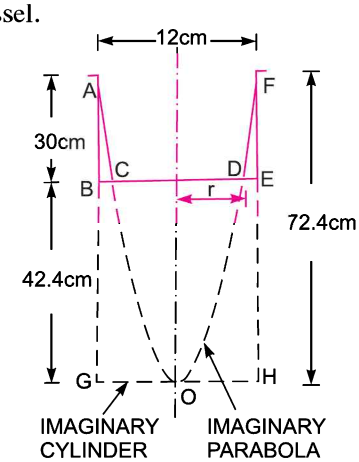


Fig. 5.17

Now volume of paraboloid

$$AOF = \frac{\pi}{4} \times D^2 \times \text{Height of parabola}$$

$$= \frac{\pi}{4} \times 12^2 \times \frac{72.4^2}{2} = 4094.12 \text{ cm}^3$$

For the imaginary parabola (*COD*), $\omega = 62.82$ rad/sec

$Z = 42.4$ cm

r = Radius at the bottom of vessel

Using the relation $Z = \dfrac{\omega^2 r^2}{2g}$, we get $42.4 = \dfrac{62.82^2 \times r^2}{2 \times 981}$

$$\therefore \quad r^2 = \frac{2 \times 981 \times 42.40}{62.82 \times 62.82} = 21.079$$

$$\therefore \quad r = \sqrt{21.079} = 4.59 \text{ cm}$$

$\therefore$ Volume of paraboloid *COD*

$$= \frac{1}{2} \times \text{Area at the top of the imaginary parabola} \times \text{Height of parabola}$$

$$= \frac{1}{2} \times \pi r^2 \times 42.4 = \frac{1}{2} \times \pi \times 4.59^2 \times 42.4 = 1403.89 \text{ cm}^3$$

$\therefore$ Volume of water left = 3392.9 – 4094.12 + 1403.89 = **702.67 cm³. Ans.**

Problem 5.24 *An open circular cylinder of 15 cm diameter and 100 cm long contains water upto a height of 70 cm. Find the speed at which the cylinder is to be rotated about its vertical axis, so that the axial depth becomes zero.*

Solution. Given :

Diameter of cylinder = 15 cm

$\therefore$ Radius, $R = \dfrac{15}{2} = 7.5$ cm

Length of cylinder = 100 cm

Initial height of water = 70 cm.

When axial depth is zero, the depth of paraboloid = 100 cm.

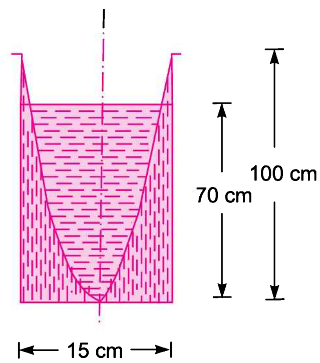


Fig. 5.18

Using the relation, $Z = \dfrac{\omega^2 R^2}{2g}$, we get

$$100 = \frac{\omega^2 \times 7.5^2}{2 \times 9.81}$$

$$\therefore \quad \omega^2 = \frac{100 \times 2 \times 9.81}{7.5 \times 7.5}$$

$$\therefore \quad \omega = \sqrt{\frac{100 \times 2 \times 9.81}{7.5 \times 7.5}} = \frac{442.92}{7.5} = 59.05 \text{ rad/s}$$

$\therefore$ Speed, N is given by $\omega = \dfrac{2\pi H}{60}$

or $N = \dfrac{60 \times \omega}{2\pi} = \dfrac{60 \times 59.05}{2\pi}$ = **563.88 r.p.m. Ans.**

Problem 5.25 *For the problem (5.24), find the difference in total pressure force (i) at the bottom of cylinder, and (ii) at the sides of the cylinder due to rotation.*

Solution. (*i*) The data is given in Problem 5.24. The difference in total pressure force at the bottom of cylinder is obtained by finding total hydrostatic force at the bottom before rotation and after rotation.

Before rotation, force = $\rho g A\bar{h}$

where ρ = 1000 kg/m^3, A = Area of bottom $= \frac{\pi}{3} D^2 = \frac{\pi}{4} \times (0.15)^2$ m^2, $\bar{h}$ = 70 cm = 0.70 m

$$\therefore \quad \text{Force} = 1000 \times 9.81 \times \frac{\pi}{4} \times (0.15)^2 \times 0.7 \text{ N} = 121.35 \text{ N}$$

After rotation, the depth of water at the bottom is not constant and hence pressure force due to the height of water, will not be constant. Consider a circular ring of radius r and width dr as shown in Fig. 5.19. Let the height of water from the bottom of the tank upto free surface of water at a radius

$$r = Z = \frac{\omega^2 r^2}{2g}.$$

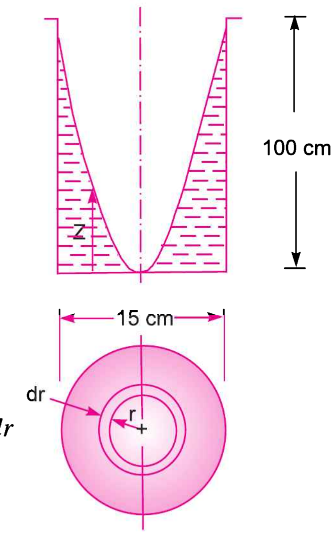


Fig. 5.19

Hydrostatic force on ring at the bottom,

$$dF = \rho g \times \text{Area of ring} \times Z$$

$$= 1000 \times 9.81 \times 2\pi r dr \times \frac{\omega^2 r^2}{2g}$$

$$= 9810 \times 2 \times \pi r \times \frac{\omega^2 r^2}{2g} \times dr$$

$\therefore$ Total pressure force at the bottom

$$= \int dF = \int_0^R 9810 \times 2 \times \pi r \times \frac{\omega^2 r^2}{2g} dr$$

$$= \int_0^{0.075} 19620 \times \pi \times \frac{\omega^2}{2g} r^3 \, dr$$

From Problem 5.24, ω = 59.05 rad/s

R = 7.5 cm = .075 m.

Substituting these values, we get total pressure force

$$= \frac{19620 \times \pi \times (59.05)^2}{2 \times 9.81} \left[\frac{r^4}{4}\right]_0^{.075}$$

$$= \frac{19620 \times \pi \times (59.05)^2}{2 \times 9.81} \times \frac{(.075)^4}{4} = 86.62 \text{ N}$$

$\therefore$ Difference in pressure forces at the bottom

121.35 − 86.62 = **34.73 N. Ans.**

(*ii*) Forces on the sides of the cylinder

Before rotation $= \rho g A\bar{h}$

where A = Surface area of the sides of the cylinder upto height of water

$= \pi D \times$ Height of water $= \pi \times .15 \times 0.70$ m^2 = 0.33 m^2

$\bar{h}$ = C.G. of the wetted area of the sides

$= \frac{1}{2} \times \text{height of water} = \frac{0.70}{2} = 0.35 \text{ m}$

$\therefore$ Force on the sides before rotation = 1000 × 9.81 × 0.33 × 0.35 = 1133 N

After rotation, the water is upto the top of the cylinder and hence force on the sides

$$= 1000 \times 9.81 \times \text{Wetted area of the sides} \times \frac{1}{2} \times \text{Height of water}$$

$$= 9810 \times \pi D \times 1.0 \times \frac{1}{2} \times 1.0 = 9810 \times \pi \times .15 \times \frac{1}{2} = 2311.43 \text{ N}$$

$\therefore$ Difference in pressure on the sides

2311.43 – 1133 = **1178.43 N. Ans.**

5.10.5 Closed Cylindrical Vessels. If a cylindrical vessel is closed at the top, which contains some liquid, the shape of paraboloid formed due to rotation of the vessel will be as shown in Fig. 5.20 for different speed of rotations.

Fig. 5.20 (*a*) shows the initial stage of the cylinder, when it is not rotated. Fig. 5.20 (*b*) shows the shape of the paraboloid formed when the speed of rotation is ω_1. If the speed is increased further say ω_2, the shape of paraboloid formed will be as shown in Fig. 5.20 (*c*). In this case the radius of the parabola at the top of the vessel is unknown. Also the height of the paraboloid formed corresponding to angular speed ω_2 is unknown. Thus to solve the two unknown, we should have two equations. One equation is

$$Z = \frac{\omega_2^2 r^2}{2g}$$

The second equation is obtained from the fact that for closed vessel, volume of air before rotation is equal to the volume of air after rotation.

Volume of air before rotation = Volume of closed vessel – Volume of liquid in vessel

Volume of air after rotation = Volume of paraboloid formed $= \frac{\pi r^2 \times Z}{2}$.

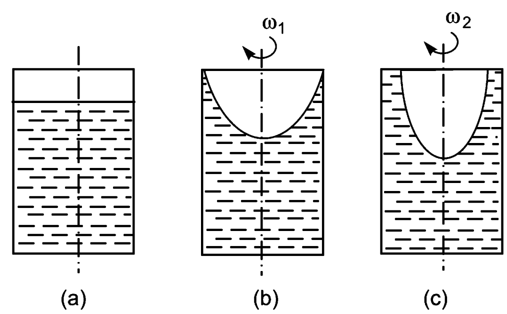


Fig. 5.20

Problem 5.26 *A vessel, cylindrical in shape and closed at the top and bottom, contains water upto a height of 80 cm. The diameter of the vessel is 20 cm and length of vessel is 120 cm. The vessel is rotated at a speed of 400 r.p.m. about its vertical axis. Find the height of paraboloid formed.*

Solution. Given :

Initial height of water = 80 cm

Diameter of vessel = 20 cm

$\therefore$ Radius, R = 10 cm

Length of vessel = 120 cm

Speed, N = 400 r.p.m.

$$\therefore \quad \omega = \frac{2\pi N}{60} = \frac{2\pi \times 400}{60} = 41.88 \text{ rad/s}$$

When the vessel is rotated, let Z = Height of paraboloid formed

r = Radius of paraboloid at the top of the vessel

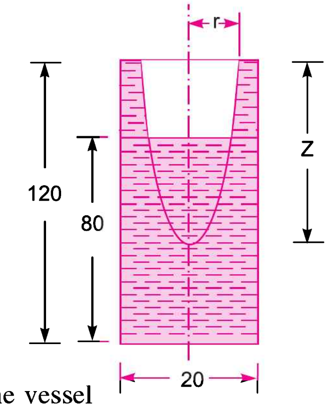


Fig. 5.21

This is the case of closed vessel.

$\therefore$ Volume of air before rotation = Volume of air after rotation

$$\text{or} \quad \frac{\pi}{4} D^2 \times L - \frac{\pi}{4} D^2 \times 80 = \pi r^2 \times \frac{Z}{2}$$

where Z = Height of paraboloid, r = Radius of parabola.

$$\text{or} \quad \frac{\pi}{4} D^2 \times 120 - \frac{\pi}{4} D^2 \times 80 = \pi r^2 \times \frac{Z}{2}$$

$$\text{or} \quad \frac{\pi}{4} \times D^2 \times (120 - 80) = \frac{\pi}{4} D^2 \times 40 = \pi r^2 \times \frac{Z}{2}$$

$$\text{or} \quad \frac{\pi}{4} \times 20^2 \times 40 = 4000 \times \pi = \pi r^2 \times \frac{Z}{2}$$

$$\therefore \quad r^2 \times Z = \frac{4000 \times \pi \times 2}{\pi} = 8000 \qquad ...(i)$$

Using relation $Z = \dfrac{\omega^2 r^2}{2g}$, we get $Z = \dfrac{41.88^2 \times r^2}{2g} = \dfrac{41.88^2 \times r^2}{2 \times 981} = 0.894\ r^2$

$$\therefore \quad r^2 = \frac{z}{0.894}$$

Substituting this value of r^2 in (i), we get

$$\frac{Z}{0.894} \times Z = 8000$$

$$\therefore \quad Z^2 = 8000 \times 0.894 = 7152$$

$$\therefore \quad Z = \sqrt{7152} = \mathbf{84.56 \text{ cm. Ans.}}$$

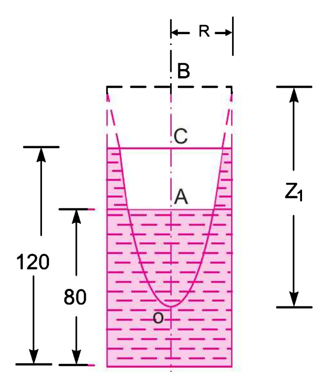


Fig. 5.22

IInd Method

Let Z_1 = Height of paraboloid, if the vessel would not have been closed at the top, corresponding to speed,

N = 400 r.p.m.

or ω = 41.88 rad/s

$$\text{Then} \quad Z_1 = \frac{\omega^2 R^2}{2g} = \frac{41.88^2 \times 10^2}{2 \times 981} = 89.34 \text{ cm.}$$

Half of Z_1 will be below the initial height of water in the vessel

i.e., $$AO = \frac{Z_1}{2} = \frac{89.34}{2} = 44.67 \text{ cm}$$

But height of paraboloid for closed vessel

$$= CO = CA + AO = (120 - 80) + 44.67 \text{ cm}$$

$$= 40 + 44.67 = \mathbf{84.67 \text{ cm. Ans.}}$$

Problem 5.27 *For the data given in Problem 5.26, find the speed of rotation of the vessel, when axial depth of water is zero.*

Solution. Given :

Diameter of vessel = 20 cm

∴ Radius, R = 10 cm

Initial height of water = 80 cm

Length of vessel = 120 cm

Let ω is the angular speed, when axial depth is zero.

When axial depth is zero, the height of paraboloid is 120 cm and radius of the parabola at the top of the vessel is r.

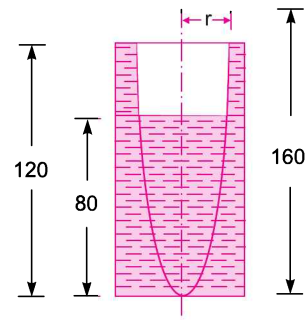


Fig. 5.23

∴ Using the relation, $$Z = \frac{\omega^2 r^2}{2g} \text{ or } 120 = \frac{\omega^2 \times r^2}{2 \times 980}$$

∴ $$\omega^2 r^2 = 2 \times 980 \times 120 = 235200 \quad \ldots(i)$$

Volume of air before rotation = Volume of air after paraboloid

∴ $$\pi R^2 \times (120 - 180) = \text{Volume of paraboloid}$$

$$= \pi r^2 \times \frac{Z}{2}$$

or $$\pi \times 10^2 \times 40 = \frac{\pi r^2 \times Z}{2} = \frac{\pi r^2}{2} \times 120$$

or $$r^2 = \frac{\pi \times 10^2 \times 40 \times 2}{\pi \times 120} = \frac{8000}{120} = 66.67$$

Substituting the value of r^2 in equation (i), we get

$$\omega^2 \times 66.67 = 235200$$

$$\omega = \sqrt{\frac{235200}{66.67}} = 59.4 \text{ rad/s}$$

∴ Speed N is given by $$\omega = \frac{2\pi N}{60}$$

or $$N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 59.4}{2\pi} = \mathbf{567.22 \text{ r.p.m. Ans.}}$$

Problem 5.28 *The cylindrical vessel of the problem 5.26 is rotated at 700 r.p.m. about its vertical axis. Find the area uncovered at the bottom of the tank.*

Solution. Given :

Initial height of water = 80 cm

∴ Diameter of vessel = 20 cm

∴ Radius, R = 10 cm

Length of vessel = 120 cm

Speed, $N = 700$ r.p.m.

$$\therefore \quad \omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 700}{60} = 73.30 \text{ rad/s.}$$

If the tank is not closed at the top and also is very long, then the height of parabola corresponding to $\omega = 73.3$ will be

$$= \frac{\omega^2 \times R^2}{2 \times g} = \frac{73.3^2 \times 10^2}{2 \times 980} = 274.12 \text{ cm}$$

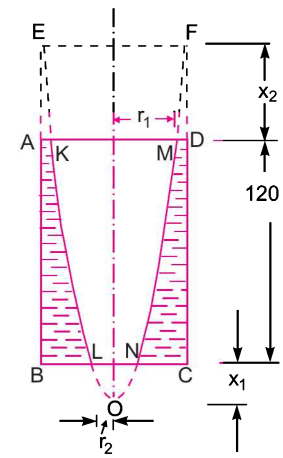


Fig. 5.24

From Fig. 5.24,

$$x_1 + 120 + x_2 = 274.12$$

or $$x_1 + x_2 = 274.12 - 120 = 154.12 \text{ cm} \quad \ldots(i)$$

From the parabola, *KOM*, we have

$$(120 + x_1) = \frac{\omega^2 r_1^2}{2g} = \frac{73.3^2 \times r_1^2}{2 \times 980} \quad \ldots(ii)$$

For the parabola, *LON*, we have

$$x_1 = \frac{\omega^2 r_2^2}{2g} = \frac{73.3^2 \times r_2^2}{2 \times 980} \quad \ldots(iii)$$

Now, volume of air before rotation = Volume of air after rotation

Volume of air before rotation $= \pi R^2 \times (120 - 80) = \pi \times 10^2 \times 40 = 12566.3 \text{ cm}^3$ ...(*iv*)

Volume of air after rotation = Volume of paraboloid *KOM* – volume of paraboloid *LON*

$$= \pi r_1^2 \times \frac{(120 + x_1)}{2} - \pi r_2^2 \times \frac{x_1}{2} \quad \ldots(v)$$

Equating (*iv*) and (*v*), we get

$$12566.3 = \frac{\pi r_1^2 (120 + x_1)}{2} - \frac{\pi r_2^2 \times x_1}{2} \quad \ldots(vi)$$

Substituting the value of r_1^2 from (*ii*) in (*vi*), we get

$$12566.3 = \pi \times \frac{(120 + x_1) \times 2 \times 980}{73.3^2} \times \frac{(120 + x_1)}{2} - \frac{\pi r_2^2 + x_1}{2}$$

$$\left\{\because \text{ From } (ii),\ r_1^2 = \frac{2 \times 980 \times (120 + x_1)}{(73.3)^2}\right\}$$

or $$12566.3 = 0.573\,(120 + x_1)^2 - \frac{\pi r_2^2 \times x_1}{2}$$

Substituting the value of x_1 from (*iii*) in the above equation

$$12566.3 = 0.573\left(120 + \frac{73.3^2 \times r_2^2}{2 \times 980}\right)^2 - \frac{\pi r_2^2}{2} \times \frac{73.3^2\, r_2^2}{2 \times 980}$$

$$= 0.573\,(120 + 2.74\, r_2^2)^2 - 4.3 \times r_2^2 \times r_2^2$$

$$= 0.573\,[120^2 + 2.74^2\, r_2^4 + 2 \times 120 \times 2.74\, r_2^2] - 4.3\, r_2^4$$

$$= 0.573\ [14400 + 7.506\ r_2^4 + 657.6\ r_2^2] - 4.3\ r_2^4$$

$$\frac{12566.3}{0.573} = 21930 = 14400 + 7.506\ r_2^4 + 657.6\ r_2^2 - 4.3\ r_2^4$$

or $$r_2^4\ (7.506 - 4.3) + 657.6\ r_2^2 + 14400 - 21930 = 0$$

or $$3.206\ r_2^4 + 657.6\ r_2^2 - 7530 = 0$$

$\therefore$ $$r_2^2 = \frac{-657.6 \pm \sqrt{657.6^2 - 4 \times (-7530) \times (3.206)}}{2 \times 3.206}$$

$$= \frac{-657.6 \pm \sqrt{432437.76 + 96564.72}}{6.412}$$

$$= \frac{-657.6 \pm 727.32}{6.412} = -215.98 \text{ or } 10.87$$

Negative value is not possible

$\therefore$ $$r_2^2 = 10.87 \text{ cm}^2$$

$\therefore$ Area uncovered at the base $= \pi r_2^2 = \pi \times 10.87 =$ **34.149 cm². Ans.**

Problem 5.29 *A closed cylindrical vessel of diameter 30 cm and height 100 cm contains water upto a depth of 80 cm. The air above the water surface is at a pressure of 5.886 N/cm². The vessel is rotated at a speed of 250 r.p.m. about its vertical axis. Find the pressure head at the bottom of the vessel : (a) at the centre, and (b) at the edge.*

Solution. Given :

Diameter of vessel $= 30$ cm

$\therefore$ Radius, $R = 15$ cm

Initial height of water, $H = 80$ cm

Length of cylinder, $L = 100$ cm

Pressure of air above water $= 5.886$ N/cm²

or $$p = 5.886 \times 10^4 \frac{\text{N}}{\text{m}^2}$$

Head due to pressure, $h = p/\rho g$

$$= \frac{5.886 \times 10^4}{1000 \times 9.81} = 6 \text{ m of water}$$

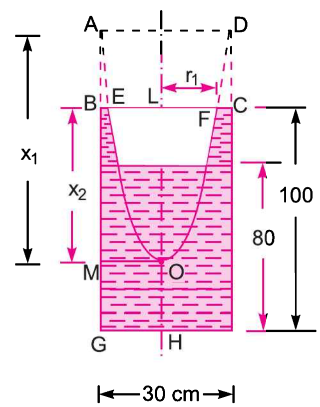


Fig. 5.25

Speed, $N = 250$ r.p.m.

$\therefore$ $$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 250}{60} = 26.18 \text{ rad/s}$$

Let x_1 = Height of paraboloid formed, if the vessel is assumed open at the top and it is very long.

Then we have $$x_1 = \frac{\omega^2 R^2}{2g} = \frac{26.18^2 \times 15^2}{2 \times 981} = 78.60 \text{ cm} \quad \ldots(i)$$

Let r_1 is the radius of the actual parabola of height x_2

Then $$x_2 = \frac{\omega^2 r_1^2}{2g} = \frac{26.18^2 \times r_1^2}{2 \times 981} = 0.35\ r_1^2 \quad \ldots(ii)$$

The volume of air before rotation

$$= \pi R^2\ (100 - 80) = \pi \times 15^2 \times 20 = 14137\ \text{cm}^3$$

Volume of air after rotation = Volume of paraboloid *EOF*

$$= \frac{1}{2} \times \pi r_1^2 \times x_2$$

But volume of air before and after rotation is same.

$$\therefore \quad 14137 = \frac{1}{2} \times \pi r_1^2 \times x_2$$

But from (*ii*), $x_2 = 0.35\ r_1^2$

$$\therefore \quad 14137 = \frac{1}{2} \times \pi r_1^2 \times 0.35\ r_1^2$$

$$\therefore \quad r_1^4 = \frac{2 \times 14137}{\pi \times 0.35} = 25714$$

$$r_1 = (25714)^{1/4} = 12.66\ \text{cm}$$

Substituting the value of r_1 in (*ii*), we get

$$x_2 = 0.35 \times 12.66^2 = 56.1\ \text{cm}$$

Pressure head at the bottom of the vessel

(*a*) At the centre. The pressure head at the centre, *i.e.*, at *H* = Pressure head due to air + *OH*

$$= 6.0 + (HL - LO) \quad \{\because OH = LH - LO\}$$

$$= 6.0 + (1.0 - 0.561) \quad \left\{\begin{array}{l} \because HL = 100\ \text{cm} = 1\ \text{m} \\ LO = x_2 = 56.1\ \text{cm} = .561\ \text{m} \end{array}\right\}$$

= 6.439 m of water. Ans.

(*b*) At the edge, *i.e.*, at *G* = Pressure head due to air + height of water above *G*

$$= 6.0 + AG = 6.0 + (GM + MA) = 6.0 + (HO + x_1)$$

$$= 6.0 + HO + 0.786 \quad \{\because x_1 = 78.6\ \text{cm} = 0.786\ \text{m}\}$$

$$= 6.0 + 0.439 + 0.786 \quad \left\{\begin{array}{l} \because HO = LH - LO = 100 - 56.1 \\ = 43.9\ \text{cm} = 0.439\ \text{m} \end{array}\right\}$$

= 7.225 m of water. Ans.

Problem 5.30 *A closed cylinder of radius R and height H is completely filled with water. It is rotated about its vertical axis with a speed of ω radians/s. Determine the total pressure exerted by water on the top and bottom of the cylinder.*

Solution. Given :

Radius of cylinder $= R$

Height of cylinder $= H$

Angular speed $= \omega$

As the cylinder is closed and completely filled with water, the rise of water level at the ends and depression of water at the centre due to rotation of the vessel, will be prevented. Thus the water will exert force on the complete top of the vessel. Also the pressure will be exerted at the bottom of the cylinder.

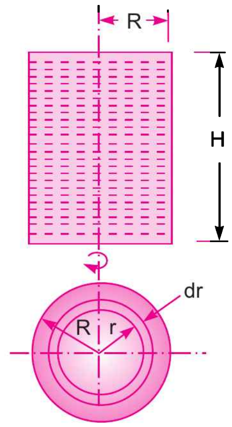


Fig. 5.26

Total Pressure exerted on the top of cylinder. The top of cylinder is in contact with water and is in horizontal plane. The pressure variation at any radius in horizontal plane is given by equation (5.21)

or $$\frac{\partial p}{\partial r} = \frac{\rho v^2}{r} = \frac{\rho \omega^2 r^2}{r} = \rho\omega^2 r \quad \{\because \; v = \omega \times r\}$$

Integrating, we get

$$\int dp = \int \rho\omega^2 r dr \quad \text{or} \quad p = \frac{\rho\omega^2 r^2}{2} = \frac{\rho}{2}\,\omega^2 r^2$$

Consider an elementary circular ring of radius r and width dr on the top of the cylinder as shown in Fig. 5.26.

Area of circular ring = $2\pi r dr$

∴ Force on the elementary ring = Intensity of pressure × Area of ring

$$= p \times 2\pi r dr$$

$$= \frac{\rho}{2}\,\omega^2 r^2 \times 2\pi r dr. \qquad \left\{\because \; p = \frac{\rho}{2}\,\omega^2 r^2\right\}$$

∴ Total force on the top of the cylinder is obtained by integrating the above equation between the limits 0 and R.

∴ Total force or $$F_T = \int_0^R \frac{\rho}{2}\,\omega^2 r^2 \times 2\pi r dr = \frac{\rho}{2}\,\omega^2 \times 2\pi \int_0^R r^3 dr$$

$$= \frac{\rho}{2}\,\omega^2 \times 2\pi \left[\frac{r^4}{4}\right]_0^R = \frac{\rho}{2}\,\omega^2 \times 2\pi \times \frac{R^4}{4}$$

$$= \frac{\rho\omega^2}{4} \times \pi R^4 \qquad \ldots(5.25)$$

Total pressure force on the bottom of cylinder, F_B

= Weight of water in cylinder + total force on the top of cylinder

$$= \rho g \times \pi R^2 \times H + \frac{\rho}{4}\,\omega^2 \times \pi R^4 = \rho g \times \pi R^2 \times H + F_T \qquad \ldots(5.26)$$

ρ = Density of water.

Problem 5.31 *A closed cylinder of diameter 200 mm and height 150 mm is completely filled with water. Calculate the total pressure force exerted by water on the top and bottom of the cylinder, if it is rotated about its vertical axis at 200 r.p.m.*

Solution. Given :

Dia. of cylinder = 200 mm = 0.20 m

Radius, R = 0.1 m

Height of cylinder, H = 150 mm = 0.15 m

Speed, N = 200 r.p.m.

∴ Angular speed, $$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 200}{60} = 20.94 \text{ rad/s}$$

Total pressure force on the top of the cylinder is given by equation (5.25)

$$F_T = \frac{\rho}{4} \times \omega^2 \times \pi \times R^4 = \frac{1000}{4} \times 20.94^2 \times \pi \times (0.1)^4 = \textbf{34.44 N. Ans.}$$

Now total pressure force on the bottom of the cylinder is given by equation (5.26) as

$$F_B = \rho g \times \pi R^2 \times H + F_T$$
$$= 1000 \times 9.81 \times \pi \times (0.1)^2 \times 0.15 + 34.44$$
$$= 46.22 + 34.44 = \textbf{80.66 N. Ans.}$$

5.10.6 Equation of Free Vortex Flow. For the free vortex, from equation (5.20), we have

$$v \times r = \text{Constant} = \text{say } c$$

or $$v = \frac{c}{r}$$

Substituting the value of v in equation (5.23), we get

$$dp = \rho \frac{v^2}{r} dr - \rho g\, dz = \rho \times \frac{c^2}{r^2 \times r} dr - \rho g\, dz = \rho \times \frac{c^2}{r^3} dr - \rho g\, dz$$

Consider two points 1 and 2 in the fluid having radius r_1 and r_2 from the central axis respectively as shown in Fig. 5.27. The heights of the points from bottom of the vessel is z_1 and z_2.

Integrating the above equation for the points 1 and 2, we get

$$\int_1^2 dp = \int_1^2 \frac{\rho c^2}{r^3} dr - \int_1^2 \rho g dz$$

or $$p_2 - p_1 = \rho c^2 \int_1^2 r^{-3}\, dr - \rho g \int_1^2 dz$$

$$= \rho c^2 \left[\frac{r^{-3+1}}{-2}\right]_1^2 - \rho g\, [z_2 - z_1] = \frac{\rho c^2}{-2}\, [r_2^{-2} - r_1^{-2}] - \rho g\, [z_2 - z_1]$$

$$= -\frac{\rho c^2}{2}\left[\frac{1}{r_2^2} - \frac{1}{r_1^2}\right] - \rho g[z_2 - z_1] = -\frac{\rho}{2}\left[\frac{c^2}{r_2^2} - \frac{c^2}{r_1^2}\right] - \rho g[z_2 - z_1]$$

$$= -\frac{\rho}{2}\left[v_2^2 - v_1^2\right] - \rho g[z_2 - z_1] \qquad \left\{\because\ v_2 = \frac{c}{r_2}, v_1 = \frac{c}{r_1}\right\}$$

$$= \frac{\rho}{2}\left[v_1^2 - v_2^2\right] - \rho g\left[z_2 - z_1\right]$$

Dividing by ρg, we get

$$\frac{p_2 - p_1}{\rho g} = \frac{v_1^2 - v_2^2}{2g} - \left[z_2 - z_1\right]$$

or $$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 \qquad \text{...(5.27)}$$

Equation (5.27) is Bernoulli's equation. Hence in case of free vortex flow, Bernoulli's equation is applicable.

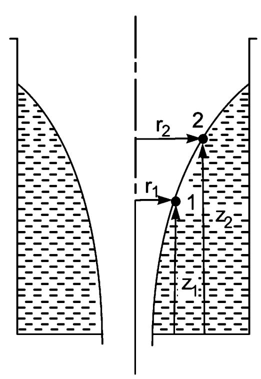


Fig. 5.27

Problem 5.32 *In a free cylindrical vortex flow, at a point in the fluid at a radius of 200 mm and at a height of 100 mm, the velocity and pressures are 10 m/s and 117.72 kN/m² absolute. Find the pressure at a radius of 400 mm and at a height of 200 mm. The fluid is air having density equal to 1.24 kg/m³.*

Solution. At Point 1 : Given :

Radius, $r_1 = 200 \text{ mm} = 0.20 \text{ m}$

Height, $z_1 = 100 \text{ mm} = 0.10 \text{ m}$

Velocity, $v_1 = 10 \text{ m/s}$

Pressure, $p_1 = 117.72 \text{ kN/m}^2 = 117.72 \times 10^3 \text{ N/m}^2$

At Point 2 : $r_2 = 400 \text{ mm} = 0.4 \text{ m}$

$z_2 = 200 \text{ mm} = 0.2 \text{ m}$

p_2 = pressure at point 2

$\rho = 1.24 \text{ kg/m}^3$

For the free vortex from equation (5.20), we have

$$v \times r = \text{constant or } v_1 r_1 = v_2 r_2$$

$$v_2 = \frac{v_1 \times r_1}{r_2} = \frac{10 \times 0.2}{0.4} = 5 \text{ m/s}$$

Now using equation (5.27), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

But $\rho = 1.24 \text{ kg/m}^3$

$$\therefore \quad \frac{117.72 \times 10^3}{1.24 \times 9.81} + \frac{10^2}{2 \times 9.81} + 0.1 = \frac{p_2}{\rho g} + \frac{5^2}{2 \times 9.81} + 0.2$$

or

$$\frac{p_2}{\rho g} = \frac{117.72 \times 10^3}{1.24 \times 9.81} + \frac{10^2}{2 \times 9.81} + 0.1 - \frac{5^2}{2 \times 9.81} - 0.2$$

$$= 9677.4 + 5.096 + 0.1 - 1.274 - 0.2 = 9676.22$$

$$p_2 = 9676.22 \times \rho g = 9676.22 \times 1.24 \times 9.81$$

$$= 117705 \text{ N/m}^2 = 117.705 \times 10^3 \text{ N/m}^2$$

$$= 117.705 \text{ kN/m}^2 \text{ (abs.)} = \mathbf{117.705 \text{ kN/m}^2}. \textbf{ Ans.}$$

(B) IDEAL FLOW (POTENTIAL FLOW)

▶ 5.11 INTRODUCTION

Ideal fluid is a fluid which is incompressible and inviscid. Incompressible fluid is a fluid for which density (ρ)remains constant. Inviscid fluid is a fluid for which viscosity (μ) is zero. Hence a fluid for which density is constant and viscosity is zero, is known as an ideal fluid.

The shear stress is given by, $\tau = \mu \frac{du}{dy}$. Hence for ideal fluid the shear stress will be zero as $\mu = 0$ for ideal fluid. Also the shear force (which is equal to shear stress multiplied by area) will be zero in case of ideal or potential flow. The ideal fluids will be moving with uniform velocity. All the fluid particles will be moving with the same velocity.

The concept of ideal fluid simplifies the typical mathematical analysis. Fluids such as water and air have low viscosity. Also when the speed of air is appreciably lower than that of sound in it, the compressibility is so low that air is assumed to be incompressible. Hence under certain conditions, certain real fluids such as water and air may be treated like ideal fluids.

▶ 5.12 IMPORTANT CASES OF POTENTIAL FLOW

The following are the important cases of potential flow :

(*i*) Uniform flow,
(*ii*) Source flow,
(*iii*) Sink flow,
(*iv*) Free-vortex flow,
(*v*) Superimposed flow.

▶ 5.13 UNIFORM FLOW

In a uniform flow, the velocity remains constant. All the fluid particles are moving with the same velocity. The uniform flow may be :

(*i*) Parallel to x-axis
(*ii*) Parallel to y-axis.

5.13.1 Uniform Flow Parallel to x-Axis. Fig. 5.27 (*a*) shows the uniform flow parallel to x-axis. In a uniform flow, the velocity remains constant. All the fluid particles are moving with the same velocity.

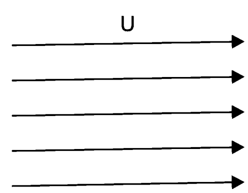


Fig. 5.27 (*a*)

Let U = Velocity which is uniform or constant along x-axis

u and v = Components of uniform velocity U along x and y-axis.

For the uniform flow, parallel to x-axis, the velocity components u and v are given as

$$u = U \text{ and } v = 0 \qquad \ldots(5.28)$$

But the velocity u in terms of stream function is given by,

$$u = \frac{\partial \psi}{\partial y}$$

and in terms of velocity potential the velocity u is given by,

$$u = \frac{\partial \phi}{\partial x}$$

$$\therefore \qquad u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \qquad \ldots(5.29)$$

Similarly, it can be shown that $v = -\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \phi}{\partial y}$ ...(5.29*A*)

But $u = U$ from equation (5.28). Substituting $u = U$ in equation (5.29), we have

$$\therefore \qquad U = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \qquad \text{...(5.30)}$$

or

$$U = \frac{\partial \psi}{\partial y} \text{ and also } U = \frac{\partial \phi}{\partial x}$$

First part gives $d\psi = U\,dy$ whereas second part gives $d\phi = U\,dx$.

Integration of these parts gives as

$$\psi = Uy + C_1 \text{ and } \phi = Ux + C_2$$

where C_1 and C_2 are constant of integration.

Now let us plot the stream lines and potential lines for uniform flow parallel to x-axis.

Plotting of Stream lines. For stream lines, the equation is

$$\psi = U \times y + C_1$$

Let $\psi = 0$, where $y = 0$. Substituting these values in the above equation, we get

$$0 = U \times 0 + C_1 \text{ or } C_1 = 0$$

Hence the equation of stream lines becomes as

$$\psi = U.\,y \qquad \text{...(5.31)}$$

The stream lines are straight lines parallel to x-axis and at a distance y from the x-axis as shown in Fig. 5.28. In equation (5.31), $U.\,y$ represents the volume flow rate (*i.e.,* m^3/s) between x-axis and that stream line at a distance y.

Note. The thickness of the fluid stream perpendicular to the plane is assumed to be unity. Then $y \times 1$ or y represents the area of flow. And $U.\,y$ represents the product of velocity and area. Hence $U.\,y$ represents the volume flow rate.

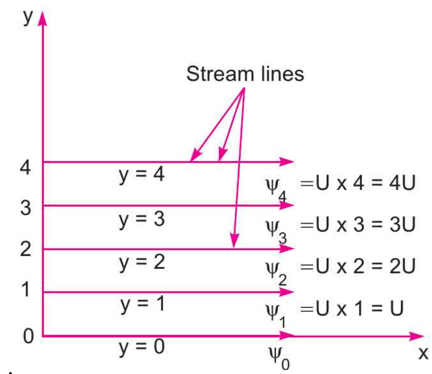


Fig. 5.28

Plotting of potential lines. For potential lines, the equation is

$$\phi = U.\,x + C_2 \qquad \text{...(5.32)}$$

Let $\phi = 0$, where $x = 0$. Substituting these values in the above equation, we get $C_2 = 0$.

Hence equation of potential lines becomes as

$$\phi = U.\,x$$

The above equation shows that potential lines are straight lines parallel to y-axis and at a distance of x from y-axis as shown in Fig. 5.29.

Fig. 5.30 shows the plot of stream lines and potential lines for uniform flow parallel to x-axis. The stream lines and potential lines intersect each other at right angles.

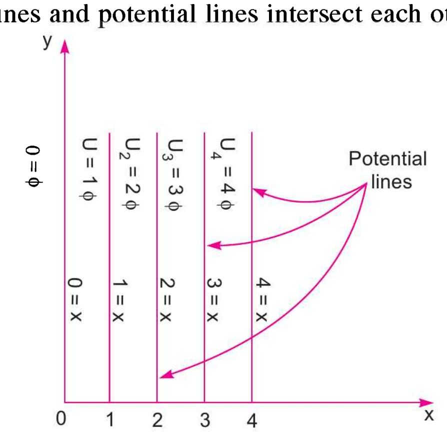


Fig. 5.29

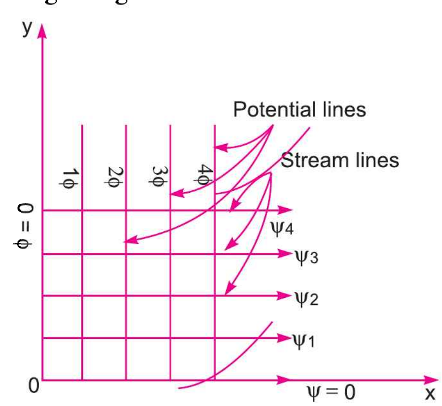


Fig. 5.30

5.13.2 Uniform Potential Flow Parallel to y-Axis.

Fig. 5.31 shows the uniform potential flow parallel to y-axis in which U is the uniform velocity along y-axis.

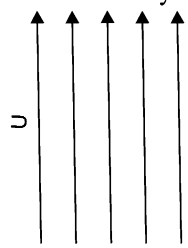


Fig. 5.31

The velocity components u, v along x-axis and y-axis are given by

$$u = 0 \text{ and } v = U \qquad \text{...(5.33)}$$

These velocity components in terms of stream function (ψ) and velocity potential function (ϕ) are given as

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \qquad \text{...(5.34)}$$

and

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \qquad \text{...(5.35)}$$

But from equation (5.33), $v = U$. Substituting $v = U$ in equation (5.35), we get

$$U = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \quad \text{or} \quad U = -\frac{\partial \psi}{\partial x} \text{ and also } U = \frac{\partial \phi}{\partial y}$$

First part gives $d\psi = -U\,dx$ whereas second part gives $d\phi = U\,dy$.

Integration of these parts gives as

$$\psi = -U\,.\,x + C_1 \text{ and } \phi = U.y + C_2 \qquad \text{...(5.36)}$$

where C_1 and C_2 are constant of integration. Let us now plot the stream lines and potential lines.

Plotting of Stream lines. For stream lines, the equation is $\psi = U.x + C_1$

Let $\psi = 0$, where $x = 0$. Then $C_1 = 0$.

Hence the equation of stream lines becomes as $\psi = -U.x$ ...(5.37)

The above equation shows that stream lines are straight lines parallel to y-axis and at a distance of x from the y-axis as shown in Fig. 5.32. The –ve sign shows that the stream lines are in the downward direction.

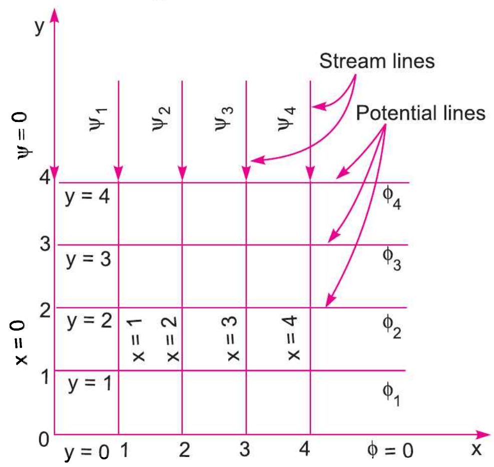


Fig. 5.32

Plotting of Potential lines. For potential lines, the equation is $\phi = U.y + C_2$

Let $\phi = 0$, where $y = 0$. Then $C_2 = 0$.

Hence equation of potential lines becomes as $\phi = U.y$ ...(5.38)

The above equation shows that potential lines are straight lines parallel to x-axis and at a distance of y from the x-axis as shown in Fig. 5.32.

▶ 5.14 SOURCE FLOW

The source flow is the flow coming from a point (source) and moving out radially in all directions of a plane at uniform rate. Fig. 5.33 shows a source flow in which the point O is the source from which the fluid moves radially outward. The strength of a source is defined as the volume flow rate per unit depth. The unit of strength of source is m^2/s. It is represented by q.

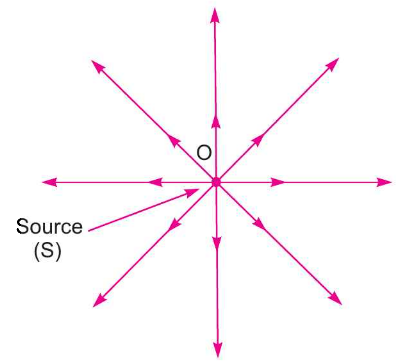


Fig. 5.33 *Source flow (Flow away from source)*

Let u_r = radial velocity of flow at a radius r from the source O

q = volume flow rate per unit depth

r = radius

The radial velocity u_r at any radius r is given by,

$$u_r = \frac{q}{2\pi r} \qquad \text{...(5.39)}$$

The above equation shows that with the increase of r, the radial velocity decreases. And at a large distance away from the source, the velocity will be approximately equal to zero. The flow is in radial direction, hence the tangential velocity $u_\theta = 0$.

Let us now find the equation of stream function and velocity potential function for the source flow. As in this case, $u_\theta = 0$, the equation of stream function and velocity potential function will be obtained from u_r.

Equation of Stream Function

By definition, the radial velocity and tangential velocity components in terms of stream function are given by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \text{ and } u_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{[See equation (5.12}A\text{)]}$$

But

$$u_r = \frac{q}{2\pi r} \qquad \text{[See equation (5.39)]}$$

$$\therefore \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{q}{2\pi r}$$

or

$$d\psi = r.\frac{q}{2\pi r}.d\theta = \frac{q}{2\pi}\,d\theta$$

Integrating the above equation w.r.t. θ, we get

$$\psi = \frac{q}{2\pi} \times \theta + C_1, \text{ where } C_1 \text{ is constant of integration.}$$

Let $\psi = 0$, when $\theta = 0$, then $C_1 = 0$.

Hence the equation of stream function becomes as

$$\psi = \frac{q}{2\pi} . \theta \qquad \ldots(5.40)$$

In the above equation, q is constant.

The above equation shows that stream function is a function of θ. For a given value of θ, the stream function ψ will be constant. And this will be a radial line. The stream lines can be plotted by having different values of θ. Here θ is taken in radians.

Plotting of stream lines

When $\theta = 0$, $\psi = 0$

$$\theta = 45° = \frac{\pi}{4} \text{ radians, } \psi = \frac{q}{2\pi} . \frac{\pi}{4} = \frac{q}{8} \text{ units}$$

$$\theta = 90° = \frac{\pi}{2} \text{ radians, } \psi = \frac{q}{2\pi} . \frac{\pi}{2} = \frac{q}{4} \text{ units}$$

$$\theta = 135° = \frac{3\pi}{4} \text{ radians, } \psi = \frac{q}{2\pi} . \frac{3\pi}{4} = \frac{3q}{8} \text{ units}$$

The stream lines will be radial lines as shown in Fig. 5.34.

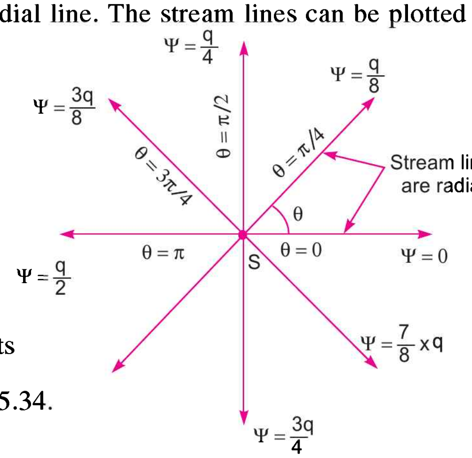


Fig. 5.34 *Stream line for source flow.*

Equation of Potential Function

By definition, the radial and tangential components in terms of velocity function are given by

$$u_r = \frac{\partial \phi}{\partial r} \text{ and } u_\theta = \frac{1}{r} . \frac{\partial \phi}{\partial \theta} \qquad \text{[See equation (5.9A)]}$$

But from equation (5.39), $u_r = \dfrac{q}{2\pi r}$

Equating the two values of u_r, we get

$$\frac{\partial \phi}{\partial r} = \frac{q}{2\pi r} \text{ or } d\phi = \frac{q}{2\pi r} \, dr$$

Integrating the above equation, we get

$$\int d\phi = \int \frac{q}{2\pi r} . dr$$

or

$$\phi = \frac{q}{2\pi} \int \frac{1}{r} dr \quad \left[\because \frac{q}{2\pi} \text{ is a constant term}\right]$$

$$= \frac{q}{2\pi} \log_e r \qquad \ldots(5.41)$$

In the above equation, q is constant.

The above equation shows, that the velocity potential function is a function of r. For a given value of r, the velocity function ϕ will be constant. Hence it will be a circle with origin at the source. The velocity potential lines will be circles with origin at the source as shown in Fig. 5.35.

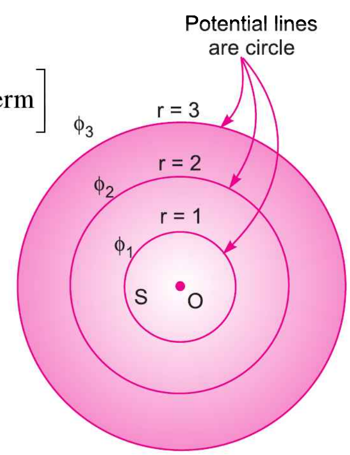


Fig. 5.35 *Potential lines for source.*

Let us now find an expression for the pressure in terms of radius.

Pressure distribution in a plane source flow

The pressure distribution in a plane source flow can be obtained with the help of Bernoulli's equation. Let us assume that the plane of the flow is horizontal. In that case the datum head will be same for two points of flow.

Let p = pressure at a point 1 which is at a radius r from the source at point 1

u_r = velocity at point 1

p_0 = pressure at point 2, which is at a large distance away from the source. The velocity will be zero at point 2. [Refer to equation (5.39)]

Applying Bernoulli's equation, we get

$$\frac{p}{\rho g} + \frac{u_r^2}{2g} = \frac{p_0}{\rho g} + 0 \quad \text{or} \quad \frac{(p - p_0)}{\rho g} = -\frac{u_r^2}{2g}$$

or
$$(p - p_0) = -\frac{\rho . u_r^2}{2}$$

But from equation(5.39), $u_r = \dfrac{q}{2\pi r}$

Substituting the value of u_r in the above equation, we get

$$(p - p_0) = -\left(\frac{\rho}{2}\right) . \left(\frac{q}{2\pi r}\right)^2$$

$$= -\frac{\rho q^2}{8\pi^2 r^2} \qquad \ldots(5.42)$$

In the above equation, ρ and q are constants.

The above equation shows that the pressure is inversely proportional to the square of the radius from the source.

▶ 5.15 SINK FLOW

The sink flow is the flow in which fluid moves radially inwards towards a point where it disappears at a constant rate. This flow is just opposite to the source flow. Fig. 5.36 shows a sink flow in which the fluid moves radially inwards towards point O, where it disappears at a constant rate. The pattern of stream lines and equipotential lines of a sink flow is the same as that of a source flow. All the equations derived for a source flow shall hold to good for sink flow also except that in sink flow equations, q is to be replaced by $(-q)$.

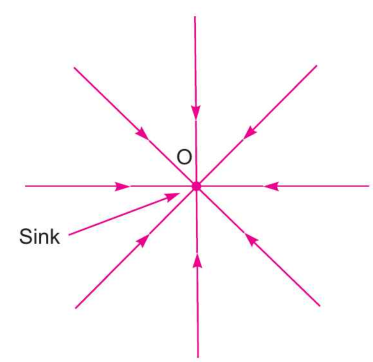


Fig. 5.36 *Sink flow (Flow toward centre)*

Problem 5.33 *Plot the stream lines for a uniform flow of : (i) 5 m/s parallel to the positive direction of the x-axis and (ii) 10 m/s parallel to the positive direction of the y-axis.*

Solution. (*i*) The stream function for a uniform flow parallel to the positive direction of the x-axis is given by equation (5.31) as

$$\psi = U \times y$$

The above equation shows that stream lines are straight lines parallel to the x-axis at a distance y from the x-axis. Here U = 5 m/s and hence above equation becomes as

$$\psi = 5y$$

For $y = 0$, stream function $\psi = 0$

For $y = 0.2$, stream function $\psi = 5 \times 0.2 = 1$ unit

For $y = 0.4$, stream function $\psi = 5 \times 0.4 = 2$ unit

The other values of stream function can be obtained by substituting the different values of y. The stream lines are horizontal as shown in Fig. 5.36 (a).

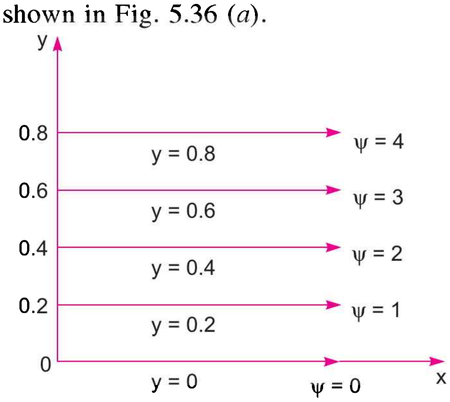


Fig. 5.36 (*a*)

(*ii*) The stream function for a uniform flow parallel to the positive direction of the y-axis is given by equation (5.37) as

$$\psi = -U \times x$$

The above equation shows that stream lines are straight lines parallel to the y-axis at a distance x from the y-axis. Here $U = 10$ m/s and hence the above equation becomes as

$$\psi = -10 \times x$$

The negative sign shows that the stream lines are in the downward direction.

For $x = 0$, the stream function $\psi = 0$

For $x = 0.1$, the stream function $\psi = -10 \times 0.1 = -1.0$ unit

For $x = 0.2$, the stream function $\psi = -10 \times 0.2 = -2.0$ unit

For $x = 0.3$, the stream function $\psi = -10 \times 0.3 = -3.0$ unit

The other values of stream function can be obtained by substituting the different values of x. The stream lines are vertical as shown in Fig. 5.36 (b).

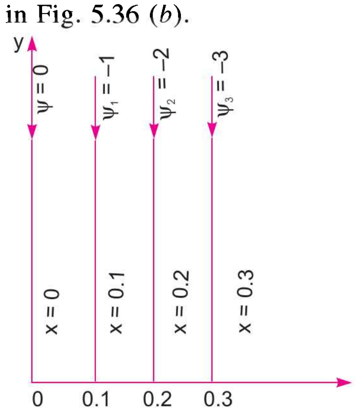


Fig. 5.36 (*b*)

Problem 5.34 *Determine the velocity of flow at radii of 0.2 m, 0.4 m and 0.8 m, when the water is flowing radially outward in a horizontal plane from a source at a strength of 12 m²/s.*

Solution. Given :

Strength of source, $q = 12$ m²/s

The radial velocity u_r at any radius r is given by equation (5.39) as

$$u_r = \frac{q}{2\pi r}$$

When $r = 0.2$ m, $$u_r = \frac{12}{2\pi \times 0.2} = \textbf{9.55 m/s. Ans.}$$

When $r = 0.4$ m, $$u_r = \frac{12}{2\pi \times 0.4} = \textbf{4.77 m/s. Ans.}$$

When $r = 0.8$ m, $$u_r = \frac{12}{2\pi \times 0.8} = \textbf{2.38 m/s. Ans.}$$

Problem 5.35 *Two discs are placed in a horizontal plane, one over the other. The water enters at the centre of the lower disc and flows radially outward from a source of strength 0.628 m²/s. The pressure, at a radius 50 mm, is 200 kN/m². Find :*

(i) pressure in kN/m² at a radius of 500 mm and

(ii) stream function at angles of 30° and 60° if $\psi = 0$ at $\theta = 0°$.

Solution. Given :

Source strength, $q = 0.628$ m²/s

Pressure at radius 50 mm, $p_1 = 200$ kN/m² $= 200 \times 10^3$ N/m²

(*i*) *Pressure at a radius 500 mm*

Let p_2 = pressure at radius 500 mm

$(u_r)_1$ = velocity at radius 50 mm

$(u_r)_2$ = velocity at radius 500 mm

The radial velocity at any radius r is given by equation (5.39) as

$$u_r = \frac{q}{2\pi r}$$

When $r = 50$ mm $= 0.05$ m, $$(u_r)_1 = \frac{0.628}{2\pi \times 0.05} = 1.998 \text{ m/s} \simeq 2 \text{ m/s}$$

When $r = 500$ mm $= 0.5$ m, $$(u_r)_2 = \frac{0.628}{2\pi \times 0.5} = 0.2 \text{ m/s}$$

Applying Bernoulli's equation at radius 0.05 m and at radius 0.5 m,

$$\frac{p_1}{\rho g} + \frac{(u_r)_1^2}{2g} = \frac{p_2}{\rho g} + \frac{(u_r)_2^2}{2g}$$

or $$\frac{p_1}{\rho} + \frac{(u_r)_1^2}{2} = \frac{p_2}{\rho} + \frac{(u_r)_2^2}{2}$$

or $$\frac{200 \times 10^3}{1000} + \frac{2^2}{2} = \frac{p_2}{1000} + \frac{0.2^2}{2}$$

or $$200 + 2 = \frac{p_2}{1000} + 0.02$$

or $$\frac{p_2}{1000} = 202 - 0.02 = 201.98$$

$\therefore$ $p_2 = 201.98 \times 1000 \text{ N/m}^2 =$ **201.98 kN/m². Ans.**

(*ii*) *Stream functions at* $\theta = 30°$ *and* $\theta = 60°$

For the source flow, the equation of stream function is given by equation (5.40) as

$$\psi = \frac{q}{2\pi}.\theta, \text{ where } \theta \text{ is in radians}$$

When $\theta = 30°$, $$\psi = \frac{0.628}{2\pi} \times \frac{30 \times \pi}{180} \qquad \left(\because \ \theta = 30° = \frac{30 \times \pi}{180} \text{ radians}\right)$$

= **0.0523 m²/s. Ans.**

When $\theta = 60°$, $$\psi = \frac{0.628}{2\pi} \times \frac{60\pi}{180} = \mathbf{0.1046 \ m^2/s. \ Ans.}$$

▶ 5.16 FREE-VORTEX FLOW

Free-vortex flow is a circulatory flow of a fluid such that its stream lines are concentric circles.
For a free-vortex flow, $u_\theta \times r$ = constant (say C)
Also, circulation around a stream line of an irrotation vortex is

$$\Gamma = 2\pi r \times u_\theta = 2\pi \times C \qquad (\because \ r \times u_\theta = C)$$

where u_θ = tangential velocity at any radius r from the centre.

$$\therefore \qquad u_\theta = \frac{\Gamma}{2\pi r}$$

The circulation Γ is taken positive if the free vortex is anticlockwise.
For a free-vortex flow, the velocity components are

$$u_\theta = \frac{\Gamma}{2\pi r} \quad \text{and} \quad u_r = 0$$

Equation of Stream Function

By definition, the stream function is given by

$$u_\theta = \frac{-\partial\psi}{\partial r} \quad \text{and} \quad u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \text{[See equation (5.12A)]}$$

In case of free-vortex flow, the radial velocity (u_r) is zero. Hence equation of stream function will be obtained from tangential velocity, u_θ. The value of u_θ is given by

$$u_\theta = \frac{\Gamma}{2\pi r}$$

Equating the two values of u_θ, we get

$$-\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r} \quad \text{or} \quad d\psi = -\frac{\Gamma}{2\pi r} dr$$

Integrating the above equation, we get

$$\int d\psi = \int -\frac{\Gamma}{2\pi r} dr = \left(-\frac{\Gamma}{2\pi}\right) \int \frac{1}{r} dr$$

or

$$\psi = \left(-\frac{\Gamma}{2\pi}\right) \log_e r \qquad \left(\because \frac{\Gamma}{2\pi} \text{ is a constant term}\right) \quad \ldots(5.43)$$

The above equation shows that stream function is a function of radius. For a given value of r, the stream function is constant. Hence the stream lines are concentric circles as shown in Fig. 5.37.

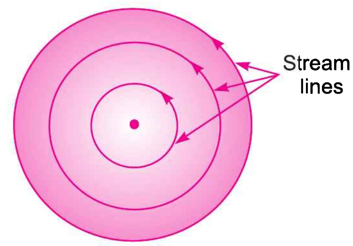


Fig. 5.37

Equation of potential function. By definition, the potential function is given by,

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad \text{and} \quad u_r = \frac{\partial \phi}{\partial r} \qquad \text{[See equation (5.9}A\text{)]}$$

Here $u_r = 0$ and $u_\theta = \dfrac{\Gamma}{2\pi r}$. Hence, the equation of potential function will be obtained from u_θ.

Equating the two values of u_θ, we get

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r} \quad \text{or} \quad d\phi = r.\frac{\Gamma}{2\pi r}.d\theta = \frac{\Gamma}{2\pi} d\theta$$

Integrating the above equation, we get

$$\int d\phi = \int \frac{\Gamma}{2\pi} d\theta \quad \text{or} \quad \phi = \frac{\Gamma}{2\pi}\int d\theta = \frac{\Gamma}{2\pi}.\theta \qquad \ldots(5.44)$$

The above equation shows that velocity potential function is a function of θ. For a given value of θ, potential function is a constant. Hence equipotential lines are radial as shown in Fig. 5.38.

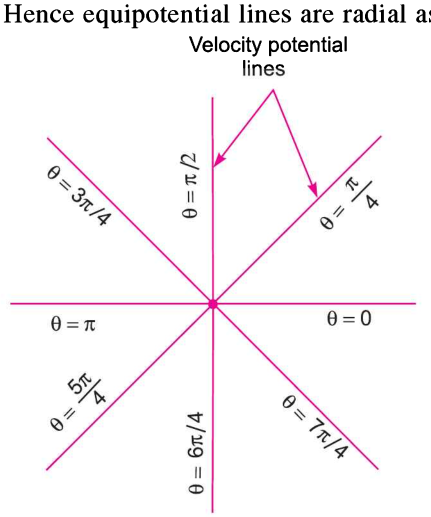


Fig. 5.38 *Potential lines are radial.*

▶ 5.17 SUPER-IMPOSED FLOW

The flow patterns due to uniform flow, a source flow, a sink flow and a free vortex flow can be super-imposed in any linear combination to get a resultant flow which closely resembles the flow around bodies. The resultant flow will still be potential and ideal. The following are the important super-imposed flow :

(*i*) Source and sink pair
(*ii*) Doublet (special case of source and sink combination)
(*iii*) A plane source in a uniform flow (flow past a half body)
(*iv*) A source and sink pair in a uniform flow
(*v*) A doublet in a uniform flow.

5.17.1 Source and Sink Pair. Fig. 5.39 shows a source and a sink of strength q and $(-q)$ placed at A and B respectively at equal distance from the point O on the x-axis. Thus the source and sink are placed symmetrically on the x-axis. The source of strength q is placed at A and sink of strength $(-q)$ is placed at B. The combination of the source and the sink would result in a flownet where stream lines will be circular arcs starting from point A and ending at point B as shown in Fig. 5.40.

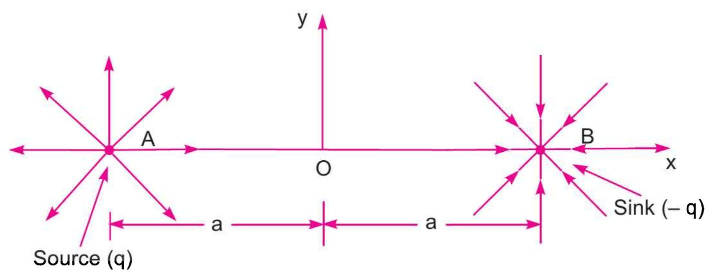


Fig. 5.39 *Source and sink pair.*

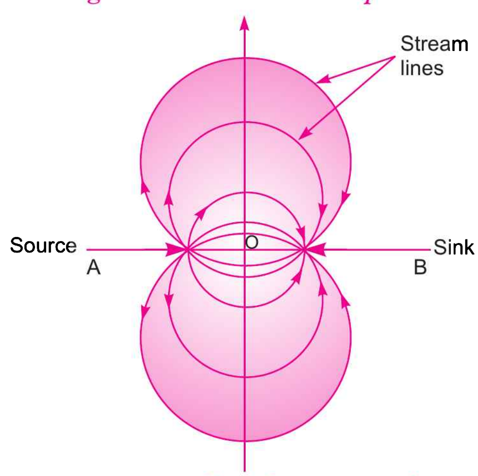


Fig. 5.40 *Stream lines for source-sink pair.*

Equation of stream function and potential function

Let P be any point in the resultant flownet of source and sink as shown in Fig. 5.41.

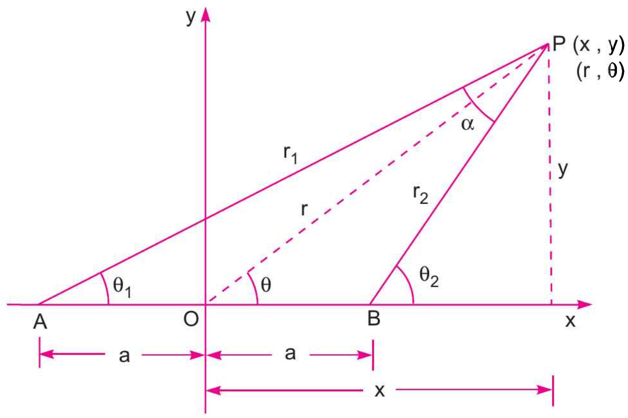


Fig. 5.41

Let r, θ = Cylindrical co-ordinates of point P with respect to origin O

x, y = Corresponding co-ordinates of point P

r_1, θ_1 = Position of point P with respect to source placed at A

r_2, θ_2 = Position of point P with respect to sink placed at B

α = Angle subtended at P by the join of source and sink *i.e.*, angle APB.

Let us find the equation for the resultant stream function and velocity potential function. The equation for stream function due to source is given by equation (5.40) as $\psi_1 = \frac{q \cdot \theta_1}{2\pi}$ whereas due to sink it is given by $\psi_2 = \frac{(-q\theta_2)}{2\pi}$. The equation for resultant stream function (ψ) will be the sum of these two stream function.

$\therefore$
$$\psi = \psi_1 + \psi_2$$

$$= \frac{q\theta_1}{2\pi} + \left(\frac{-q\theta_2}{2\pi}\right) = \frac{-q}{2\pi}(\theta_2 - \theta_1)$$

$$= \frac{-q}{2\pi} \cdot \alpha \quad [\because \ \alpha = \theta_2 - \theta_1. \text{ In triangle } ABP, \theta_1 + \alpha + (180^\circ - \theta_2) = 180^\circ \ \therefore \ \alpha = \theta_2 - \theta_1]$$

$$= \frac{-q \cdot \alpha}{2\pi} \quad \ldots(5.45)$$

The equation for potential function due to source is given by equation (5.41) as $\phi_1 = \frac{q}{2\pi} \log_e r_1$ and due to sink it is given as $\phi_2 = \frac{-q}{2\pi} \log_e r_2$. The equation for resultant potential function (ϕ) will be the sum of these two potential function.

$\therefore$
$$\phi = \phi_1 + \phi_2$$

$$= \frac{q}{2\pi} \log_e r_1 + \left(\frac{-q}{2\pi}\right) \log_e r_2$$

$$= \frac{q}{2\pi}[\log_e r_1 - \log_e r_2] = \frac{q}{2\pi}\log_e\left(\frac{r_1}{r_2}\right) \qquad ...(5.46)$$

To prove that resultant stream lines will be circular arc passing through source and sink

The resultant stream function is given by equation (5.45) as

$$\psi = \frac{-q\,.\,\alpha}{2\pi} \quad \text{or} \quad \frac{-q}{2\pi}(\theta_2 - \theta_1) \qquad (\because \alpha = \theta_2 - \theta_1)$$

For a given stream line ψ = constant. In the above equation the term $\frac{q}{2\pi}$ is also constant. This means that $(\theta_2 - \theta_1)$ or angle α will also be constant for various positions of P in the plane.

To satisfy this, the locus of P must be a circle with AB as chord, having its centre on y-axis, as shown in Fig. 5.40.

Consider the equation (5.45) again as

$$\psi = \frac{-q}{2\pi}\alpha = \frac{-q}{2\pi}(\theta_2 - \theta_1) \qquad (\because \alpha = \theta_2 - \theta_1)$$

$$= \frac{q}{2\pi}(\theta_1 - \theta_2)$$

or
$$(\theta_1 - \theta_2) = \frac{2\pi\psi}{q}$$

Taking tangent to both sides, we get

$$\tan(\theta_1 - \theta_2) = \tan\left(\frac{2\pi\psi}{q}\right) \quad \text{or} \quad \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\,.\,\tan\theta_2} = \tan\left(\frac{2\pi\psi}{q}\right) \qquad ...(i)$$

But
$$\tan\theta_1 = \frac{y}{x+a} \quad \text{and} \quad \tan\theta_2 = \frac{y}{x-a} \qquad ...(5.46A)$$

Substituting the values of $\tan\theta_1$ and $\tan\theta_2$ in equation (*i*),

$$\frac{\frac{y}{(x+a)} - \frac{y}{(x-a)}}{1 + \frac{y}{(x+a)}\cdot\frac{y}{(x-a)}} = \tan\left(\frac{2\pi\psi}{q}\right)$$

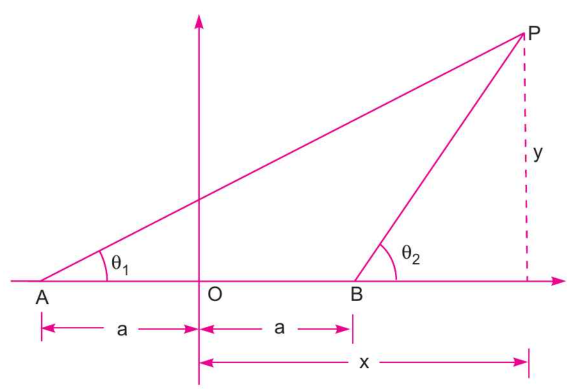


Fig. 5.41 (*a*)

or
$$\frac{y(x-a) - y(x+a)}{x^2 - a^2 + y^2} = \tan\left(\frac{2\pi\psi}{q}\right)$$

or
$$\frac{-2ay}{x^2 - a^2 + y^2} = \tan\left(\frac{2\pi\psi}{q}\right)$$

or
$$\frac{-2ay}{x^2 - a^2 + y^2} = \frac{1}{\cot\left(\frac{2\pi\psi}{q}\right)}$$

or
$$x^2 - a^2 + y^2 = -2ay\cot\left(\frac{2\pi\psi}{q}\right)$$

or $$x^2 - a^2 + y^2 + 2ay \cot\left(\frac{2\pi\psi}{q}\right) = 0$$

or $$x^2 + y^2 + 2ay \cot\left(\frac{2\pi\psi}{q}\right) - a^2 = 0$$

or $$x^2 + y^2 + 2ay \cot\left(\frac{2\pi\psi}{q}\right) + a^2 \cot^2\left(\frac{2\pi\psi}{q}\right) - a^2 \cot^2\left(\frac{2\pi\psi}{q}\right) - a^2 = 0$$

$$\left[\text{Adding and subtracting } a^2 \cot^2\left(\frac{2\pi\psi}{q}\right)\right]$$

or $$x^2 + \left[y + a\cot\left(\frac{2\pi\psi}{q}\right)\right]^2 = a^2 + a^2 \cot^2\left(\frac{2\pi\psi}{q}\right)$$

$$= a^2\left[1 + \cot^2\left(\frac{2\pi\psi}{q}\right)\right]$$

$$= a^2 \operatorname{cosec}^2\left(\frac{2\pi\psi}{q}\right) \qquad \left[\because\ 1 + \cot^2\left(\frac{2\pi\psi}{q}\right) = \operatorname{cosec}^2\left(\frac{2\pi\psi}{q}\right)\right]$$

or $$x^2 + \left[y + a\cot\left(\frac{2\pi\psi}{q}\right)\right]^2 = \left[a \operatorname{cosec}\left(\frac{2\pi\psi}{q}\right)\right]^2 \qquad \ldots(5.47)$$

The above is the equation of a circle* with centre on y-axis at a distance of $\pm\, a \cot\left(\frac{2\pi\psi}{q}\right)$ from the origin. The radius of the circle will be $a \operatorname{cosec}\left(\frac{2\pi\psi}{q}\right)$.

Similarly, it can be shown that the potential lines for the source-sink pair will be eccentric non-intersecting circles with their centres on the x-axis as shown in Fig. 5.41 (b).

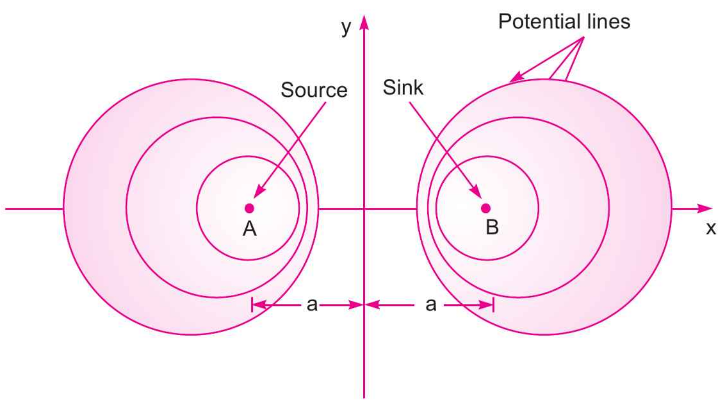


Fig. 5.41 (*b*) *Potential lines for source sink pair (Potential lines are eccentric non-intersecting circles with their centres on x-axis).*

*The equation $x^2 + y^2 = a^2$ is the equation of a circle with centre at origin and of radius 'a'.

Problem 5.36 *A source and a sink of strength 4 m²/s and 8 m²/s are located at (– 1, 0) and (1, 0) respectively. Determine the velocity and stream function at a point P (1, 1) which is lying on the flownet of the resultant stream line.*

Solution. Given :

Source strength, $q_1 = 4\ m^2/s$

Sink strength, $q_2 = 8\ m^2/s$

Distance of the source and sink from origin, $a = 1$ unit.

The position of the source, sink and point P in the flow field is shown in Fig. 5.42.

From Fig. 5.42, it is clear that angle θ_2 will be 90° and angle θ_1 can be calculated from right angled triangle ABP.

The equation for stream function due to source is given by equation (5.40) as $\psi_1 = \frac{q_1 \times \theta_1}{2\pi}$, whereas due to sink it is given by $\psi_2 = \frac{-q_2 \times \theta_2}{2\pi}$. The resultant stream function ψ is given as

$$\psi = \psi_1 + \psi_2$$

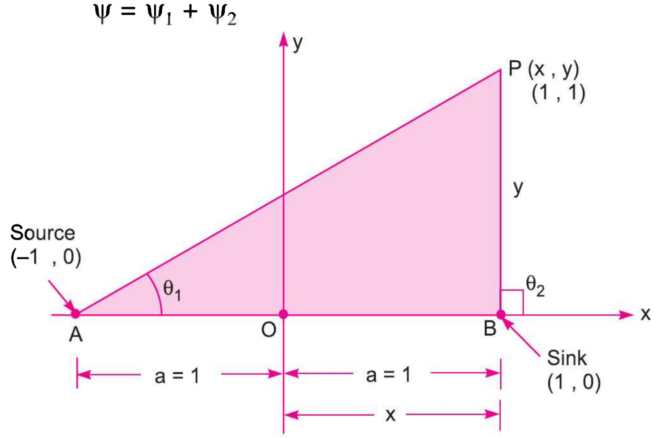


Fig. 5.42

$$= \frac{q_1 \times \theta_1}{2\pi} + \left(\frac{-q_2 \times \theta_2}{2\pi}\right) = \frac{q_1 \times \theta_1}{2\pi} - \frac{q_2 \times \theta_2}{2\pi} \quad \ldots(i)$$

Let us find the values of θ_1 and θ_2 in radians. From the geometry, it is clear that the traingle ABP is a right angled triangle with angle $\theta_2 = 90° = \frac{90}{180} \times \pi = \frac{\pi}{2}$ radians.

Also $$\tan \theta_1 = \frac{BP}{AB} = \frac{1}{2} = 0.5$$

or $$\theta_1 = \tan^{-1} 0.5 = 26.56° = 26.56 \times \frac{\pi}{180} \text{ radians} = 0.463$$

Substituting these values in equation (i),

$$\psi = \frac{q_1}{2\pi} \times 0.463 - \frac{q_2}{2\pi} \times \frac{\pi}{2}$$

$$= \frac{\pi}{2\pi} \times 0.463 - \frac{8}{2\pi} \times \frac{\pi}{2} \qquad (\because\ q_1 = 4 \text{ m}^2/\text{s},\ q_2 = 8 \text{ m}^2/\text{s})$$

$$= 0.294 - 2.0 = \mathbf{-\ 1.706\ m^2/s.\ Ans.}$$

To find the velocity at the point P, let us first find the stream function in terms of x and y co-ordinates. The stream function in terms of θ_1 and θ_2 is given by equation (*i*) above as

$$\psi = \frac{q_1 \times \theta_1}{2\pi} - \frac{q_2 \times \theta_2}{2\pi}$$

The values of θ_1 and θ_2 in terms of x, y and a are given by equation (5.46*A*) as

$$\tan \theta_1 = \frac{y}{x + a} \quad \text{and} \quad \tan \theta_2 = \frac{y}{(x - a)}$$

or

$$\theta_1 = \tan^{-1} \frac{y}{x + a} \quad \text{and} \quad \theta_2 = \tan^{-1} \frac{y}{(x - a)}$$

Substituting these values of θ_1 and θ_2 in equation (*i*), we get

$$\psi = \frac{q_1}{2\pi} \tan^{-1} \frac{y}{x + a} - \frac{q_2}{2\pi} \tan^{-1} \frac{y}{x - a}$$

The velocity component $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$.

$$\therefore \qquad u = \frac{\partial \psi}{\partial y}$$

$$= \frac{\partial}{\partial y} \left[\frac{q_1}{2\pi} \tan^{-1} \frac{y}{x + a} - \frac{q_2}{2\pi} \tan^{-1} \frac{y}{x - a} \right]$$

$$= \frac{q_1}{2\pi} \times \frac{1}{1 + \left(\dfrac{y}{x + a} \right)^2} \times \frac{1}{(x + a)} - \frac{q_2}{2\pi} \times \frac{1}{1 + \left(\dfrac{y}{x - a} \right)^2} \times \frac{1}{(x - a)}$$

$$= \frac{q_1}{2\pi} \frac{(x + a)^2}{(x + a)^2 + y^2} \times \frac{1}{(x + a)} - \frac{q_2}{2\pi} \times \frac{(x - a)^2}{(x - a)^2 + y^2} \times \frac{1}{(x - a)}$$

$$= \frac{q_1}{2\pi} \frac{(x + a)}{(x + a)^2 + y^2} - \frac{q_2}{2\pi} \frac{(x - a)}{(x - a)^2 + y^2}$$

At the point $P(1, 1)$, the component u is obtained by substituting $x = 1$ and $y = 1$ in the above equation. The value of a is also equal to one.

$$\therefore \qquad u = \frac{q_1}{2\pi} \frac{1 + 1}{(1 + 1)^2 + 1^2} - \frac{q_2}{2\pi} \frac{(1 - 1)}{(1 - 1)^2 + 1^2}$$

$$= \frac{q_1}{2\pi} \frac{2}{5} - \frac{q_2}{2\pi} \times 0 = \frac{q_1}{2\pi} \times \frac{2}{5} = \frac{4}{2\pi} \times \frac{2}{5} = 0.2544 \text{ m/s}$$

Now $$v = -\frac{\partial \psi}{\partial x}$$

$$= -\frac{\partial}{\partial x}\left[\frac{q_1}{2\pi}\tan^{-1}\frac{y}{x+a} - \frac{q_2}{2\pi}\tan^{-1}\frac{y}{x-a}\right]$$

$$= -\left[\frac{q_1}{2\pi}\frac{1}{1+\left(\frac{y}{x+a}\right)^2}\times\frac{y(-1)}{(x+a)^2}\times 1 - \frac{q_2}{2\pi}\times\frac{1}{1+\left(\frac{y}{x-a}\right)^2}\times\frac{y(-1)}{(x-a)^2}\times 1\right]$$

$$= -\left[\frac{q_1}{2\pi}\frac{(x+a)^2}{(x+a)^2+y^2}\times\frac{(-y)}{(x+a)^2} - \frac{q_2}{2\pi}\frac{(x-a)^2}{(x-a)^2+y^2}\times\frac{(-y)}{(x-a)^2}\right]$$

$$= \frac{q_1}{2\pi}\frac{y}{(x+a)^2+y^2} - \frac{q_2}{2\pi}\frac{y}{(x-a)^2+y^2}$$

At the point $P(1, 1)$,

$$v = \frac{q_1}{2\pi}\times\frac{1}{(1+1)^2+1^2} - \frac{q_2}{2\pi}\times\frac{1}{(1-1)^2+1^2} \qquad (\because a = 1)$$

$$= \frac{q_1}{2\pi}\times\frac{1}{5} - \frac{q_2}{2\pi}\times\frac{1}{1}$$

$$= \frac{q_1}{2\pi}\times\frac{1}{5} - \frac{q_2}{2\pi} = \frac{4}{2\pi}\times\frac{1}{5} - \frac{8}{2\pi} = 0.1272 - 1.272 = -1.145 \text{ m/s}^2$$

$\therefore$ The resultant velocity, $V = \sqrt{u^2+v^2} = \sqrt{0.2544^2 + (-1.145)^2}$ = **1.174 m/s. Ans.**

Problem 5.37 *For the above problem, determine the pressure at P(1, 1) if the pressure at infinity is zero and density of fluid is 1000 kg/m³.*

Solution. Given :

Pressure at infinity, $p_0 = 0$

Density of fluid, $\rho = 1000$ kg/m³

The velocity* of fluid at infinity will be zero. If V_0 = velocity at infinity, then $V_0 = 0$.

The resultant velocity of fluid at $P(1, 1)$ = 1.174 m/s (calculated above)

or $V = 1.174$ m/s.

Let p = pressure at $P(1, 1)$

Applying Bernoulli's theorem at point at infinity and at point P, we get

$$\frac{p_0}{\rho g} + \frac{V_0^2}{2g} = \frac{p}{\rho g} + \frac{V^2}{2g}$$

or $$0 + 0 = \frac{p}{\rho g} + \frac{V^2}{2g} \quad\text{or}\quad 0 = \frac{p}{\rho g} + \frac{V^2}{2g} \quad\text{or}\quad 0 = \frac{p}{\rho} + \frac{V^2}{2}$$

or $$\frac{p}{\rho} = -\frac{V^2}{2} = -\frac{1.174^2}{2} \qquad (\because V = 1.174 \text{ m/s})$$

---

* From equation (5.39), the velocity at a distance 'r' from source or sink is given by $u_r = \frac{q}{2\pi r}$. At infinity, r is very very large hence velocity is zero.

or $$p = -\frac{1.174^2}{2} \times \rho = -\frac{1.174^2 \times 1000}{2} = \mathbf{-\ 689.14\ N/m^2.\ Ans.}$$

5.17.2 Doublet. It is a special case of a source and sink pair (both of them are of equal strength) when the two approach each other in such a way that the distance $2a$ between them approaches zero and the product $2a \cdot q$ remains constant. This product $2a \cdot q$ is known as doublet strength and is denoted by μ.

$\therefore$ Doublet strength, $\mu = 2a \cdot q$ ...(5.48)

Let q and $(-q)$ may be the strength of the source and the sink respectively as shown in Fig. 5.43. Let $2a$ be the distance between them and P be any point in the combined field of source and sink.

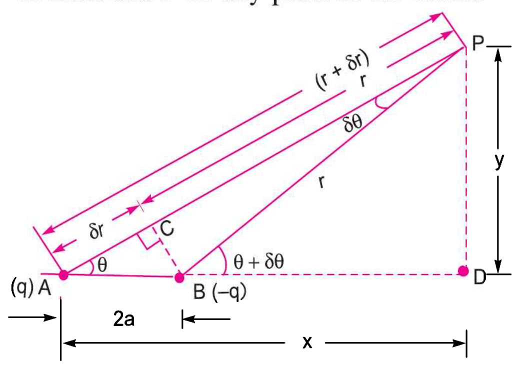


Fig. 5.43

Let θ is the angle made by P at A whereas $(\theta + \delta\theta)$ is the angle at B.

Now the stream function at P,

$$\psi = \frac{q\theta}{2\pi} - \frac{q}{2\pi}(\theta + \delta\theta) = -\frac{q}{2\pi}\delta\theta \quad \text{...(5.49)}$$

From B, draw $BC \perp$ on AP. Let $AC = \delta r$, $CP = r$ and $AP = r + \delta r$. Also angle $BPC = \delta\theta$. The angle $\delta\theta$ is very small. The distance BC can be taken equal to $r \times \delta\theta$. In triangle ABC, angle $BCA = 90°$ and hence distance BC is also equal to $2a \cdot \sin\theta$. Equating the two values of BC, we get

$$r \times \delta\theta = 2a \cdot \sin\theta$$

$$\therefore \quad \delta\theta = \frac{2a \cdot \sin\theta}{r}$$

Substituting the value of $\delta\theta$ in equation (5.49), we get

$$\psi = -\frac{q}{2\pi} \times \frac{2a\sin\theta}{r}$$

$$= -\frac{\mu}{2\pi} \times \frac{\sin\theta}{r} \quad [\because\ 2a \cdot q = \mu \text{ from equation (5.48)}] \text{ ...(5.50)}$$

In Fig. 5.43, when $2a \to 0$, the angle $\delta\theta$ subtended by point P with A and B becomes very small. Also $\delta r \to 0$ and AP becomes equal to r. Then

$$\sin\theta = \frac{PD}{AP} = \frac{y}{r}$$

Also $AP^2 = AD^2 + PD^2$ or $r^2 = x^2 + y^2$

Substituting the value of $\sin\theta$ in equation (5.50), we get

$$\psi = -\frac{\mu}{2\pi} \times \frac{y}{r} \times \frac{1}{r} = -\frac{\mu y}{2\pi r^2} = -\frac{\mu y}{2\pi\left(x^2 + y^2\right)} \qquad (\because \quad r^2 = x^2 + y^2) \qquad \text{...(5.50A)}$$

or $$x^2 + y^2 = -\frac{\mu y}{2\pi\psi} \quad \text{or} \quad x^2 + y^2 + \frac{\mu y}{2\pi\psi} = 0$$

The above equation can be written as

$$x^2 + y^2 + 2 \times y \times \frac{\mu}{4\pi\psi} + \left(\frac{\mu}{4\pi\psi}\right)^2 - \left(\frac{\mu}{4\pi\psi}\right)^2 = 0 \qquad \left[\text{Adding and subtracting } \left(\frac{\mu}{4\pi\psi}\right)^2\right]$$

or $$x^2 + \left(y + \frac{\mu}{4\pi\psi}\right)^2 = \left(\frac{\mu}{4\pi\psi}\right)^2 \qquad \text{...(5.51)}$$

The above is the equation of a circle with centre $\left(0, \frac{\mu}{4\pi\psi}\right)$ and radius $\frac{\mu}{4\pi\psi}$. The centre of the circle lies on y-axis at a distance of $\frac{\mu}{4\pi\psi}$ from x-axis. As the radius of the circle is also equal to $\frac{\mu}{4\pi\psi}$, hence the circle will be tangent to the x-axis. Hence stream lines of the doublet will be the family of circles tangent to the x-axis as shown in Fig. 5.44.

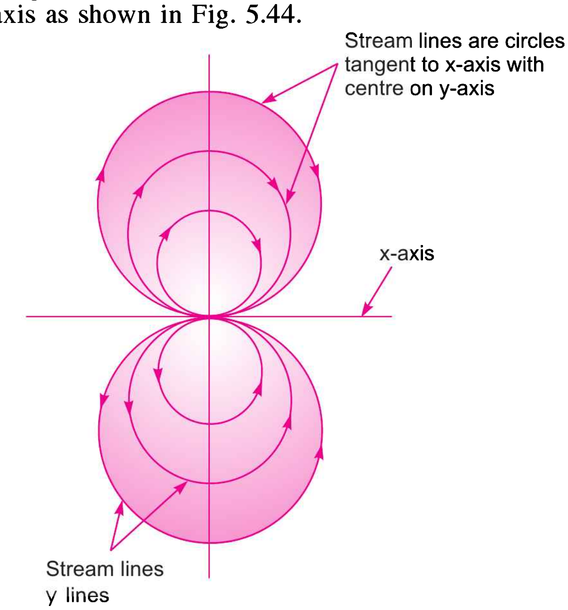


Fig. 5.44 *Stream lines for a doublet.*

Potential function at P

Refer to Fig. 5.43. The potential function at P is given by

$$\phi = \frac{q}{2\pi} \log_e (r + \delta r) + \left(-\frac{q}{2\pi}\right) \log_e r \qquad \text{[Refer to equation (5.41)]}$$

$$= \frac{q}{2\pi}\log_e (r + \delta r) - \frac{q}{2\pi}\log_e r = \frac{q}{2\pi}\log_e\left(\frac{r+\delta r}{r}\right) = \frac{q}{2\pi}\log_e\left(1+\frac{\delta r}{r}\right)^*$$

$$= \frac{q}{2\pi}\left[\frac{\delta r}{r} + \left(\frac{\delta r}{r}\right)^2 \times \frac{1}{2} + ...\right]$$

$$= \frac{q}{2\pi}\cdot\frac{\delta r}{r} \qquad \left[\text{As } \frac{\delta r}{r} \text{ is a small quantity. Hence } \left(\frac{\delta r}{r}\right)^2 \text{ becomes negligible}\right]$$

But in Fig. 5.43, from triangle *ABC*, we get $\frac{\delta r}{2a} = \cos\theta$

$\therefore \qquad \delta r = 2a\cos\theta$

Substituting the value of δr, we get

$$\phi = \frac{q}{2\pi} \times \frac{2a\cos\theta}{r}$$

$$= \frac{\mu}{2\pi} \times \frac{\cos\theta}{r} \qquad [\because 2a \times q = \mu \text{ from equation } (i)] \quad ...(5.52)$$

In Fig. 5.43, when $2a \to 0$, the angle $\delta\theta$ becomes very small.

Also $\delta r \to 0$ and *AP* becomes equal to *r*. Then

$$\cos\theta = \frac{AD}{AP} = \frac{x}{r}$$

Also $AP^2 = AD^2 + PD^2$ or $r^2 = x^2 + y^2$

Substituting the value of cos θ in equation (5.52), we get

$$\phi = \frac{\mu}{2\pi} \times \left(\frac{x}{r}\right) \times \frac{1}{r} = \frac{\mu}{2\pi} \times \frac{x}{r^2}$$

$$= \frac{\mu}{2\pi} \times \frac{x}{(x^2+y^2)} \qquad [\because r^2 = x^2 + y^2]$$

or $$x^2 + y^2 = \frac{\mu}{2\pi} \times \frac{x}{\phi} \quad \text{or} \quad x^2 + y^2 - \frac{\mu}{2\pi} \times \frac{x}{\phi} = 0$$

The above equation can be written as

$$x^2 - \frac{\mu}{2\pi}\frac{x}{\phi} + \left(\frac{\mu}{4\pi\phi}\right)^2 - \left(\frac{\mu}{4\pi\phi}\right)^2 + y^2 = 0 \qquad \left[\text{Adding and subtracting } \left(\frac{\mu}{4\pi\phi}\right)^2\right]$$

or $$\left(x - \frac{\mu}{4\pi\phi}\right)^2 + y^2 = \left(\frac{\mu}{4\pi\phi}\right)^2 \qquad ...(5.53)$$

The above is the equation of a circle with centre $\left(\frac{\mu}{4\pi\phi}, 0\right)$ and radius $\left(\frac{\mu}{4\pi\phi}\right)$. The centre of the circle lies on *x*-axis at a distance of $\frac{\mu}{4\pi\phi}$ from *y*-axis. As the radius of the circle is equal to the distance of the centre of the circle from the *y*-axis, hence the circle will be tangent to the *y*-axis.

* Expansion of $\log_e (1 + x) = x + \frac{x^2}{2} + ...$

Hence the potential lines of a doublet will be a family of circles tangent to the y-axis with their centres on the x-axis as shown in Fig. 5.45.

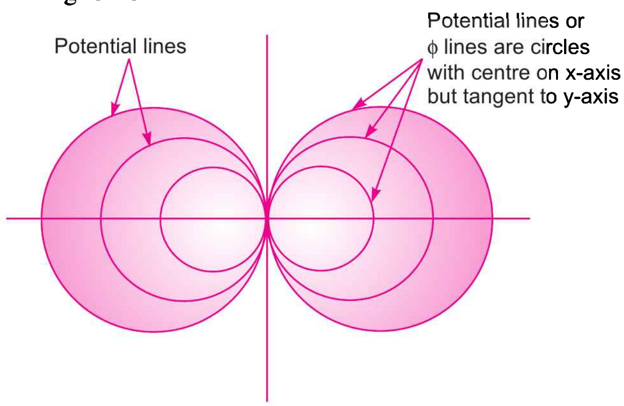


Fig. 5.45 *Potential lines for a doublet.*

Problem 5.38 *A point P(0.5, 1) is situated in the flow field of a doublet of strength 5 m^2/s. Calculate the velocity at this point and also the value of the stream function.*

Solution. Given : Point $P(0.5, 1)$. This means $x = 0.5$ and $y = 1.0$

Strength of doublet, $\mu = 5\ m^2/s$

(*i*) *Velocity at point P*

The velocity at the given point can be obtained if we know the stream function (ψ). But stream function is given by equation (5.50*A*) as

$$\psi = -\frac{\mu}{2\pi} \times \frac{y}{(x^2 + y^2)}$$

The velocity components u and v are obtained from the stream function as

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left[-\frac{\mu}{2\pi} \times \frac{y}{(x^2 + y^2)}\right]$$

$$= -\frac{\mu}{2\pi}\frac{\partial}{\partial y}\left[\frac{y}{(x^2 + y^2)}\right] \qquad \left(\because \frac{\mu}{2\pi} \text{ is a constant term}\right)$$

$$= -\frac{\mu}{2\pi}\left[\frac{x^2 - y^2}{(x^2 + y^2)^2}\right]$$

$$\left[\because \frac{\partial}{\partial y}\left[y(x^2 + y^2)^{-1}\right] = y[-1][x^2 + y^2]^{-2}[2y] + (x^2 + y^2)^{-1}.1\right.$$

$$\left. = \frac{-2y^2}{(x^2 + y^2)^2} + \frac{1}{(x^2 + y^2)} = \frac{-2y^2 + x^2 + y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}\right]$$

and
$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[-\frac{\mu}{2\pi} \times \frac{y}{(x^2 + y^2)}\right]$$

$$= \frac{\mu}{2\pi}\frac{\partial}{\partial x}\left[\frac{y}{(x^2+y^2)}\right] = \frac{\mu}{2\pi}\left[\frac{-2xy}{(x^2+y^2)^2}\right]$$

Substituting the values of $\mu = 5\ m^2/s$, $x = 0.5$ and $y = 1.0$, we get the velocity components as

$$u = -\frac{\mu}{2\pi}\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right] = -\frac{5}{2\pi}\left[\frac{0.5^2-1^2}{(0.5^2+1^2)^2}\right] = -\frac{5}{2\pi}\frac{0.75}{1.25^2} = -0.382$$

and $$v = \frac{\mu}{2\pi}\left[\frac{-2xy}{(x^2+y^2)^2}\right] = \frac{5}{2\pi}\left[\frac{-2\times 0.5\times 1}{(0.5^2+1^2)^2}\right] = \frac{5}{2\pi}\left[\frac{-1}{1.25^2}\right] = -0.509$$

$\therefore$ Resultant velocity, $V = \sqrt{u^2+v^2} = \sqrt{(-0.382)^2+(-0.509)^2}$ = **0.636 m/s. Ans.**

(*ii*) *Value of stream function at point P*

$$\psi = -\frac{\mu}{2\pi}\frac{y}{(x^2+y^2)} = -\frac{5}{2\pi}\times\frac{1.0}{(0.5^2+1^2)} = -\frac{5}{2\pi}\times\frac{1}{1.25}$$

$$= \mathbf{-0.636\ m^2/s.\ Ans.}$$

Solution in polar co-ordinates

The above question can also be done in r, θ (*i.e.*, polar) co-ordinates. The stream function in r, θ co-ordinates is given by equation (5.50) as

$$\psi = -\frac{\mu}{2\pi}\times\frac{\sin\theta}{r} \quad ...(i)$$

and velocity components in radial and tangential directions are given as

$$u_r = \frac{1}{r}\times\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[-\frac{\mu}{2\pi}\frac{\sin\theta}{r}\right]$$

$$= \frac{1}{r}\times\left(-\frac{\mu}{2\pi}\right)\times\frac{1}{r}\frac{\partial}{\partial\theta}(\sin\theta)$$

$$\left[\because \frac{\mu}{2\pi} \text{ is a constant term and also } r \text{ is constant w.r.t. } \theta\right]$$

$$= -\frac{\mu}{2\pi}\times\frac{1}{r^2}\cos\theta \quad ...(ii)$$

and $$u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\partial}{\partial r}\left[-\frac{\mu}{2\pi}\frac{\sin\theta}{r}\right]$$

$$= -\left(-\frac{\mu}{2\pi}\sin\theta\right)\frac{\partial}{\partial r}\left[\frac{1}{r}\right] = \frac{\mu}{2\pi}\sin\theta\,(-1)\,.\,\frac{1}{r^2}$$

$$\left[\because \frac{\mu\sin\theta}{2\pi} \text{ is a constant w.r.t. } r\right]$$

$$= -\frac{\mu}{2\pi}\times\frac{\sin\theta}{r^2} \quad ...(iii)$$

Now $r = \sqrt{x^2 + y^2} = \sqrt{0.5^2 + 1^2} = \sqrt{1.25}$

$\therefore$ $\sin\theta = \dfrac{y}{r} = \dfrac{1}{\sqrt{1.25}} = 0.894$ and $\cos\theta = \dfrac{x}{r} = \dfrac{0.5}{\sqrt{1.25}} = 0.447$

Substituting the values of r, $\sin\theta$ and $\cos\theta$ in above equations (*i*), (*ii*) and (*iii*), we get

$$\psi = -\frac{\mu}{2\pi}\frac{\sin\theta}{r} = -\frac{5}{2\pi}\times\frac{0.894}{\sqrt{1.25}} = \mathbf{-\ 0.636\ m^2/s.\ Ans.}$$

$$u_r = -\frac{\mu}{2\pi}\times\frac{1}{r^2}\times\cos\theta = -\frac{5}{2\pi}\times\frac{1}{(1.25)}\times 0.447 = -\ 0.2845\ \text{m/s}$$

and $$u_\theta = -\frac{\mu}{2\pi}\times\frac{\sin\theta}{r^2} = -\frac{5}{2\pi}\times\frac{0.894}{1.25} = -\ 0.569\ \text{m/s}$$

$\therefore$ Resultant velocity, $$V = \sqrt{u_r^2 + u_\theta^2}$$

$$= \sqrt{(-\,0.2845)^2 + (-\,0.569)^2} = \mathbf{0.636\ m/s.\ Ans.}$$

5.17.3 A Plane Source in a Uniform Flow (Flow Past a Half-Body). Fig. 5.46 (*a*) shows a uniform flow of velocity U and Fig. 5.46 (*b*) shows a source flow of strength q. When this uniform flow is flowing over the source flow, a resultant flow will be obtained as shown in Fig. 5.46. This resultant flow is also known as the flow past a half-body. Let the source is placed on the origin O. Consider a point $P(x, y)$ lying in the resultant flow field with polar co-ordinates r and θ as shown in Fig. 5.46.

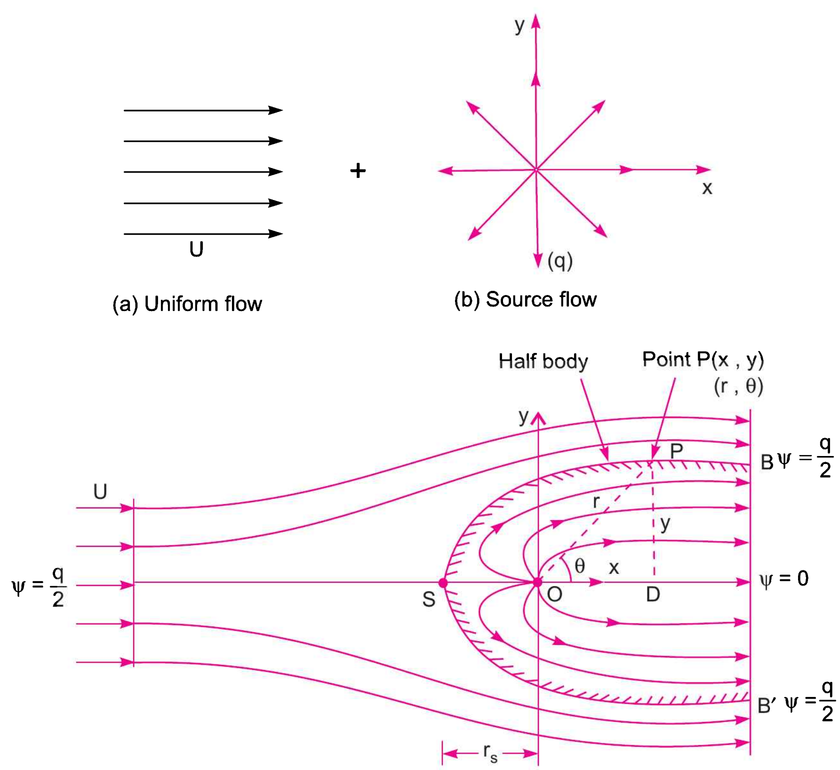


Fig. 5.46 *Flow pattern resulting from the combination of a uniform flow and a source.*

The stream function (ψ) and potential function (ϕ) for the resultant flow are obtained as given below :

$$\psi = \text{Stream function due to uniform flow + stream function due to source}$$

$$= U \, . \, y + \frac{q}{2\pi} \theta \qquad \text{...(5.54)}$$

$$= U \, . \, r \sin \theta + \frac{q}{2\pi} \theta \qquad (\because \; y = r \sin \theta) \text{ ...(5.54A)}$$

and ϕ = Velocity potential function due to uniform flow + Velocity potential function due to source

$$= U \, . \, x + \frac{q}{2\pi} \log_e r = U \, . \, r \cos \theta + \frac{q}{2\pi} \log_e r \qquad \text{...(5.54B)}$$

The following are the important points for the resultant flow pattern :

(*i*) *Stagnation point.* On the left side of the source, at the point S lying on the x-axis, the velocity of uniform flow and that due to source are equal and opposite to each other. Hence the net velocity of the combined flow field is zero. This point is known as stagnation point and is denoted by S. The polar co-ordinates of the stagnation point S are r_S and π, where r_S is radial distance of point S from O.

The net velocity (or resultant velocity) is zero at the stagnation point S.

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}\left(U . r \sin \theta + \frac{q}{2\pi}\theta\right) \qquad \left[\because \; \psi = U . r \sin \theta + \frac{q}{2\pi}\theta\right]$$

$$\therefore \qquad = \frac{1}{r}\left[U . r \cos \theta + \frac{q}{2\pi}\right] = U \, . \, \cos \theta + \frac{q}{2\pi r}$$

At the stagnation point, $\theta = \pi$ radians (180°) and $r = r_S$ and net velocity is zero. This means $u_r = 0$ and $v_\theta = 0$. Substituting these values in the above equation, we get

$$0 = U \, . \, \cos 180° + \frac{q}{2\pi r_S} \qquad [\because \; u_r = 0, \theta = 180° \text{ and } r = r_S]$$

$$= - U + \frac{q}{2\pi r_S} \quad \text{or} \quad U = \frac{q}{2\pi r_S}$$

or

$$r_S = \frac{q}{2\pi U} \qquad \text{...(5.55)}$$

From the above equation it is clear that position of stagnation point depends upon the free stream velocity U and source strength q. At the stagnation point, the value of stream function is obtained from equation (5.54*A*) as

$$\psi = U \, . \, r \sin \theta + \frac{q}{2\pi} \, . \, \theta$$

For the stagnation point, the above equation becomes as

$$\therefore \qquad \psi_s = U . r_s \sin 180° + \frac{q}{2\pi} \times \theta$$

$$[\because \; \text{At stagnation point, } \theta = \pi \text{ radians} = 180° \text{ and } r = r_S]$$

$$= 0 + \frac{q}{2} = \frac{q}{2} \qquad \text{...(5.56)}$$

The above relation gives the equation of stream line passing through stagnation point. We know that no fluid mass crosses a stream line. Hence a stream line is a *virtual solid surface.*

(*ii*) *Shape of resultant flow*. At the stagnation point S, the net velocity is zero. The fluid particles that issue from the source cannot proceed further to the left of stagnation point. They are carried along the contour BSB' that separates the source flow from uniform flow. The curve BSB' can be regarded as the **solid boundary** of a round nosed body such as a bridge pier around which the uniform flow is forced to pass. The contour BSB' is called the half body, because it has only the leading point, it trails to infinity at down stream end.

The value of stream function of the stream line passing through stagnation point S and passing over the solid boundary (*i.e.*, curve BSB') is $\psi_S = \frac{q}{2}$.

Thus the composite flow consists of :

(1) flow over a plane half-body (*i.e.*, flow over curve BSB') outside $\psi = \frac{q}{2}$ and

(2) source flow within the plane half-body.

The plane half-body is described by the dividing stream line, $\psi = \frac{q}{2}$.

But the stream function at any point in the combined flow field is given by equation (5.54) as

$$\psi = U \,.\, y + \frac{q}{2\pi}\,\theta$$

If we take $\psi = \frac{q}{2}$ in the above equation, we will get the equation of the dividing stream line.

$\therefore$ Equation of the dividing stream line (*i.e.*, equation of curve BSB') will be

$$\frac{q}{2} = U \,.\, y + \frac{q}{2\pi}\,.\,\theta \text{ or } U \,.\, y = \frac{q}{2} - \frac{q}{2\pi}\,\theta = \frac{q}{2}\left(1 - \frac{\theta}{\pi}\right)$$

or
$$y = \frac{q}{2U}\left(1 - \frac{\theta}{\pi}\right) \qquad \text{...(5.57)}$$

From the above equation, the main dimensions of the plane half-body may be obtained. From this equation, it is clear that y is maximum, when $\theta = 0$.

Hence At $\theta = 0$, $\quad y$ is maximum and $y_{max} = \frac{q}{2U}$ ... the maximum ordinate

At $\theta = \frac{\pi}{2}$, $\quad y = \frac{q}{2U}\left(1 - \frac{\pi}{2}\,.\,\frac{1}{\pi}\right) = \frac{q}{4U}$ ... the ordinate above the origin

At $\theta = \pi$, $\quad y = \frac{q}{2U}\left(1 - \frac{\pi}{\pi}\right) = 0$ ... the leading point of the half-body

At $\theta = \frac{3\pi}{2}$, $\quad y = \frac{q}{2U}\left(1 - \frac{3\pi}{2\pi}\right) = -\frac{q}{4U}$ ... the ordinate below the origin.

The main dimensions are shown in Fig. 5.47.

(*iii*) *Resultant velocity at any point*

The velocity components at any point in the flow field are given by

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\frac{d}{d\theta}\left[U.r\sin\theta + \frac{q}{2\pi}\,\theta\right]$$

$$= \frac{1}{r}\left[U.r\cos\theta + \frac{q}{2\pi}\right] = U.\cos\theta + \frac{q}{2\pi r}$$

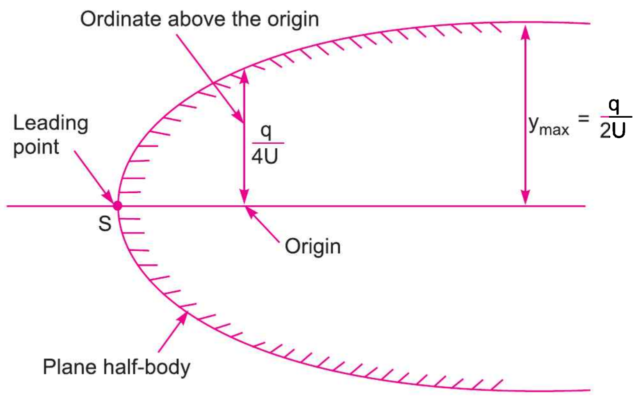


Fig. 5.47

The above equation gives the radial velocity at any point in the flow field. This radial velocity is due to uniform flow and due to source. Due to source the radial velocity is $\frac{q}{2\pi r}$. Hence the velocity due to source diminishes with increase in radial distance from the source. At large distance from the source the contribution of source is negligible and hence free stream uniform flow is not influenced by the presence of source.

$$u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\partial}{\partial r}\left[U.r\sin\theta + \frac{q}{2\pi}\theta\right]$$

$$= -[U \,.\, \sin\theta + 0] = -U\sin\theta \qquad \left[\because \frac{q}{2\pi}\theta \text{ is constant w. r. t. } r\right]$$

$\therefore$ Resultant velocity, $V = \sqrt{u_r^2 + u_\theta^2}$

(*iv*) *Location of stagnation point*

At the stagnation point, the velocity components are zero. Hence equating the radial and tangential velocity components to zero, we get

$$u_r = 0 \quad \text{or} \quad U\cos\theta + \frac{q}{2\pi r} = 0 \quad \text{or} \quad U\cos\theta = -\frac{q}{2\pi r}$$

or $$r\cos\theta = -\frac{q}{2\pi U} \quad \text{But} \quad r\cos\theta = x$$

$\therefore$ $$x = -\frac{q}{2\pi U}$$

When $u_\theta = 0$ or $-U\sin\theta = 0$ or $\sin\theta = 0$ as U cannot be zero

or $\theta = 0$ or π But $y = r\sin\theta$ $\therefore$ $y = 0$

Hence stagnation point is at $\left(-\frac{q}{2\pi U}, 0\right)$, the leading point of the half-body.

(*v*) *Pressure at any point in flow field*

Let p_0 = pressure at infinity where velocity is U

p = pressure at any point P in the flow field, where velocity is V

Now applying the Bernoulli's equation at a point at infinity and at a point P in the flow field, we get

$$\frac{p_0}{\rho g} + \frac{U^2}{2g} = \frac{p}{\rho g} + \frac{V^2}{2g} \quad \text{or} \quad \frac{U^2}{2g} - \frac{V^2}{2g} = \frac{p}{\rho g} - \frac{p_0}{\rho g} = \frac{p - p_0}{\rho g}$$

The pressure co-efficient is defined as

$$C_p = \frac{p - p_0}{\frac{1}{2}\rho U^2}$$

$$= \frac{\rho g\left[\frac{U^2}{2g} - \frac{V^2}{2g}\right]}{\frac{1}{2}\rho U^2} \qquad \left[\because \; p - p_0 = \rho g\left(\frac{U^2}{2g} - \frac{V^2}{2g}\right)\right]$$

$$= \frac{U^2 - V^2}{U^2} = 1 - \left(\frac{V}{U}\right)^2 \qquad ...(5.58)$$

Problem 5.39 *A uniform flow with a velocity of 3 m/s is flowing over a plane source of strength 30 m^2/s. The uniform flow and source flow are in the same plane. A point P is situated in the flow field. The distance of the point P from the source is 0.5 m and it is at an angle of 30° to the uniform flow. Determine : (i) stream function at point P, (ii) resultant velocity of flow at P and (iii) location of stagnation point from the source.*

Solution. Given : Uniform velocity, U = 3 m/s ; source strength, q = 30 m^2/s ; co-ordinates of point P are r = 0.5 m and θ = 30°.

(*i*) *Stream function at point P*

The stream function at any point in the combined flow field is given by equation (5.54*A*)

$$\psi = U \,.\, r \sin \theta + \frac{q}{2\pi}\,\theta$$

at point P, r = 0.5 m and θ = 30° or $\frac{30}{180} \times \pi$ radians.

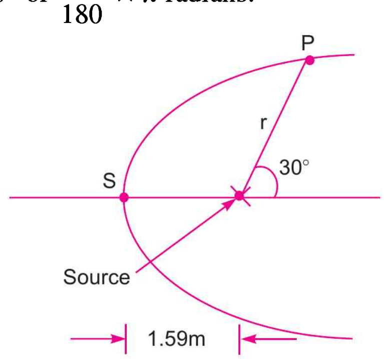


Fig. 5.48

∴ Stream function at point P,

$$\psi = 3 \times 0.5 \times \sin 30° + \frac{30}{2\pi} \times \left(\frac{30}{180} \times \pi\right)$$

$$= 0.75 + 2.5 = \mathbf{3.25\ m^2/s.\ Ans.}$$

(*ii*) *Resultant velocity at P*

The velocity components anywhere in the flow are given by

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[U.r\sin\theta + \frac{q}{2\pi}\theta\right]$$

$$= \frac{1}{r}\left[U.r\cos\theta + \frac{q}{2\pi}\right] = U.\cos\theta + \frac{q}{2\pi r}$$

$$= 3 \times \cos 30° + \frac{30}{2\pi \times 0.5} \qquad (\because \text{ At } P,\ r = 0.5,\ \theta = 30°,\ q = 30)$$

$$= 2.598 + 9.55 = 12.14$$

and

$$u_\theta = \frac{-\partial\psi}{\partial r} = -\frac{\partial}{\partial r}\left[U.r\sin\theta + \frac{q}{2\pi}.\theta\right]$$

$$= -U\sin\theta + 0 = -U\sin\theta$$

$$= -3 \times \sin 30° = -1.5$$

$\therefore$ Resultant velocity,
$$V = \sqrt{u_r^2 + u_\theta^2}$$

$$= \sqrt{12.14^2 + (-1.5)^2} = \mathbf{12.24\ m/s.\ Ans.}$$

(*iii*) *Location of stagnation point*

The horizontal distance of the stagnation point S from the source is given by equation (5.55) as

$$r_S = \frac{q}{2\pi U} = \frac{30}{2\pi \times 3} = \mathbf{1.59\ m.\ Ans.}$$

The stagnation point will be at a distance of 1.59 m to the left side of the source on the x-axis.

Problem 5.40 *A uniform flow with a velocity of 20 m/s is flowing over a source of strength 10 m²/s. The uniform flow and source flow are in the same plane. Obtain the equation of the dividing stream line and sketch the flow pattern.*

Solution. Given : Uniform velocity, U = 20 m/s ; Source strength, q = 10 m²/s

(*i*) *Equation of the dividing stream line*

The stream function at any point in the combined flow field is given by equation (5.54*A*)

$$\psi = U\ .\ r\sin\theta + \frac{q}{2\pi}\theta$$

$$= 20 \times r\sin\theta + \frac{10}{2\pi}\theta \qquad (\because\ U = 20 \text{ m/s and } q = 10 \text{ m}^2\text{/s})$$

The value of the stream function for the dividing stream line is $\psi = \frac{q}{2}$. Hence substituting $\psi = \frac{q}{2}$ in the above equation, we get the equation of the dividing stream line.

$$\therefore \qquad \frac{q}{2} = 20r\sin\theta + \frac{10}{2\pi}\theta$$

or

$$\frac{10}{2} = 20r\sin\theta + \frac{10}{2\pi}\theta \qquad (\because\ q = 10)$$

or $$5 = 20r \sin\theta + \frac{10}{2\pi}\theta = 20y + \frac{10}{2\pi}\theta \qquad (\because \quad r\sin\theta = y)$$

∴ $$20y = 5 - \frac{10}{2\pi}\theta$$

or $$y = \frac{5}{20} - \frac{10}{2\pi} \times \frac{\theta}{20} = 0.25 - \frac{\theta}{4\pi} \qquad ...(i)$$

The above relation gives the equation of the dividing stream line.

From the above equation, for different values of θ the value of y is obtained as :

Value of θ	*Value of y from (i)*	*Remarks*
0	0.25 m	Max. half width of body
$\frac{\pi}{2}$	0.125 m	The +ve ordinate above the origin
π	0	The leading point
$\frac{3\pi}{2}$	– 0.125 m	The –ve ordinate below the origin
2π	– 0.25 m	The max. –ve ordinate

(*ii*) *Sketch of flow pattern*

For sketching the flow pattern, let us first find the location of the stagnation point. The horizontal distance of the stagnation point S from the source is given by the equation,

$$r_s = \frac{q}{2\pi U} = \frac{10}{2\pi \times 20} = 0.0795 \text{ m}$$

Hence the stagnation point lies on the x-axis at a distance of 0.0795 m or 79.5 mm from the source towards left of the source. The flow pattern is shown in Fig. 5.49.

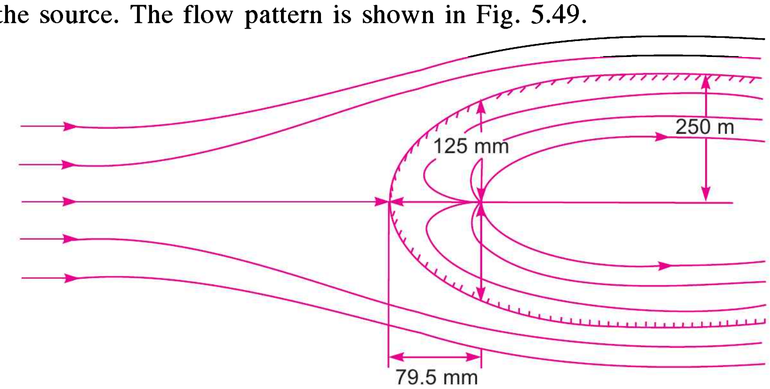


Fig. 5.49

Problem 5.41 *A uniform flow with a velocity of 2 m/s is flowing over a source placed at the origin. The stagnation point occurs at (– 0.398, 0). Determine :*

(i) Strength of the source, (ii) Maximum width of Rankine half-body and

(iii) Other principal dimensions of the Rankine half-body.

Solution. Given :

Uniform velocity, $U = 2$ m/s

Co-ordinates of stagnation point = (– 0.398, 0)

This means r_s = 0.398 and stagnation point lies on x-axis at a distance of 0.398 m towards left of origin. The source is placed at origin.

(*i*) *Strength of the source*

Let q = strength of the source

We know that $$r_s = \frac{q}{2\pi U}$$

or $$q = 2\pi U \times r_s = 2\pi \times 2 \times 0.398 = 5.0014 \text{ m}^2\text{/s} \simeq \mathbf{5\ m^2/s.\ Ans.}$$

(*ii*) *Maximum width of Rankine half-body*

The main dimensions of the Rankine half-body are obtained from equation (5.57) as

$$y = \frac{q}{2U}\left(1 - \frac{\theta}{\pi}\right) \quad ...(i)$$

The value of y is maximum, when θ = 0.

$$\therefore \quad y_{max} = \frac{q}{2U}\left(1 - \frac{0}{\pi}\right) = \frac{q}{2U} = \frac{5}{2 \times 2} = 1.25 \text{ m}$$

$\therefore$ Maximum width of Rankine body = $2 \times y_{max} = 2 \times 1.25$ = **2.5 m. Ans.**

(*iii*) *Other Principal dimensions of Rankine half-body*

Using equation (5.57), we get

$$y = \frac{q}{2U}\left(1 - \frac{\theta}{\pi}\right)$$

At $\theta = \frac{\pi}{2}$, $$y = \frac{q}{2U}\left[1 - \frac{\left(\frac{\pi}{2}\right)}{\pi}\right] = \frac{q}{2U}\left[1 - \frac{1}{2}\right] = \frac{q}{4U} = \frac{5}{4 \times 2} = 0.625 \text{ m}$$

The above value gives the upper ordinate at the origin, where source is placed.

$\therefore$ Width of body at origin = 2×0.625 = 1.25 m

At the stagnation point, the width of the body is zero.

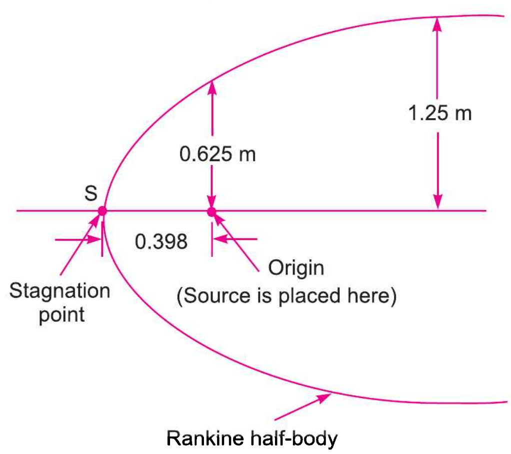


Fig. 5.50

5.17.4 A Source and Sink Pair in a Uniform Flow (Flow Past a Rankine Oval Body). Fig. 5.51 (*a*) shows a uniform flow of velocity U and Fig. 5.51 (*b*) shows a source sink pair of equal strength. When this uniform flow is flowing over the source sink pair, a resultant flow will be obtained as shown in Fig. 5.51 (*c*). This resultant flow is also known as the flow past a Rankine oval body.

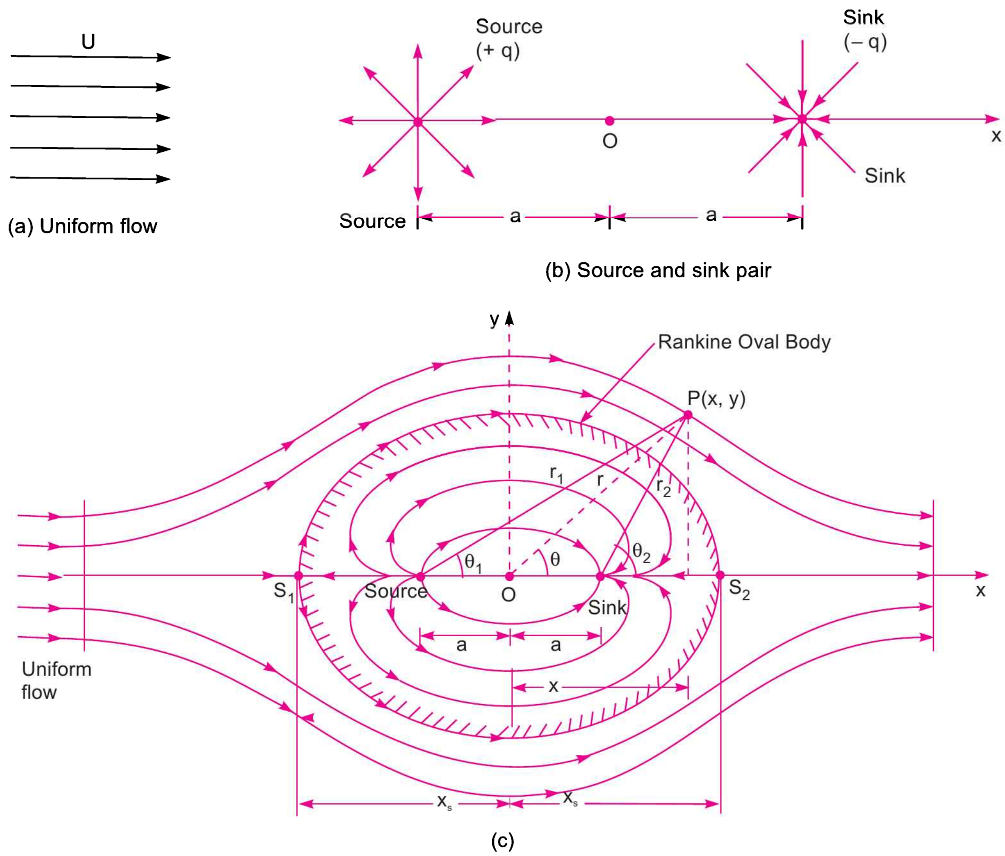


Fig. 5.51

Let U = Velocity of uniform flow along x-axis

q = Strength of source

$(-q)$ = Strength of sink

$2a$ = Distance between source and sink which is along x-axis.

The origin O of the x-y co-ordinates is mid-way between source and sink. Consider a point $P(x, y)$ lying in the resultant flow field. The stream function (ψ) and velocity potential function (ϕ) for the resultant flow field are obtained as given below :

ψ = Stream function due to uniform flow + stream function due to source + stream function due to sink

$$= \psi_{\text{uniform flow}} + \psi_{\text{source}} + \psi_{\text{sink}}$$

$$= U \times y + \frac{q}{2\pi}\,\theta_1 + \frac{(-q)}{2\pi} \times \theta_2$$

(where θ_1 is the angle made by P with source along x-axis and θ_2 with sink)

$$= U \times y + \frac{q\theta_1}{2\pi} - \frac{q\theta_2}{2\pi} = U \times y + \frac{q}{2\pi}(\theta_1 - \theta_2)$$

$$= U \times r \sin \theta + \frac{q}{2\pi}(\theta_1 - \theta_2) \qquad (\because \; y = r \sin \theta) \quad ...(5.59)$$

and ϕ = potential function due to uniform flow + potential function due to source + potential function due to sink

$$= \phi_{\text{uniform flow}} + \phi_{\text{source}} + \phi_{\text{sink}}$$

$$= U \times x + \frac{q}{2\pi} \log_e r_1 + \frac{(-q)}{2\pi} \log_e r_2$$

$$= U \times r \cos \theta + \frac{q}{2\pi} [\log_e r_1 - \log_e r_2] \qquad (\because \; x = r \cos \theta)$$

$$= U \times r \cos \theta + \frac{q}{2\pi}\left[\log_e \frac{r_1}{r_2}\right] \quad ...(5.60)$$

The following are the important points for the resultant flow pattern :

(*a*) There will be two stagnation points S_1 and S_2, one to the left of the source and other to the right of the sink. At the stagnation points, the resultant velocity (*i.e.*, velocity due to uniform flow, velocity due to source and velocity due to sink) will be zero. The stagnation point S_1 is to the left of the source and stagnation point S_2 will be to the right of the sink on the x-axis.

Let x_S = Distance of the stagnation points from origin O along x-axis.

Let us calculate this distance x_S.

For the stagnation point S_1,

(*i*) Velocity due to uniform flow = U

(*ii*) Velocity due to source = $\dfrac{q}{2\pi(x_S - a)}$ $\quad \left[\begin{array}{l} \because \text{ The velocity at any radius due to source} = \dfrac{q}{2\pi r} \\ \text{For } S_1\text{, the radius from source} = (x_S - a) \end{array}\right]$

(*iii*) Velocity due to sink = $\dfrac{-q}{2\pi(x_S + a)}$ $\quad$ [$\because$ At S_1, the radius from sink = $(x_S + a)$]

At point S_1, the velocity due to uniform flow is in the positive x-direction whereas due to source and sink are in the –ve x-direction.

$\therefore$ The resultant velocity at $S_1 = U - \dfrac{q}{2\pi(x_S - a)} - \dfrac{(-q)}{2\pi(x_S + a)}$

But the resultant velocity at stagnation point S_1 should be zero.

$$\therefore \qquad U - \frac{q}{2\pi(x_S - a)} + \frac{q}{2\pi(x_S + a)} = 0$$

or

$$U = \frac{q}{2\pi(x_S - a)} - \frac{q}{2\pi(x_S + a)}$$

$$= \frac{q}{2\pi}\left[\frac{1}{(x_S - a)} - \frac{1}{(x_S + a)}\right] = \frac{q}{2\pi}\left[\frac{(x_S + a) - (x_S - a)}{(x_S - a)(x_S + a)}\right] = \frac{q}{2\pi}\frac{2a}{\left(x_S^2 - a^2\right)}$$

or $$x_S^2 - a^2 = \frac{q \cdot a}{\pi U}$$

or $$x_S^2 = a^2 + \frac{qa}{\pi U} = a^2\left[1 + \frac{q}{\pi a U}\right]$$

∴ $$x_S = a\sqrt{\left(1 + \frac{q}{\pi a U}\right)} \qquad ...(5.61)$$

The above equation gives the location of the stagnation point on the x-axis.

(*b*) The stream line passing through the stagnation points is having zero velocity and hence can be replaced by a solid body. This solid body is having a shape of oval as shown in Fig. 5.51. There will be two flow fields, one within the oval contour and the other outside the solid body. The flow field within the oval contour will be due to source and sink whereas the flow field outside the body will be due to uniform flow only.

The shape of solid body is obtained from the stream line having stream function equal to zero. But the stream function is given by equation as

$$\psi = U \times r \sin\theta + \frac{q}{2\pi}(\theta_1 - \theta_2)$$

For the shape of solid body, $\psi = 0$

∴ $$0 = U \times r \sin\theta + \frac{q}{2\pi}(\theta_1 - \theta_2)$$

or $$U \times r \sin\theta = -\frac{q}{2\pi}(\theta_1 - \theta_2) = \frac{q}{2\pi}(\theta_2 - \theta_1)$$

∴ $$r = \frac{q}{2\pi}\frac{(\theta_2 - \theta_1)}{U \sin\theta} \qquad ...(5.62)$$

From the above equation, the distances of the surface of the solid body from the origin can be obtained or the shape of the solid body can be obtained. The maximum width of the body (y_{max}) will be equal to OM as shown in Fig. 5.52.

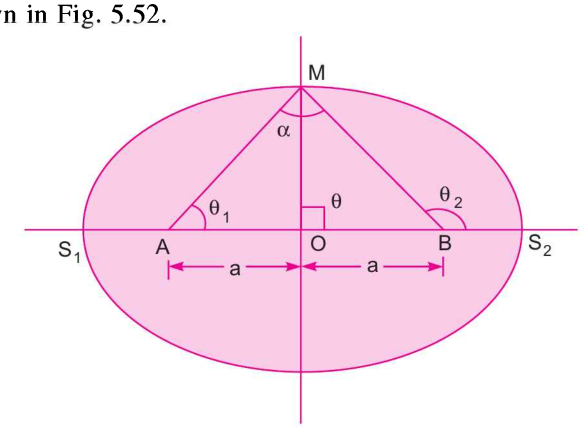


Fig. 5.52

From triangle AOM, we have

$$\tan \theta_1 = \frac{OM}{AO}$$

or $$OM = AO \tan \theta_1 = a \tan \theta_1$$

or $$y_{max} = a \tan \theta_1 \qquad (\because OM = y_{max}) \quad ...(5.63)$$

Let us find the value of θ_1.

When the point P lies on M, then $r = OM$, $\theta = 90° = \frac{\pi}{2}$

and $$\theta_2 = 180° - \theta_1 = \pi - \theta_1 \qquad \text{[Refer to Fig. 5.52]}$$

$[\because \; AM = BM \quad \therefore \quad \text{Angle } ABM = \text{Angle } BAM = \theta_1]$

Substituting these values in equation (5.62), we get

$$OM = \frac{q}{2\pi} \frac{((\pi - \theta_1) - \theta_1)}{U \sin \frac{\pi}{2}} = \frac{q}{2\pi} \frac{(\pi - 2\theta_1)}{U}$$

or $$y_{max} = \frac{q(\pi - 2\theta_1)}{2\pi U} \qquad \text{[where } OM = y_{max}\text{]}$$

or $$2\pi U y_{max} = q(\pi - 2\theta_1) \quad \text{or} \quad \frac{2\pi U y_{max}}{q} = \pi - 2\theta_1$$

or $$2\theta_1 = \pi - \frac{2\pi U y_{max}}{q} \quad \text{or} \quad \theta_1 = \frac{\pi}{2} - \frac{\pi U y_{max}}{q}$$

Substituting this value of θ_1 in equation (5.63), we get

$$y_{max} = a \tan \left[\frac{\pi}{2} - \frac{\pi U y_{max}}{q}\right] = a \cot \left[\frac{\pi U y_{max}}{q}\right] \quad ...(5.64)$$

From the above equation, the value of y_{max} is obtained by hit and trial method till L.H.S. = R.H.S. In this equation $\left(\frac{\pi U y_{max}}{q}\right)$ is in radians.

The length and width of the Rankine oval is obtained as :

Length, $$L = 2 \times x_S$$

$$= 2 \times a \sqrt{\left(1 + \frac{q}{\pi a U}\right)} \qquad \left[\because \; x_S = a \sqrt{\left(1 + \frac{q}{\pi a U}\right)}\right] \quad ...(5.65)$$

and Width, $$B = 2 \times y_{max}$$

$$= 2a \cot \left(\frac{\pi U y_{max}}{q}\right). \quad ...(5.66)$$

Problem 5.42 *A uniform flow of velocity 6 m/s is flowing along x-axis over a source and a sink which are situated along x-axis. The strength of source and sink is 15 m²/s and they are at a distance of 1.5 m apart. Determine :*

(i) Location of stagnation points, *(ii) Length and width of the Rankine oval*
(iii) Equation of profile of the Rankine body.

Solution. Given : Uniform flow velocity, $U = 6$ m/s
Strength of source and sink, $q = 15$ m^2/s
Distance between source and sink, $2a = 1.5$ m

$$\therefore \quad a = \frac{1.5}{2} = 0.75 \text{ m}$$

(*i*) *Location of stagnation points* (Refer to Fig. 5.51)

For finding the location of the stagnation points, the equation (5.61) is used.

$$\therefore \quad x_s = a\sqrt{\left(1+\frac{q}{\pi a U}\right)} = 0.75\sqrt{\left(1+\frac{15}{\pi \times 0.75 \times 6}\right)} = 1.076 \text{ m}$$

The above equation gives the distance of the stagnation points from the origin. There will be two stagnation points.

The distance of stagnation points from the source and sink = $x_s - a = 1.076 - 0.75 =$ **0.326 m. Ans.**

(*ii*) *Length and width of the Rankine oval*

Length, $L = 2 \times x_s = 2 \times 1.076 = 2.152$ m.

Width, $B = 2 \times y_{max}$ ...(*i*)

Let us now find the value of y_{max}

Using equation (5.64), we get

$$y_{max} = a \cot\left(\frac{\pi U y_{max}}{q}\right) = 0.75 \cot\left(\frac{\pi \times 6 \times y_{max}}{15}\right) = 0.75 \cot (0.4\pi\, y_{max})$$

$$= 0.75 \cot\left(0.4\pi\, y_{max} \times \frac{180}{\pi}\right)^{\circ}$$

$$\left[\because \ (0.4\pi\, y_{max}) \text{ is in radians and hence } (0.4\pi\, y_{max}) \times \frac{180}{\pi} \text{ will be in degrees}\right]$$

$$= 0.75 \cot (72 \times y_{max})^{\circ}$$

The above equation will be solved by hit and trial method. The value of $x_s = 1.076$. But x_s is equal to length of major axis of Rankine body and y_{max} is the length of minor axis of the Rankine body. The length of minor axis will be less than length of major axis. Let us first assume $y_{max} = 0.8$ m. Then

y_{max}	*L.H.S.*	*R.H.S.*
0.8	0.8	0.75 cot (72 × 0.8)° = 0.75 cot 51.6° = 0.475
0.7	0.7	0.75 cot (72 × 0.7)° = 0.75 cot 50.4° = 0.577
0.6	0.6	0.75 cot (72 × 0.6)° = 0.75 cot 43.2° = 0.798
0.65	0.65	0.75 cot (72 × 0.65)° = 0.75 cot 46.8° = 0.704
0.67	0.67	0.75 cot (72 × 0.67)° = 0.75 cot 48.24° = 0.669 ≃ 0.67

From above it is clear that, when $y_{max} = 0.67$, then L.H.S. = R.H.S.

$$\therefore \quad y_{max} = 0.67 \text{ m}$$

Substituting this value in equation (*i*), we get

Width, $B = 2 \times y_{max} = 2 \times 0.67 =$ **1.34 m. Ans.**

(*iii*) *Equation of profile of the Rankine body*

The equation of profile of the Rankine body is given by equation (5.62) as

$$r = \frac{q}{2\pi} \frac{(\theta_2 - \theta_1)}{U \sin \theta} = \frac{15}{2\pi} \frac{(\theta_2 - \theta_1)}{6 \times \sin \theta} = \frac{\mathbf{0.398\,(\theta_2 - \theta_1)}}{\mathbf{\sin \theta}}. \textbf{ Ans.}$$

5.17.5 A Doublet in a Uniform Flow (Flow Past a Circular Cylinder). Fig. 5.53 (*a*) shows a uniform flow of velocity U in the positive x-direction and Fig. 5.53 (*b*) shows a doublet at the origin. Doublet is a special case of a source and a sink combination in which both of equal strength approach each other such that distance between them tends to be zero. When the uniform flow is flowing over the doublet, a resultant flow will be obtained as shown in Fig. 5.53 (*c*). This resultant flow is **known as the flow past a Rankine oval of equal axes or flow past a circular cylinder.**

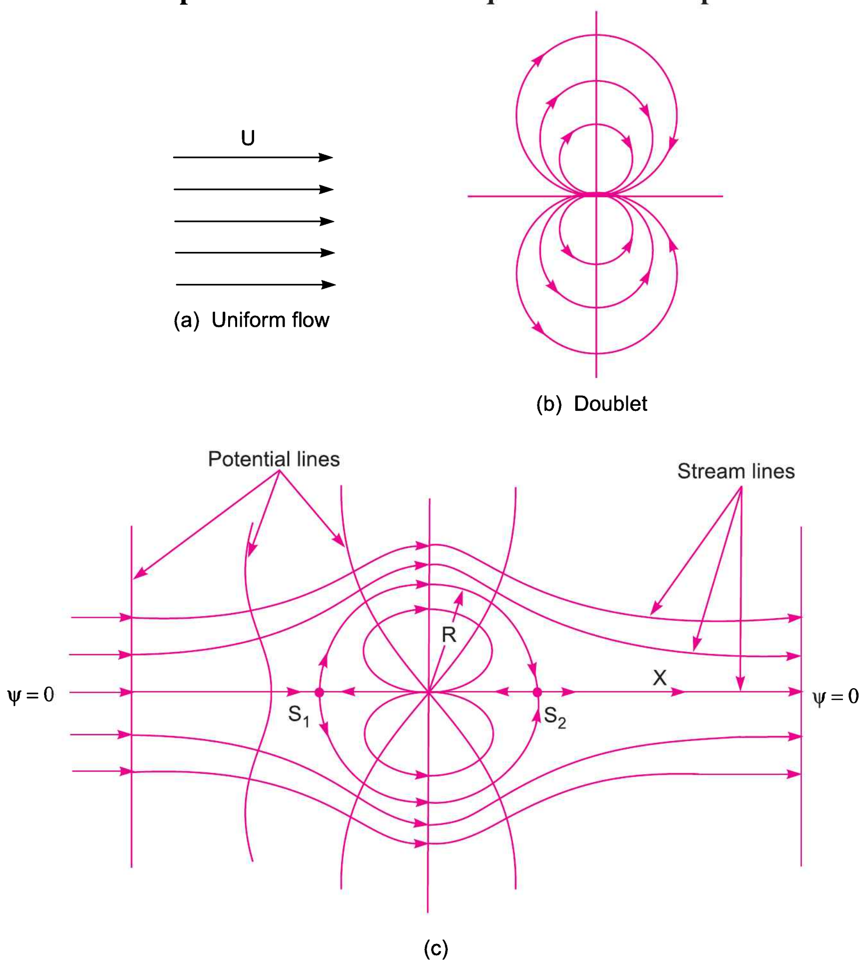


Fig. 5.53

The stream function (ψ) and velocity potential function (ϕ) for the resultant flow is obtained as given below :

ψ = stream function due to uniform flow + stream function due to doublet

$$= U \times y + \left(\frac{-\mu}{2\pi r} \sin \theta\right)$$

$$\left[\text{Stream function due to doublet is given by equation (5.50) as} = -\frac{\mu}{2\pi r} \sin \theta\right]$$

$$= U \times r \times \sin\theta - \frac{\mu}{2\pi r}\sin\theta \qquad (\because \quad y = r\sin\theta)$$

$$= \left(U \times r - \frac{\mu}{2\pi r}\right)\sin\theta \qquad ...(5.67)$$

and ϕ = Potential function due to uniform flow + potential function due to doublet

$$= U \times x + \frac{\mu}{2\pi} \times \frac{\cos\theta}{r}$$

$\left[\text{From equation (5.52), potential function due to doublet} = \frac{\mu}{2\pi} \times \frac{\cos\theta}{r}\right]$

$$= U \times r\cos\theta + \frac{\mu}{2\pi} \times \frac{\cos\theta}{r} \qquad (\because x = r\cos\theta)$$

$$= \left(U \times r + \frac{\mu}{2\pi r}\right)\cos\theta \qquad ...(5.68)$$

Shape of Rankine oval of equal axes

To get the profile of the Rankine oval of equal axes, the stream line ψ is taken as zero. Hence substituting $\psi = 0$ in equation (5.67), we get

$$0 = \left(U \times r - \frac{\mu}{2\pi r}\right)\sin\theta$$

This means either $\sin\theta = 0$ or $U \times r - \frac{\mu}{2\pi r} = 0$

(*i*) If $\sin\theta = 0$, then $\theta = 0$ and $\pm\pi$ *i.e.*, a horizontal line through the origin of the doublet. This horizontal line is the x-axis.

(*ii*) If $U \times r - \frac{\mu}{2\pi r} = 0$, then $U \times r = \frac{\mu}{2\pi r}$ or $r^2 = \frac{\mu}{2\pi U}$

or $r = \sqrt{\frac{\mu}{2\pi U}}$ = a constant as μ and U are constant.

Let this constant is equal to R.

$$\therefore \qquad r = \sqrt{\frac{\mu}{2\pi U}} = R$$

This gives that the closed body profile is a circular cylinder of radius R with centre on doublet. The dividing stream line corresponds to $\psi = 0$. This stream line is a circle of radius R. The stream line $\psi = 0$ has two stagnation points S_1 and S_2. At S_1, the uniform flow splits into two streams that flow along the circle with radius $R = \sqrt{\frac{\mu}{2\pi U}}$, the two branches meet again at the stagnation point S_2 and the flow continues in the downward direction. The uniform flow occurs outside the circle whereas the flow field due to doublet lies entirely within the circle. The stream function for the composite flow is given by equation (5.67) as

$$\psi = \left(U \times r - \frac{\mu}{2\pi r}\right) \sin\theta = U\left(r - \frac{\mu}{2\pi U r}\right) \sin\theta$$

$$= U\left(r - \frac{R^2}{r}\right) \sin\theta \qquad \left(\because \frac{\mu}{2\pi U} = R^2\right) \quad ...(5.69)$$

Velocity Components ($\mathbf{u}_r$ and $\mathbf{u}_\theta$)

The velocity components at any point in the flow field are given by,

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[U\left(r - \frac{R^2}{r}\right)\sin\theta\right] = \frac{1}{r}U\left(r - \frac{R^2}{r}\right)\cos\theta$$

$$= U\left(1 - \frac{R^2}{r^2}\right)\cos\theta \quad ...(5.70)$$

and

$$u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\partial}{\partial r}\left[U\left(r - \frac{R^2}{r}\right)\sin\theta\right] = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta$$

$$= -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta \quad ...(5.71)$$

$\therefore$ Resultant velocity, $V = \sqrt{u_r^2 + u_\theta^2}$ ...(5.72)

On the surface of the cylinder, r = R

$$u_r = U\left[1 - \frac{R^2}{R^2}\right]\cos\theta \qquad [\because \text{ In equation (5.70), } r = R]$$

$$= 0$$

and

$$u_\theta = -U\left[1 + \frac{R^2}{R^2}\right]\sin\theta = -2U\sin\theta \quad ...(5.73)$$

–ve sign shows the clockwise direction of tangential velocity at that point. The value of u_θ is maximum, when $\theta = 90°$ and $270°$.

At $\theta = 0°$ or $180°$, the value of $u_\theta = 0$. Hence on the surface of the cylinder, the resultant velocity is zero, when $\theta = 0°$ or $180°$. These two points on the surface of cylinder [*i.e.*, at $\theta = 0°$ and $180°$] where resultant velocity is zero, are known as stagnation points. They are denoted by S_1 and S_2. Stagnation point S_1 corresponds to $\theta = 180°$ and S_2 corresponds to $\theta = 0°$.

Pressure distribution on the surface of the cylinder

Let p_0 = pressure at a point in the uniform flow far away from the cylinder and towards the left of the cylinder [*i.e.*, approaching uniform flow]

U = velocity of uniform flow at that point

p = pressure at a point on the surface of the cylinder

V = resultant velocity at that point on the surface of the cylinder. This velocity is equal to u_θ as u_r is zero on the surface of the cylinder.

$\therefore \qquad V = u_\theta = -2U \sin\theta$

Applying Bernoulli's equation at the above two points,

$$\frac{p_0}{\rho g} + \frac{U^2}{2g} = \frac{p}{\rho g} + \frac{V^2}{2g}$$

or $$\frac{p_0}{\rho g} + \frac{U^2}{2g} = \frac{p}{\rho g} + \frac{[-2U\sin\theta]^2}{2g} \qquad [\because \; V = u_\theta = -2U \sin\theta]$$

or $$\frac{p_0}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{4U^2 \sin^2\theta}{2}$$

or $$\frac{p - p_0}{\rho} = \frac{U^2}{2} - \frac{4U^2\sin^2\theta}{2} = \frac{1}{2} U^2 (1 - 4\sin^2\theta)$$

or $$\frac{p - p_0}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta$$

But $\frac{p - p_0}{\frac{1}{2}\rho U^2}$ is a dimensionless term and is known as dimensionless pressure co-efficient and is denoted by C_p.

$$\therefore \qquad C_p = \frac{p - p_0}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta$$

Value of pressure co-efficient for different values of θ

Value of θ	*Value of C_p*
0	$1 - 4\sin^2\theta = 1 - 0 = 1$
30°	$1 - 4\sin^2 30° = 1 - 4 \times \left(\frac{1}{2}\right)^2 = 1 - \frac{4}{4} = 1 - 1 = 0$
90°	$1 - 4\sin^2 90° = 1 - 4 \times 1 = 1 - 4 = -3$
150°	$1 - 4\sin^2 150° = 1 - 4 \times \frac{1}{4} = 1 - 1 = 0$
180°	$1 - 4\sin^2 180° = 1 - 0 = 1$

At $\theta = 0$ and 180°, there are stagnation points S_2 and S_1 respectively.

At $\theta = 30°$ and 150°, the pressure co-efficient is zero.

At $\theta = 90°$, the pressure co-efficient is -3 (*i.e.*, least pressure)

The variation of pressure co-efficient along the surface of the cylinder for different values of θ are shown in Fig. 5.54.

The positive pressure is acting normal to the surface and towards the surface of the cylinder whereas the negative pressure is acting normal to the surface and away from the surface of the cylinder as shown in Fig. 5.55.

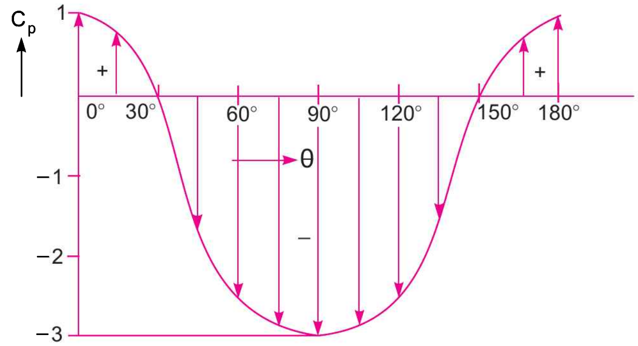


Fig. 5.54

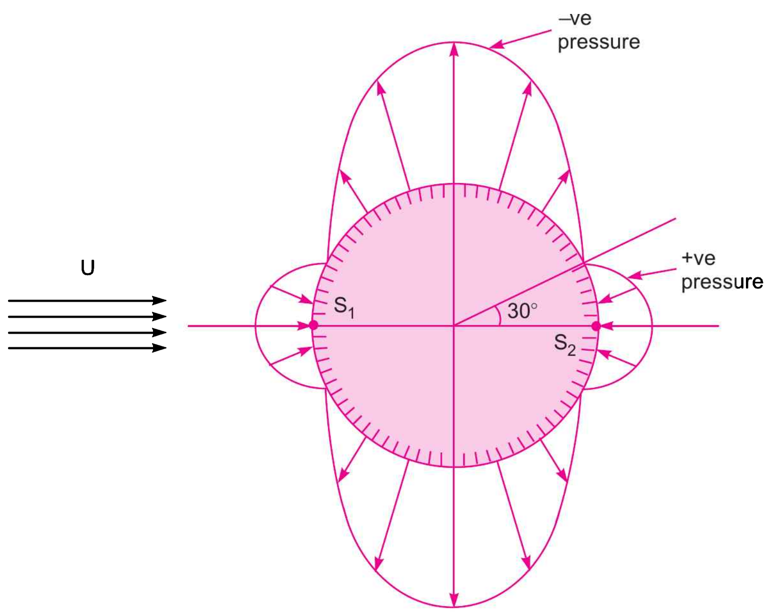


Fig. 5.55

Problem 5.43 *A uniform flow of 12 m/s is flowing over a doublet of strength 18 m^2/s. The doublet is in the line of the uniform flow. Determine :*

(i) shape of the Rankine oval *(ii) radius of the Rankine circle*

(iii) value of stream line function at Rankine circle

(iv) resultant velocity at a point on the Rankine circle at an angle of 30° from x-axis

(v) value of maximum velocity on the Rankine circle and location of the point where velocity is max.

Solution. Given : $U = 12$ m/s ; $\mu = 18$ m^2/s

(*i*) *Shape of the Rankine oval*

When a uniform flow is flowing over a doublet and doublet and uniform flow are in line, then the shape of the Rankine oval will be a circle of radius = $\sqrt{\frac{\mu}{2\pi U}}$. **Ans.**

(*ii*) *Radius of the Rankine circle*

$$R = r = \sqrt{\frac{\mu}{2\pi U}} = \sqrt{\frac{18}{2\pi \times 12}} = \mathbf{0.488\ m.\ Ans.}$$

(*iii*) *Value of stream line function at the Rankine circle*

The value of stream line function (ψ) at the Rankine circle is zero *i.e.*, $\psi = 0$.

(*iv*) *Resultant velocity on the surface of the circle, when* $\theta = 30°$

On the surface of the cylinder, the radial velocity (u_r) is zero. The tangential velocity (u_θ) is given by equation (5.73) as

$$u_\theta = -2U \sin\theta = -2 \times 12 \times \sin 30° = \mathbf{-12\ m/s.\ Ans.}$$

–ve sign shows the clockwise direction of tangential velocity at that point.

$\therefore$ Resultant velocity, $V = \sqrt{u_r^2 + u_\theta^2} = \sqrt{0^2 + (-12)^2} = \mathbf{12\ m/s.\ Ans.}$

(*v*) *Maximum velocity and its location*

The resultant velocity at any point on the surface of the cylinder is equal to u_θ. But u_θ is given by,

$$u_\theta = -2U \sin\theta$$

This velocity will be maximum, when $\theta = 90°$.

$\therefore$ Max. velocity $= -2U = -2 \times 12 = \mathbf{-24\ m/s.\ Ans.}$

Problem 5.44 *A uniform flow of 10 m/s is flowing over a doublet of strength 15 m²/s. The doublet is in the line of the uniform flow. The polar co-ordinates of a point P in the flow field are 0.9 m and 30°. Find : (i) stream line function and (ii) the resultant velocity at the point.*

Solution. Given : $U = 10$ m/s ; $\mu = 15$ m^2/s ; $r = 0.9$ m and $\theta = 30°$.

Let us first find the radius (R) of the Rankine circle. This is given by

$$R = \sqrt{\frac{\mu}{2\pi U}} = \sqrt{\frac{15}{2\pi \times 10}} = 0.488 \text{ m}$$

The polar co-ordinates of the point P are 0.9 m and 30°.

Hence $r = 0.9$ m and $\theta = 30°$.

As the value of r is more than the radius of the Rankine circle, hence point P lies outside the cylinder.

(*i*) *Value of stream line function at the point P*

The stream line function for the composite flow at any point is given by equation (5.69) as

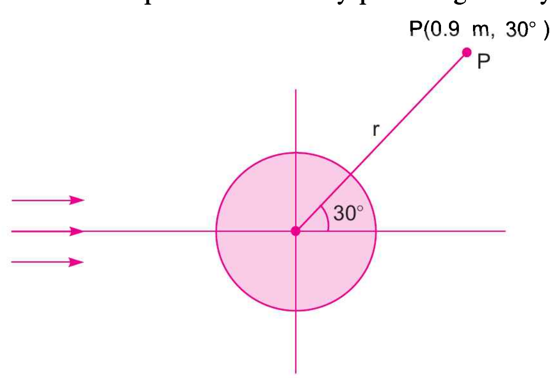


Fig. 5.56

$$\psi = U\left(r - \frac{R^2}{r}\right)\sin\theta$$

$$= 10\left(0.9 - \frac{0.488^2}{0.9}\right)\sin 30°(\because\ r = 0.9 \text{ m}, R = 0.488 \text{ and } \theta = 30°)$$

$$= 10(0.9 - 0.2646) \times \frac{1}{2} = \mathbf{3.177\ m^2/s.\ Ans.}$$

(*ii*) *Resultant velocity at the point P*

The radial velocity and tangential velocity at any point in the flow field are given by equations (5.70) and (5.71) respectively.

$\therefore$ $$u_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta = 10\left(1 - \frac{0.488^2}{0.9^2}\right)\cos 30° = 611 \text{ m/s}$$

+ve sign shows the radial velocity is outward.

and $$u_\theta = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta = -10\left(1 + \frac{0.488^2}{0.9^2}\right)\sin 30° = -6.47 \text{ m/s}$$

–ve sign shows the clockwise direction of tangential velocity.

$\therefore$ Resultant velocity,

$$V = \sqrt{u_r^2 + u_\theta^2}$$

$$= \sqrt{6.11^2 + (-6.47)^2} = \sqrt{37.33 + 44.86}$$

$$= \mathbf{8.89\ m/s.\ Ans.}$$

HIGHLIGHTS

1. If the fluid characteristics like velocity, pressure, density etc. do not change at a point with respect to time, the fluid flow is called steady flow. If they change w.r.t. time, the fluid flow is called unsteady flow.

   Or $\left(\frac{\partial v}{\partial t}\right) = 0$ for steady flow and $\left(\frac{\partial v}{\partial t}\right) \neq 0$ for unsteady flow.

2. If the velocity in a fluid flow does not change with respect to space (length of direction of flow), the flow is said uniform otherwise non-uniform. Thus,

   $\left(\frac{\partial v}{\partial s}\right) = 0$ for uniform flow and $\left(\frac{\partial v}{\partial s}\right) \neq 0$ for non-uniform flow.

3. If the Reynolds number in a pipe is less than 2000, the flow is said to be laminar and if Reynold number is more than 4000, the flow is said to be turbulent.
4. For compressible flow, $\rho \neq$ constant

   For incompressible flow, $\rho =$ constant.
5. Rate of discharge for incompressible fluid (liquid), $Q = A \times v$.
6. Continuity equation is written as $A_1v_1 = A_2v_2 = A_3v_3$.

7. Continuity equation in differential form,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \text{ for three-dimensional flow}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ for two-dimensional flow.}$$

8. The components of acceleration in x, y and z direction are

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}.$$

9. The components of velocity in x, y and z direction in terms of velocity potential (ϕ) are

$$u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y} \text{ and } w = -\frac{\partial \phi}{\partial z}.$$

10. The stream function (ψ) is defined only for two-dimensional flow. The velocity components in x and y directions in terms of stream function are $u = -\frac{\partial \psi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$.

11. Angular deformation or shear strain rate is given as

$$\text{Shear strain rate} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

12. Rotational components of a fluid particle are

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right];\ \omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right];\ \omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]$$

13. Vorticity is two times the value of rotation.

14. Flow of a fluid along a curved path is known as vortex flow. If the particles are moving round in curved path with the help of some external torque the flow is called forced vortex flow. And if no external torque is required to rotate the fluid particles, the flow is called free-vortex flow.

15. The relation between tangential velocity and radius :

for forced vortex, $v = \omega \times r$,

for free vortex, $v \times r =$ constant.

16. The pressure variation along the radial direction for vortex flow along a horizontal plane, $\frac{\partial p}{\partial r} = \rho\frac{v^2}{r}$

and pressure variation in the vertical plane $\frac{\partial p}{\partial z} = -\rho g$.

17. For the forced vortex flow, $Z = \frac{v^2}{2g} = \frac{\omega^2 r^2}{2g} = \frac{\omega^2 R^2}{2g}$

where Z = height of paraboloid formed

ω = angular velocity.

18. For a forced vortex flow in a open tank.
Fall of liquid level at centre = Rise of liquid level at the ends.

19. In case of closed cylinder, the volume of air before rotation is equal to the volume of air after rotation.

20. If a close cylindrical vessel completely filled with water is rotated about its vertical axis, the total pressure forces acting on the top and bottom are

$$F_T = \frac{\rho}{4}\,\omega^2 \pi R^4$$

and $$F_B = F_T + \text{weight of water in cylinder}$$

where F_T = Pressure force on top of cylinder
F_B = Pressure force on the bottom of cylinder
ω = Angular velocity
R = Radius of the vessel
ρ = Density of fluid $= \frac{w}{g}$.

21. For a free vortex flow the equation is $\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$.

EXERCISE

(A) THEORETICAL PROBLEMS

1. What are the methods of describing fluid flow ?

2. Explain the terms :
(*i*) Path line, (*ii*) Streak line,
(*iii*) Stream line, and (*iv*) Stream tube.

3. Distinguish between :
(*i*) Steady flow and un-steady flow, (*ii*) Uniform and non-uniform flow,
(*iii*) Compressible and incompressible flow,
(*iv*) Rotational and irrotational flow, (*v*) Laminar and turbulent flow.

4. Define the following and give one practical example for each :
(*i*) Laminar flow, (*ii*) Turbulent flow,
(*iii*) Steady flow, and (*iv*) Uniform flow.

5. Define the equation of continuity. Obtain an expression for continuity equation for a three-dimensional flow. *(R.G.P.V, S 2002)*

6. What do you understand by the terms : (*i*) Total acceleration, (*ii*) Convective acceleration, and (*iii*) Local acceleration ? *(Delhi University, Dec. 2002)*

7. (*a*) Define the terms :
(*i*) Velocity potential function, and (*ii*) Stream function.
(*b*) What are the conditions for flow to be irrotational ?

8. What do you mean by equipotential line and a line of constant stream function ?

9. (*a*) Describe the use and limitations of the flow nets.
(*b*) Under what conditions can one draw flow net ?

10. Define the terms :
(*i*) Vortex flow, (*ii*) Forced vortex flow, and (*iii*) Free vortex flow.

11. Differentiate between forced vortex and free vortex flow.

12. Derive an expression for the depth of paraboloid formed by the surface of a liquid contained in a cylindrical tank which is rotated at a constant angular velocity ω about its vertical axis.
13. Derive an expression for the difference of pressure between two points in a free vortex flow. Does the difference of pressure satisfy Bernoulli's equation ? Can Bernoulli's equation be applied to a forced vortex flow ?
14. Derive, from first principles, the condition for irrotational flow. Prove that, for potential flow, both the stream function and velocity potential function satisfy the Laplace equation.
15. Define velocity potential function and stream function.
16. Under what conditions can one treat real fluid flow as irrotational (as an approximation).
17. Define the following :
    (*i*) Steady flow, (*ii*) Non-uniform flow,
    (*iii*) Laminar flow, and (*iv*) Two-dimensional flow.
18. (*a*) Distinguish between rotational flow and irrotational flow. Give one example of each
    (*b*) Cite two examples of unsteady, non-uniform flow. How can the unsteady flow be transformed to steady flow ? *(J.N.T. University, S 2002)*
19. Explain uniform flow with source and sink. Obtain expressions for stream and velocity potential functions.
20. A point source is a point where an incompressible fluid is imagined to be created and sent out evenly in all directions. Determine its velocity potential and stream function.
21. (*i*) Explain doublet and define the strength of the doublet
    (*ii*) Distinguish between a source and a sink.
22. Sketch the flow pattern of an ideal fluid flow past a cylinder with circulation.
23. Show that in case of forced vortex flow, the rise of liquid level at the ends is equal to the fall of liquid level at the axis of rotation.
24. Differentiate between :
    (*i*) Stream function and velocity potential function
    (*ii*) Stream line and streak line and
    (*iii*) Rotational and irrotational flows.

(B) NUMERICAL PROBLEMS

1. The diameters of a pipe at the sections 1 and 2 are 15 cm and 20 cm respectively. Find the discharge through the pipe if velocity of water at section 1 is 4 m/s. Determine also the velocity at section 2. [**Ans.** 0.07068 m^3/s, 2.25 m/s]
2. A 40 cm diameter pipe, conveying water, branches into two pipes of diameters 30 cm and 20 cm respectively. If the average velocity in the 40 cm diameter pipe is 3 m/s. Find the discharge in this pipe. Also determine the velocity in 20 cm pipe if the average velocity in 30 cm diameter pipe is 2 m/s. [**Ans.** 0.3769 m^3/s, 7.5 m/s]
3. A 30 cm diameter pipe carries oil of sp. gr. 0.8 at a velocity of 2 m/s. At another section the diameter is 20 cm. Find the velocity at this section and also mass rate of flow of oil. [**Ans.** 4.5 m/s, 113 kg/s]
4. The velocity vector in a fluid flow is given by $V = 2x^3\mathbf{i} - 5x^2y\mathbf{j} + 4t\mathbf{k}$.
   Find the velocity and acceleration of a fluid particle at (1, 2, 3) at time, $t = 1$. [**Ans.** 10.95 units, 16.12 units]
5. The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation :

(i) $u = 4x^2$, $v = 4xyz$ (ii) $u = 4x^2 + 3xy$, $w = z^3 - 4xy - 2yz$.

$$\left[\textbf{Ans. } (i)\ w = -8xz - 2xz^2 + f(x, y) \quad (ii)\ v = -8xy - \frac{y^2}{2} + 3yz^2 + f(x, z)\right]$$

Calculate the unknown velocity components so that they satisfy the following equations :
(i) $u = 2x^2$, $v = 2xyz$, $w = ?$ (ii) $u = 2x^2 + 2xy$, $w = z^3 - 4xz + 2yz$, $v = ?$ [**Ans.** (i) $w = -1xz - x^2z$]

6. A fluid flow is given by : $V = xy^2 i - 2yz^2 j - \left(zy^2 - \frac{2z^3}{3}\right) k$.

Prove that it is a case of possible steady incompressible fluid flow.
Calculate the velocity and acceleration at the point [1, 2, 3]. [**Ans.** 36.7 units, 874.50 units]

7. Find the convective acceleration at the middle of a pipe which converges uniformly from 0.6 m diameter to 0.3 m diameter over 3 m length. The rate of flow is 40 lit/s. If the rate of flow changes uniformly from 40 lit/s to 80 lit/s in 40 seconds, find the total acceleration at the middle of the pipe at 20th second. [**Ans.** .0499 m/s^2 ; .11874 m/s^2]

8. The velocity potential function, ϕ, is given by $\phi = x^2 - y^2$. Find the velocity components in x and y direction. Also show that ϕ represents a possible case of fluid flow. [**Ans.** $u = 2x$ and $v = -2y$]

9. For the velocity potential function, $\phi = x^2 - y^2$, find the velocity components at the point (4, 5). [**Ans.** $u = 8$, $v = -10$ units]

10. A stream function is given by : $\psi = 2x - 5y$. Calculate the velocity components and also magnitude and direction of the resultant velocity at any point. [**Ans.** $u = 5$, $v = 2$, Resultant = 5.384 and $\theta = 21° 48'$]

11. If for a two-dimensional potential flow, the velocity potential is given by : $\phi = 4x(3y - 4)$, determine the velocity at the point (2, 3). Determine also the value of stream function ψ at the point (2, 3).

$$\left[\textbf{Ans. } 40 \text{ units, } \psi = 6x^2 - 4\left(\frac{3}{2}y^2 - 4y\right), -18\right]$$

12. The stream function for a two-dimensional flow is given by $\psi = 8xy$, calculate the velocity at the point $p(4, 5)$. Find the velocity potential function ϕ. [**Ans.** $u = -32$ units, $v = 40$ units, $\phi = 4y^2 - 4x^2$]

13. Sketch the stream lines represented by $\psi = xy$. Also find out the velocity and its direction at point (2, 3). [**Ans.** 3.60 units and $\theta = 56° 18.6'$ or $123° 42'$]

14. For the velocity components given as : $u = ay \sin xy$, $v = ax \sin xy$.
Obtain an expression for the velocity potential function. [**Ans.** $\phi = a \cos xy$]

15. A fluid flow is given by : $V = 10x^3 i - 8x^3 yj$.
Find the shear strain rate and state whether the flow is rotational or irrotational. [**Ans.** $-8xy$, rotational]

16. The velocity components in a two-dimensional flow are :

$$u = 8x^2y - \frac{8}{3}y^3 \text{ and } v = -8xy^3 + \frac{8}{3}x^3.$$

Show that these velocity components represent a possible case of an irrotational flow.

$$\left[\textbf{Ans. } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\ \omega_z = 0\right]$$

17. An open circular cylinder of 20 cm diameter and 100 cm long contains water upto a height of 80 cm. It is rotated about its vertical axis. Find the speed of rotation when :
(i) no water spills, (ii) axial depth is zero. [**Ans.** (i) 267.51 r.p.m., (ii) 422.98 r.p.m.]

18. A cylindrical vessel 15 cm in diameter and 40 cm long is completely filled with water. The vessel is open at the top. Find the quantity of water left in the vessel, when it is rotated about its vertical axis with a speed of 300 r.p.m. [**Ans.** 4566.3 cm^2]

19. An open circular cylinder of 20 cm diameter and 120 cm long contains water upto a height of 80 cm. It is rotated about its vertical axis at 400 r.p.m. Find the difference in total pressure force (*i*) at the bottom of the cylinder, and (*ii*) at the sides of the cylinder due to rotation. [**Ans.** (*i*) 14.52 N, (*ii*) 2465.45 N]

20. A closed cylindrical vessel of diameter 15 cm and length 100 cm contains water upto a height of 80 cm. The vessel is rotated at a speed of 500 r.p.m. about its vertical axis. Find the height of paraboloid formed. [**Ans.** 56.06 cm]

21. For the data given in question 20, find the speed of rotation of the vessel, when axial depth is zero. [**Ans.** 891.7 r.p.m.]

22. If the cylindrical vessel of question 20, is rotated at 950 r.p.m. about its vertical axis, find the area uncovered at the base of the tank. [**Ans.** 20.4 cm^2]

23. A closed cylindrical vessel of diameter 20 cm and height 100 cm contains water upto a height of 70 cm. The air above the water surface is at a pressure of 78.48 kN/m^2. The vessel is rotated at a speed of 300 r.p.m. about its vertical axis. Find the pressure head at the bottom of the vessel ; (*a*) at the centre, and (*b*) at the edge. [**Ans.** (*a*) 8.4485 m (*b*) 8.9515 m]

24. A closed cylinder of diameter 30 cm and height 20 cm is completely filled with water. Calculate the total pressure force exerted by water on the top and bottom of the cylinder, if it is rotated about its vertical axis at 300 r.p.m. [**Ans.** $F_T = 392.4$ N, $F_B = 531$ N]

25. In a free cylindrical vortex flow of water, at a point at a radius of 150 mm the velocity and pressure are 5 m/s and 14.715 N/cm^2. Find the pressure at a radius of 300 mm. [**Ans.** 15.65 N/cm^2]

26. Do the following velocity components represent physically possible flows ?

$$u = x^2 + z^2 + 5,\ v = y^2 + z^2,\ w = 4xyz.$$ [**Ans.** No.]

27. State if the flow represented by $u = 3x + 4y$ and $v = 2x - 3y$ is rotational or irrotational. [**Ans.** Rotational]

28. A vessel, cylindrical in shape and closed at the top and bottom, contains water upto a height of 700 mm. The diameter of the vessel is 200 mm and length of vessel is 1.1 m. Find the speed of rotation of the vessel if the axial depth of water is zero.

29. Define rotational and irrotational flow. The stream function and velocity potential for a flow are given by :

$$\psi = 2xy,\ \phi = x^2 - y^2.$$

Show that the conditions of continuity and irrotational flow are satisfied.

30. For the steady incompressible flow, are the following values of u and v possible ?

(*i*) $u = 4xy + y^2$, $v = 6xy + 3x$ and (*ii*) $u = 2x^2 + y^2$, $v = -4xy$. [**Ans.** (*i*) No, (*ii*) Yes]

31. Define two-dimensional stream function and velocity potential. Show that following stream function :

$$\psi = 6x - 4y + 7xy + 9$$

represents an irrotational flow. Find its velocity potential. [**Ans.** $\phi = 4x + 6y - 3.5x^2 + 3.5y^2 + C$]

32. Check if $\phi = x^2 - y^2 + y$ represents the velocity potential for 2-dimensional irrotational flow. If it does, then determine the stream function ψ. [**Ans.** Yes, $\psi = -2xy + x$]

33. If stream function for steady flow is given by $\psi = (y^2 - x^2)$, determine whether the flow is rotational or irrotational. Then determine the velocity potential ϕ. [**Ans.** Irrotational, $\phi = -2xy + C$]

34. A pipe (1) 450 mm in diameter branches into two pipes (2) and (3) of diameters 300 mm and 200 mm respectively as shown in Fig. 5.57. If the average velocity in 450 mm diameter pipe is 3 m/s, find : (*i*) discharge through 450 mm dia. pipe and (*ii*) velocity in 200 mm diameter pipe if the average velocity in 300 mm pipe is 2.5 m/s. *(J.N.T.U., Hyderabad, S 2002)*

[**Hint.** Given : $d_1 = 450$ mm $= 0.45$ m, $d_2 = 300$ mm $= 0.3$ m

$d_3 = 200$ mm $= 0.2$ m, $V_1 = 3$ m/s, $V_2 = 2.5$ m/s (*i*) $$Q_1 = A_1V_1 = \frac{\pi}{4}\ (0.45^2) \times 3 = \mathbf{0.477\ m^3/s.}$$

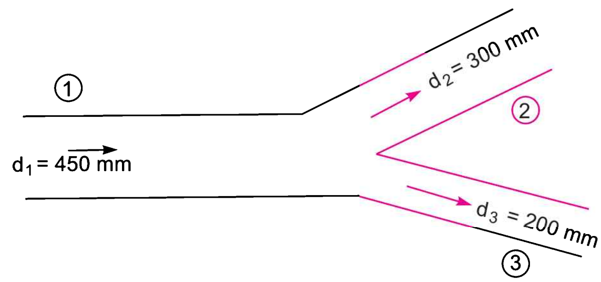


Fig. 5.57

(*ii*) $$Q_2 = A_2V_2 = \frac{\pi}{4}\ (.3^2) \times 2.5 = 0.176\ \text{m}^3/\text{s}$$

But $$Q_1 = Q_2 + Q_3 \quad \therefore \quad Q_3 = Q_1 - Q_2 = 0.477 - 0.176 = 0.301$$

Also $$Q_3 = A_3 \times V_3 = \frac{\pi}{4}\ (0.2^2) \times V_3$$

$\therefore$ $$V_3 = \frac{Q_3}{\frac{\pi}{4}\left(0.2^2\right)} = \frac{0.301}{0.0314} = \mathbf{9.6\ m/s.}]$$