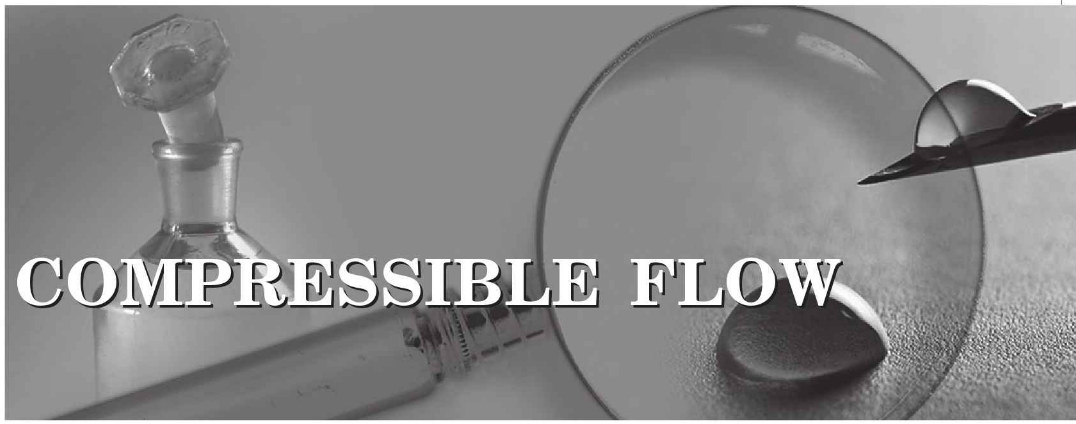
15

CHAPTER



COMPRESSIBLE FLOW

▶ 15.1 INTRODUCTION

Compressible flow is defined as that flow in which the density of the fluid does not remain constant during flow. This means that the density changes from point to point in compressible flow. But in case of incompressible flow, the density of the fluid is assumed to be constant. In the previous chapters, the fluid was assumed incompressible, and the basic equations such as equation of continuity, Bernoulli's equation and impulse momentum equations were derived on the assumption that fluid is incompressible. This assumption is true for flow of liquids, which are incompressible fluids. But in case of flow of fluids, such as

(*i*) flow of gases through orifices and nozzles,

(*ii*) flow of gases in machines such as compressors, and

(*iii*) projectiles and airplanes flying at high altitude with high velocities, the density of the fluid changes during the flow. The change in density of a fluid is accompanied by the changes in pressure and temperature and hence the thermodynamic behaviour of the fluids will have to be taken into account.

▶ 15.2 THERMODYNAMIC RELATIONS

The thermodynamic relations have been discussed in Chapter 1, which are as follows :

15.2.1 Equation of State. Equation of state is defined as the equation which gives the relationship between the pressure, temperature and specific volume of a gas. For a perfect gas the equation of state is

$$p\forall = RT \qquad ...(15.1)$$

where p = Absolute pressure in kgf/m^2 or N/m^2

$\forall$ = Specific volume or volume per unit mass

T = Absolute temperature = 273 + $t°$ (centigrade)

R = Gas constant in kgf-m/kg °K or (J/kg K)

= 29.2 kgf-m/kg °K or 287 J/kg K for air.

In equation (15.1), $\forall$ is the specific volume which is the reciprocal of density or

$$\forall = \frac{1}{\rho}.$$

Substituting this value of ∀ in equation (15.1), we get

$$\frac{p}{\rho} = RT \qquad \text{...(15.2)}$$

Note. In the equation of state given by equation (15.2), the dimensions of p, ρ and R should be used with care. The following points must be remembered :

1. If the value of R is given as 29.2 kgf-m/kg °K for air, the corresponding value of p and ρ should be taken in kgf/m^2 and kg/m^3. The mass rate of flow of the gas will be in kg/sec.
2. If the value of R is given as 287 J/kg K, the corresponding value of p and ρ should be in N/m^2 and kg/m^3. The mass rate of flow will be in kg/sec.

Value of $\frac{p}{\rho}$ in Bernoulli's Equation*. (*i*) If the value of p is taken in N/m^2, the corresponding value of ρ is in kg/m^3. And as mentioned above (point number 2), the value of R should be 287 J/kg K.

(*ii*) If the value of p is taken in kgf/m^2 in Bernoulli's equation, the corresponding value of ρ should be in msl/m^3. But as mentioned in point number 1, if the value of R is taken 29.2, the corresponding values of p and ρ are in kgf/m^2 and kg/m^3. Hence the mass density in equation of state is in kg/m^3 while in Bernoulli's equation it is in msl/m^3. The density calculated from equation of state must be converted into msl/m^3.

Note. It is better to use pressure in N/m^2, density in kg/m^3 and value of R = 287 J/kg K. The value of density calculated from equation of state will be in the same dimensions as used in Bernoulli's equation.

15.2.2 Expansion and Compression of Perfect Gas.

When the expansion or compression of a perfect gas takes place, the pressure, temperature and density are changed. The change in pressure, temperature and density of a gas is brought about by the two processes which are known as

1. Isothermal process, and 2. Adiabatic process.

1. Isothermal Process. This is the process in which a gas is compressed or expanded while the temperature is kept constant. The gas obeys Boyle's law, according to which we have

$$p\forall = \text{Constant, where } \forall = \text{Specific volume}$$

or

$$\frac{p}{\rho} = \text{Constant} \qquad \left(\because \ \forall = \frac{1}{\rho}\right) \qquad \text{...(15.3)}$$

2. Adiabatic Process. If the compression or expansion of a gas takes place in such a way that the gas neither gives heat, nor takes heat from its surrounding, then the process is said to be adiabatic. According to this process,

$$p\forall^k = \text{Constant}$$

where k = Ratio of the specific heat at constant pressure to the specific heat at constant volume

$$= \frac{C_p}{C_v} = 1.4 \text{ for air.}$$

The above relation is also written as $\frac{p}{\rho^k}$ = Constant. ...(15.4)

If the adiabatic process is reversible (or frictionless), it is known as isentropic process. And if the pressure and density are related in such a way that k is not equal to $\frac{C_p}{C_v}$ but equal to some positive value then the process is known as polytropic. According to which

$$\frac{p}{\rho^n} = \text{Constant} \qquad \text{...(15.5)}$$

where $n \neq k$ but equal to some positive constant.

* Please, refer to equations (15.10) and (15.11).

▶ 15.3 BASIC EQUATIONS OF COMPRESSIBLE FLOW

The basic equations of the compressible flows are

1. Continuity Equation,
2. Bernoulli's Equation or Energy Equation,
3. Momentum Equation,
4. Equation of state.

15.3.1 Continuity Equation. This is based on law of conservation of mass which states that matter cannot be created nor destroyed. Or in other words, the matter or mass is constant. For one-dimensional steady flow, the mass per second = ρAV

where ρ = Mass density, A = Area of cross-section, V = Velocity

As mass or mass per second is constant according to law of conservation of mass. Hence

$$\rho AV = \text{Constant.} \quad \text{...(15.6)}$$

Differentiating equation (15.6), $d(\rho AV) = 0$ or $\rho d(AV) + AVd\rho = 0$

or $\rho[AdV + VdA] + AVd\rho = 0$ or $\rho AdV + \rho VdA + AVd\rho = 0$

Dividing by ρAV, we get $\dfrac{dV}{V} + \dfrac{dA}{A} + \dfrac{d\rho}{\rho} = 0.$ ...(15.7)

Equation (15.7) is also known as continuity equation in differential form.

15.3.2 Bernoulli's Equation. Bernoulli's equation has been derived for incompressible fluids in Chapter 6. The same procedure is followed. The flow of a fluid particle along a stream-line in the direction of S is considered. The resultant force on the fluid particle in the direction of S is equated to the mass of the fluid particle and its acceleration. As the flow of compressible fluid is steady, the same Euler's equation as given by equation (6.3) is obtained as

$$\frac{dp}{\rho} + VdV + gdZ = 0 \quad \text{...(15.8)}$$

Integrating the above equation, we get

$$\int \frac{dp}{\rho} + \int VdV + \int gdZ = \text{Constant}$$

or

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gZ = \text{Constant} \quad \text{...(15.9)}$$

In case of incompressible flow, the density ρ is constant and hence integration of $\dfrac{dp}{\rho}$ is equal to $\dfrac{p}{\rho}$.

But in case of compressible flow, the density ρ is not constant. Hence ρ cannot be taken outside the integration sign. With the change of ρ, the pressure p also changes for compressible fluids. This change of ρ and p takes place according to equations (15.3) or (15.4) depending upon the type of process during compressible flow. The value of ρ from these equations in terms of p is obtained and is substituted in $\int \dfrac{dp}{\rho}$ and then the integration is done. The Bernoulli's equation will be different for isothermal process and for adiabatic process.

(A) Bernoulli's Equation for Isothermal Process. For isothermal process, the relation between pressure (p) and density (ρ) is given by equation (15.3) as

$$\frac{p}{\rho} = \text{Constant} = C_1 \text{ (say)} \qquad ...(i)$$

$$\therefore \qquad \rho = \frac{p}{C_1}$$

Hence
$$\int \frac{dp}{\rho} = \int \frac{dp}{p/C_1} = \int \frac{C_1 dp}{p} = C_1 \int \frac{dp}{p} \qquad (\because C_1 \text{ is constant})$$

$$= C_1 \log_e p = \frac{p}{\rho} \log_e p \qquad \left(\because C_1 = \frac{p}{\rho} \text{ from equation } (i)\right)$$

Substituting the value $\int \frac{dp}{\rho}$ in equation (15.9), we get

$$\frac{p}{\rho} \log_e p + \frac{V^2}{2} + gZ = \text{Constant}$$

Dividing by 'g',
$$\frac{p}{\rho g} \log_e p + \frac{V^2}{2g} + Z = \text{Constant.} \qquad ...(15.10)$$

Equation (15.10) is the Bernoulli's equation for compressible flow undergoing isothermal process. For the two points 1 and 2, this equation is written as

$$\frac{p_1}{\rho_1 g} \log_e p_1 + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho_2 g} \log_e p_2 + \frac{V_2^2}{2g} + Z_2 \qquad ...(15.11)$$

(B) Bernoulli's Equation for Adiabatic Process. For the adiabatic process, the relation between pressure (p) and density (ρ) is given by equation (15.4) as

$$\frac{p}{\rho^k} = \text{Constant} = \text{say } C_2 \qquad ...(ii)$$

$$\therefore \qquad \rho^k = \frac{p}{C_2} \quad \text{or} \quad \rho = \left(\frac{p}{C_2}\right)^{1/k}$$

Hence
$$\int \frac{dp}{\rho} = \int \frac{dp}{\left(\frac{p}{C_2}\right)^{1/k}} = \int \frac{C_2^{1/k}}{p^{1/k}} dp = C_2^{1/k} \int \frac{1}{p^{1/k}} dp$$

$$= C_2^{1/k} \int p^{-1/k} dp = C_2^{1/k} \frac{p^{\left(-\frac{1}{k}+1\right)}}{\left(-\frac{1}{k}+1\right)}$$

$$= \frac{C_2^{1/k} p^{\left(\frac{k-1}{k}\right)}}{\left(\frac{k-1}{k}\right)} = \left(\frac{k}{k-1}\right) C_2^{1/k} p^{\left(\frac{k-1}{k}\right)}$$

$$= \left(\frac{k}{k-1}\right)\left(\frac{p}{\rho^k}\right)^{1/k} p^{\left(\frac{1-k}{k}\right)} \qquad \left(\because \; C_2^{1/k} = \frac{p}{\rho^k} \text{ from } (ii)\right)$$

$$= \left(\frac{k}{k-1}\right)\frac{p^{1/k}}{\rho^{k \times 1/k}} p^{\left(\frac{k-1}{k}\right)} = \left(\frac{k}{k-1}\right)\frac{p^{\frac{1}{k}+\frac{k-1}{k}}}{\rho} = \left(\frac{k}{k-1}\right)\frac{p}{\rho}$$

Substituting the value of $\int \frac{dp}{\rho} = \left(\frac{k}{k-1}\right)\frac{p}{\rho}$ in equation (15.9), we get

$$\left(\frac{k}{k-1}\right)\frac{p}{\rho} + \frac{V^2}{2} + gZ = \text{Constant}$$

Dividing by 'g' $\left(\frac{k}{k-1}\right)\frac{p}{\rho g} + \frac{V^2}{2g} + Z = \text{Constant.}$ ...(15.12)

Equation (15.12) is the Bernoulli's equation for compressible flow undergoing adiabatic process. For the two points 1 and 2, this equation is written as

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} + Z_1 = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} + Z_2 \quad \text{...(15.13)}$$

Problem 15.1 *A gas is flowing through a horizontal pipe at a temperature of 4°C. The diameter of the pipe is 8 cm and at a section 1-1 in this pipe, the pressure is 30.3 N/cm² (gauge). The diameter of the pipe changes from 8 cm to 4 cm at the section 2-2, where pressure is 20.3 N/cm² (gauge). Find the velocities of the gas at these sections assuming an isothermal process. Take R = 287.14 Nm/kg K, and atmospheric pressure = 10 N/cm².*

Solution. Given :

For the section 1-1,

Temperature, $t_1 = 4°C$

∴ Absolute temperature, $T_1 = 4 + 273 = 277°K$

Diameter pipe, $D_1 = 8 \text{ cm} = 0.08 \text{ m}$

∴ Area of pipe, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.08)^2 = .005026 \text{ m}^2$

Pressure, $p_1 = 30.3 \text{ N/cm}^2$ (gauge)

$= 30.3 + 10 = 40.3 \text{ N/cm}^2 \text{ (absolute)} = 40.3 \times 10^4 \text{ N/m}^2$ (abs.)

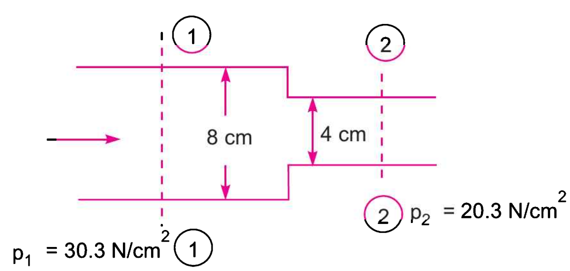


Fig. 15.1

For the section 2-2,

Diameter of pipe, $D_2 = 4 \text{ cm} = .04 \text{ m}$

$\therefore$ Area, $A_2 = \frac{\pi}{4} (.04)^2 = .0012565 \text{ m}^2$

Pressure, $p_2 = 20.3 + 10 = 30.3 \text{ N/cm}^2 \text{ (abs.)} = 30.3 \times 10^4 \text{ N/m}^2 \text{ (abs.)}$

Gas constant, $R = 287.14 \text{ N-m/kg°K}$

Ratio of specific heat, $k = 1.4$.

Applying continuity equation at sections (1) and (2), we get

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

or

$$\frac{V_2}{V_1} = \frac{\rho_1 A_1}{\rho_2 A_2} = \frac{\rho_1 \times .005026}{\rho_2 \times .0012565} = 4 \times \frac{\rho_1}{\rho_2} \quad ...(i)$$

For isothermal process using equation (15.3),

$$\frac{p_1}{\rho_1} = \frac{p_2}{\rho_2} \text{ or } \frac{\rho_1}{\rho_2} = \frac{p_1}{p_2} = \frac{40.3 \times 10^4}{30.3 \times 10^4} = 1.33$$

Substituting the value of $\frac{\rho_1}{\rho_2} = 1.33$ in equation (*i*), we get

$$\frac{V_2}{V_1} = 4 \times 1.33 = 5.32$$

$$\therefore \quad V_2 = 5.32 \, V_1 \quad ...(ii)$$

Applying Bernoulli's equation at sections 1-1 and 2-2 for isothermal process which is given by equation (15.11), we get

$$\frac{p_1}{\rho_1 g} \log_e p_1 + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho_2 g} \log_e p_2 + \frac{V_2^2}{2g} + Z_2$$

For horizontal pipe, $Z_1 = Z_2$

$$\therefore \quad \frac{p_1}{\rho_1 g} \log_e p_1 + \frac{V_1^2}{2g} = \frac{p_2}{\rho_2 g} \log_e p_2 + \frac{V_2^2}{2g}$$

or

$$\frac{p_1}{\rho_1 g} \log_e p_1 - \frac{p_2}{\rho_2 g} \log_e p_2 = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

But for isothermal process, $\frac{p_1}{\rho_1} = \frac{p_2}{\rho_2}$

$$\therefore \quad \frac{p_1}{\rho_1 g} \log_e p_1 - \frac{p_1}{\rho_1 g} \log_e p_2 = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

or

$$\frac{p_1}{\rho_1 g} \left[\log_e \frac{p_1}{p_2} \right] = \frac{(5.32 \, V_1)^2}{2g} - \frac{V_1^2}{2g} \quad (\because \text{ From } (ii), V_2 = 5.32 \, V_1)$$

or $$\frac{p_1}{\rho_1 g} \log_e \left(\frac{40.3 \times 10^4}{30.3 \times 10^4}\right) = \frac{V_1^2}{2g} (5.32^2 - 1) = 27.30 \frac{V_1^2}{2g}$$

or $$\frac{p_1}{\rho_1 g} \log_e 1.33 = 27.30 \frac{V_1^2}{2g}$$

or $$\frac{p_1}{\rho_1 g} \times 0.285 = 27.30 \frac{V_1^2}{2g}$$

or $$\frac{p_1}{\rho_1} = \frac{27.30}{2 \times .285} V_1^2 = 47.894\ V_1^2 \quad ...(iii)$$

Now from equation of state, *i.e.*, from equation (15.2), we have

$$\frac{p}{\rho} = RT \text{ or at section 1, } \frac{p_1}{\rho_1} = RT_1$$

or $$\frac{p_1}{\rho_1} = RT_1 = 287.14 \times 277 = 79537.4$$

Substituting this value of $\frac{p_1}{\rho_1} = 79537.4$ in equation (*iii*), we get $79537.4 = 47.894\ V_1^2$

$\therefore$ $$V_1 = \sqrt{\frac{79537.4}{47.894}} = \mathbf{40.75\ m/s.\ Ans.}$$

From equation (*ii*), $V_2 = 5.32 \times V_1 = 5.32 \times 40.75 =$ **216.79 m/s. Ans.**

Problem 15.2 *A gas is flowing through a horizontal pipe which is having area of cross-section as 40 cm², where pressure is 40 N/cm² (gauge) and temperature is 15°C. At another section the area of cross-section is 20 cm² and pressure is 30 N/cm² (gauge). If the mass rate of flow of gas through the pipe is 0.5 kg/s, find the velocities of the gas at these sections, assuming an isothermal change. Take R = 292 N-m/kg°K, and atmospheric pressure = 10 N/cm².*

Solution. Given :

	Section 1		Section 2
Area,	$A_1 = 40\text{ cm}^2 = 40 \times 10^{-4}\text{ m}^2$	Area,	$A_2 = 20\text{ cm}^2 = 20 \times 10^{-4}\text{ m}^2$
Pressure,	$p_1 = 40\text{ N/cm}^2$ (gauge)	Pressure,	$p_2 = 30\text{ N/cm}^2$ (gauge)
	$= 40 + 10 = 50\text{ N/cm}^2$ (abs.)		$= 30 + 10 = 40\text{ N/cm}^2$ (abs.)
	$= 50 \times 10^4\text{ N/m}^2$		$= 40 \times 10^4\text{ N/m}^2$
Temperature,	$t_1 = 15°C$		
$\therefore$	$T_1 = 15 + 273 = 288°K$		

Mass rate of flow $= 0.5$ kg/s.

Gas constant, $R = 292$ N-m/kg°K

From equation of state, *i.e.*, equation (15.2), $\frac{p_1}{\rho_1} = RT_1$

$\therefore \quad \rho_1 = \dfrac{p_1}{RT_1} = \dfrac{50 \times 10^4}{292 \times 288} \dfrac{kg}{m^3} = 5.945 \dfrac{kg}{m^3}$

Mass rate of flow is given by $= \rho_1 A_1 V_1$ or $0.5 = 5.945 \times 40 \times 10^{-4} \times V_1$

$\therefore \quad V_1 = \dfrac{0.5}{5.945 \times 40 \times 10^{-4}} = 21.02$ m/s

For isothermal process, temperature is constant and hence temperature at section 2 is also 288°K.

$\therefore \quad T_2 = 288°K$

Using equation (15.2), we get $\dfrac{p_2}{\rho_2} = RT_2$

$\therefore \quad \rho_2 = \dfrac{p_2}{RT_2} = \dfrac{40 \times 10^4}{292 \times 288} = 4.756$ kg/m^3

Now mass rate of flow $= \rho_2 A_2 V_2$

$\therefore \quad 0.5 = 4.756 \times 20 \times 10^{-4} \times V_2$

$\therefore \quad V_2 = \dfrac{0.5}{4.756 \times 20 \times 10^{-4}} =$ **52.565 m/s. Ans.**

Problem 15.3 *A gas with a velocity of 300 m/s is flowing through a horizontal pipe at a section where pressure is 6×10^4 N/m^2 (absolute) and temperature 40°C. The pipe changes in diameter and at this section the pressure is 9×10^4 N/m^2. Find the velocity of the gas at this section if the flow of the gas is adiabatic.*

Take R = 287 J/kg°K and k = 1.4.

Solution. Given :

Section 1	Section 2
$V_1 = 300$ m/s	$p_2 = 9 \times 10^4$ N/m^2
$p_1 = 6 \times 10^4$ N/m^2	$V_2 =$ velocity at section 2
$t_1 = 40°C$	$R = 287$ J/kg°K
$\therefore \quad T_1 = 273 + 40 = 313°K$	

Adiabatic flow, $k = 1.4$

Applying Bernoulli's equation at sections 1 and 2, given by equation (15.13), we get

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} \qquad (\because \quad Z_1 = Z_2)$$

or

$$\left(\frac{k}{k-1}\right)\left[\frac{p_1}{\rho_1 g} - \frac{p_2}{\rho_2 g}\right] = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

Dividing by 'g', we get $\left(\dfrac{k}{k-1}\right)\left[\dfrac{p_1}{\rho_1} - \dfrac{p_2}{\rho_2}\right] = \dfrac{V_2^2}{2} - \dfrac{V_1^2}{2}$

or $$\left(\frac{1.4}{1.4-1.0}\right)\frac{p_1}{\rho_1}\left[1-\frac{p_2}{\rho_2}\times\frac{\rho_1}{p_1}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2} \quad \text{...}(i)$$

For adiabatic flow, using equation (15.4), we get

$$\frac{p_1}{\rho_1^k}=\frac{p_2}{\rho_2^k} \quad \text{or} \quad \frac{p_1}{p_2}=\left(\frac{\rho_1}{\rho_2}\right)^k \text{ or } \left(\frac{\rho_1}{\rho_2}\right)=\left(\frac{p_1}{p_2}\right)^{1/k}$$

Substituting the value of $\frac{\rho_1}{\rho_2}$ in equation (i), we get

$$\frac{1.4}{0.4}\frac{p_1}{\rho_1}\left[1-\frac{p_2}{p_1}\times\left(\frac{p_1}{p_2}\right)^{1/k}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2}$$

or $$3.5\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)\times\left(\frac{p_2}{p_1}\right)^{-1/k}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2}$$

or $$3.5\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{1-\frac{1}{k}}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2} \text{ or } 3.5\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2}$$

Substituting the value of p_2 and p_1, we get

$$3.5\frac{p_1}{\rho_1}\left[1-\left(\frac{9\times10^4}{6\times10^4}\right)^{\frac{1.4-1}{1.4}}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2}$$

or $$3.5\frac{p_1}{\rho_1}\left[1-1.5^{2/7}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2} \text{ or } 3.5\frac{p_1}{\rho_1}\left[1-1.5^{.2857}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2}$$

or $$-0.4298\frac{p_1}{\rho_1}=\frac{V_2^2}{2}-\frac{V_1^2}{2}=\frac{V_2^2}{2}-\frac{300^2}{2}$$

or $$-.4298\frac{p_1}{\rho_1}=\frac{V_2^2}{2}-45000 \quad \text{...}(ii)$$

From equation of state, we have $\frac{p_1}{\rho_1}=RT_1=287\times313=89831$

Substituting the value of $\frac{p_1}{\rho_1}=89831$ in equation (ii),

$$-0.4298\times89831=\frac{V_2^2}{2}-45000$$

or $$\frac{V_2^2}{2}=45000-.4298\times89831=6390.6$$

$\therefore$ $$V_2=\sqrt{2\times6390.6}=\mathbf{113.0\ m/s.\ Ans.}$$

15.3.3 Momentum Equations. The momentum per second of a flowing fluid (or momentum flux) is equal to the product of mass per second and the velocity of the flow. Mathematically, the momentum per second of a flowing fluid (compressible or incompressible) is

$$= \rho AV \times V, \quad \text{where } \rho AV = \text{Mass per second.}$$

The term ρAV is constant at every section of flow due to continuity equation. This means the momentum per second at any section is equal to the product of a constant quantity and the velocity. This also implies that momentum per second is independent of compressible effect. Hence the momentum equation for incompressible and compressible fluid is the same. The momentum equation for compressible fluid for any direction may be expressed as,

Net force in the direction of S = Rate of change of momentum in the direction of S

= Mass per second [change of velocity]

$$= \rho AV[V_2 - V_1] \qquad \text{...(15.14)}$$

where V_2 = Final velocity in the direction of S,

V_1 = Initial velocity in the direction of S.

▶15.4 VELOCITY OF SOUND OR PRESSURE WAVE IN A FLUID

The disturbance in a solid, liquid or gas is transmitted from one point to the other. The velocity with which the disturbance is transmitted depends upon the distance between the molecules of the medium. In case of solids, molecules are closely packed and hence the disturbance is transmitted instantaneously. In case of liquids and gases (or fluids) the molecules are relatively apart. The disturbance will be transmitted from one molecule to the next molecule. But in case of fluids, there is some distance between two adjacent molecules. Hence each molecule will have to travel a certain distance before it can transmit the disturbance. Thus the velocity of disturbance in case of fluids will be less than the velocity of the disturbance in solids.

The distance between the molecules is related with the density, which in turn depends upon pressure in case of fluids. Hence the velocity of disturbance depends upon the changes of pressure and density of the fluid.

15.4.1 Expression for Velocity of Sound Wave in a Fluid. The disturbance creates the pressure waves in a fluid. These pressure waves travel with a velocity of sound waves in all directions. But for the sake of simplicity, one-dimensional case will be considered.

Fig. 15.2 shows the model for one-dimensional propagation of the pressure waves. It is a right long pipe of uniform cross-sectional area, fitted with a piston. Let the pipe is filled with a compressible fluid, which is at rest initially. The piston is moved towards right and a disturbance is created in the fluid. This disturbance is in the form of pressure wave, which travels in the fluid with a velocity of sound wave.

Let A = Cross-sectional area of the pipe

V = Velocity of piston

p = Pressure of the fluid in pipe before the movement of the piston

ρ = Density of fluid before the movement of the piston

dt = A small interval of time with which piston is moved

C = Velocity of pressure wave or sound wave travelling in the fluid

Distance travelled by the piston in time dt

$$= \text{Velocity of piston} \times dt = V \times dt$$

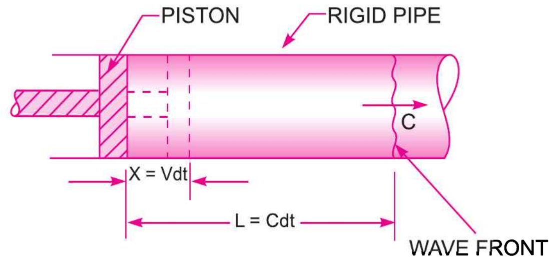


Fig. 15.2 *Propagation of pressure wave.*

Distance travelled by the pressure wave in time dt

$$= \text{Velocity of pressure wave} \times dt = C \times dt$$

As the value of C will be very large, hence $C \times dt$ will be more than $V \times dt$. For the time interval 'dt', the pressure wave has travelled a distance L and piston has moved through x. Thus in the length of the tube equal to $(L - x)$, the fluid will be compressed. Due to compression of the fluid, the pressure and density of the fluid will change.

Let $p + dp$ = Pressure after compression

$\rho + d\rho$ = Density after compression or the density of fluid in the length $(L - x)$

Now mass of fluid for a length 'L' before compression

$$= \rho \times \text{Volume of fluid upto length } L$$

$$= \rho \times A \times L = \rho \times A \times C \times dt \qquad (\because \; L = Cdt) \quad ...(i)$$

Mass of fluid after compression for length $(L - x)$

$$= \text{Density after compression} \times \text{Area} \times \text{Length}$$

$$= (\rho + d\rho)\, A \times (L - x)$$

$$= (\rho + d\rho)\, A \times (Cdt - Vdt) \qquad (\because \; L = Cdt,\, x = Vdt) \quad ...(ii)$$

From the continuity equation, we have

Mass of fluid before compression

$$= \text{Mass of fluid after compression}$$

$$\therefore \quad \rho ACdt = (\rho + d\rho)\, A \times (Cdt - Vdt) \text{ or } \rho ACdt = (\rho + d\rho)\, A \times dt\, (C - V)$$

Dividing by $A \times dt$, $\rho C = (\rho + d\rho)\,(C - V) = \rho C - \rho V + Cd\rho - Vd\rho$

$$\therefore \quad Cd\rho = \rho C - \rho C + \rho V + Vd\rho = \rho V + Vd\rho. \quad ...(iii)$$

But the velocity of the piston, V, is very small as compared to the velocity of the pressure wave C. Also the value of $d\rho$ is very small. Hence the term $(V \times d\rho)$ will be very-very small and can be neglected. Hence equation (*iii*) becomes,

$$Cd\rho = \rho V \quad ...(iv)$$

Now when the piston is moved with a velocity V for time dt, the fluid which is at rest initially will move with a velocity equal to the velocity of the piston. Also the pressure of the fluid will increase from p to $p + dp$ due to the movement of the piston. Hence applying the impulse momentum equation, we get

Net force on the fluid = Rate of change of momentum

or $(p + dp)\, A - p \times A$ = Mass per second [change of velocity of fluid]

or
$$dp \times A = \frac{\text{Total mass}}{\text{Time}} [V - 0] = \frac{\rho AL}{dt} [V - 0]$$
$$= \frac{\rho ACdt}{dt} [V - 0] \qquad (\because \quad L = Cdt)$$
$$= \rho AC [V - 0] = \rho ACV \quad \text{or} \quad dp = \frac{\rho ACV}{A} = \rho CV$$
or
$$C = \frac{dp}{\rho V} \qquad \text{...}(v)$$

Multiplying equations (*iv*) and (*v*), we get
$$C^2 d\rho = \rho V \times \frac{dp}{\rho V} = dp$$
$$C^2 = \frac{dp}{d\rho}$$
$\therefore$
$$C = \sqrt{\frac{dp}{d\rho}} \qquad \text{...(15.15)}$$

Hence equation (15.15) gives the velocity of sound wave which is the square root of the ratio of change of pressure to the change of density of a fluid due to disturbance.

15.4.2 Velocity of Sound in Terms of Bulk Modulus. Bulk modulus K is defined as
$$K = \frac{\text{Increase in pressure}}{\frac{\text{Decrease in volume}}{\text{Original volume}}} = \frac{dp}{-\left(\frac{d\forall}{\forall}\right)} \qquad \text{...}(vi)$$

where $d\forall$ = Decrease in volume, $\forall$ = Original volume.

Negative sign is taken, as with the increase of pressure, volume decreases.

Now we know mass of a fluid is constant. Hence
$$\rho \times \text{Volume} = \text{Constant} \qquad (\because \quad \text{Mass} = \rho \times \text{Volume})$$
or
$$\rho \times \forall = \text{Constant}$$

Differentiating the above equation (ρ and $\forall$ are variables),
$$\rho d\forall + \forall d\rho = 0 \quad \text{or} \quad \rho d\forall = -\forall d\rho \text{ or } -\frac{d\forall}{\forall} = \frac{d\rho}{\rho}$$

Substituting the value $\left(-\frac{d\forall}{\forall}\right)$ in equation (*vi*), we get
$$K = \frac{dp}{\frac{d\rho}{\rho}} = \rho \frac{dp}{d\rho} \text{ or } \frac{dp}{d\rho} = \frac{K}{\rho}$$

From equation (15.15), the velocity of sound wave is
$$C = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{K}{\rho}} \qquad \text{...(15.16)}$$

Equation (15.16) gives the velocity of sound wave in terms of bulk modulus and density. This equation is applicable for liquids and gases.

15.4.3 Velocity of Sound for Isothermal Process. Isothermal process is given by equation (15.3), as

$$\frac{p}{\rho} = \text{Constant or } p\rho^{-1} = \text{Constant}$$

Differentiating the above equation (p and ρ both are variable),

$$p\,(-\,1)\,\rho^{-2}\,d\rho + \rho^{-1}\,dp = 0$$

Dividing by ρ^{-1}, we get $-\,p\rho^{-1}\,d\rho + dp = 0$ or $\frac{-p}{\rho}\,d\rho + dp = 0$

$$\therefore \qquad dp = \frac{p}{\rho}\,d\rho \text{ or } \frac{dp}{d\rho} = \frac{p}{\rho} = RT \qquad \left(\because \text{ From equation of state } \frac{p}{\rho} = RT\right)$$

Substituting the value of $\frac{dp}{d\rho}$ in equation (15.15), $C = \sqrt{\frac{p}{\rho}} = \sqrt{RT}$ ...(15.17)

15.4.4 Velocity of Sound for Adiabatic Process. Adiabatic process is given by equation (15.4), as

$$\frac{p}{\rho^k} = \text{Constant or } p\rho^{-k} = \text{Constant}$$

Differentiating the above equation, we get

$$p\,(-\,k)\,\rho^{-k-1}\,d\rho + \rho^{-k}\,dp = 0$$

Dividing by ρ^{-k}, we get $-\,pk\rho^{-1}\,d\rho + dp = 0$ or $dp = \frac{pk}{\rho}\,d\rho$

$$\therefore \qquad \frac{dp}{d\rho} = \frac{p}{\rho}\,k = RTk \qquad \left(\because \frac{p}{\rho} = RT\right)$$

$$= kRT$$

Substituting the value of $\frac{dp}{d\rho}$ in equation (15.15), we get $C = \sqrt{kRT}$. ...(15.18)

Note 1. For the propagation of the minor disturbances through air, the process is assumed to be adiabatic. The velocity of the disturbances (pressure waves) through air is very high and hence there is no time for any appreciable heat transfer.

2. Isothermal process is considered for the calculation of the velocity of the sound waves (or pressure waves) only when it is given in the numerical problem that process is isothermal. If no process is mentioned, it is assumed to be adiabatic.

▶15.5 MACH NUMBER

In Chapter 12, Art. 12.8.5, Mach number has been defined as the square root of the ratio of the inertia force of a flowing fluid to the elastic force.

$\therefore$ Mach number $= M = \sqrt{\dfrac{\text{Inertia force}}{\text{Elastic force}}} = \sqrt{\dfrac{\rho A V^2}{KA}}$

$$= \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C} \quad \left[\because \sqrt{\frac{K}{\rho}} = C \text{ from equation (15.16)}\right]$$

Thus Mach number $= M$

$$= \frac{\text{Velocity of fluid or body moving in fluid}}{\text{Velocity of sound in the fluid}}$$

$$= \frac{V}{C}. \qquad \text{...(15.19)}$$

For the compressible fluid flow, Mach number is an important non-dimensional parameter. On the basis of the Mach number, the flow is defined as :

1. Sub-sonic flow, 2. Sonic flow, and 3. Super-sonic flow.

1. Sub-sonic Flow. A flow is said sub-sonic flow if the Mach number is less than 1.0 (or $M < 1$) which means the velocity of flow is less than the velocity of sound wave (or $V < C$).

2. Sonic Flow. A flow is said to be sonic flow if the Mach number (M) is equal to 1.0. This means that when the velocity of flow V is equal to the velocity of sound C, the flow is said to be sonic flow.

3. Super-sonic Flow. A flow is said to be super-sonic flow if the Mach number is greater than 1.0 (or $M > 1$). This means that when velocity of flow V is greater than the velocity of sound wave, the flow is said to be super-sonic flow.

Problem 15.4 *Find the sonic velocity for the following fluids :*

(i) Crude oil of sp. gr. 0.8 and bulk modulus 153036 N/cm^2.

(ii) Mercury having a bulk modulus of 2648700 N/cm^2.

Solution. Given :

(*i*) For Crude oil, sp. gr. = 0.8

$\therefore$ Density, $\rho = 0.8 \times 1000 = 800$ kg/m^3

Bulk modulus, $K = 153036$ N/cm^2 $= 153036 \times 10^4$ N/m^2

Using equation (15.16) for sonic velocity, as

$$C = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{153036 \times 10^4}{800}} = 1383.09 \simeq \textbf{1383 m/s. Ans.}$$

(*ii*) For Mercury, sp. gr. = 13.6

$\therefore$ Density of Mercury, $\rho = 13.6 \times 1000$ kg/m^3

Bulk modulus, $K = 2648700$ N/cm^2 $= 2648700 \times 10^4$ N/m^2

The sonic velocity, $C = \sqrt{\dfrac{K}{\rho}} = \sqrt{\dfrac{2648700 \times 10^4}{13.6 \times 1000}} = \textbf{1395.55 m/s. Ans.}$

Problem 15.5 *Find the speed of the sound wave in air at sea-level where the pressure and temperature are 10.1043 N/cm^2 (abs.) and 15°C respectively. Take R = 287 J/kg°K and k = 1.4.*

Solution. Given :

Pressure, $p = 10.1043 \text{ N/cm}^2 = 10.1043 \times 10^4 \text{ N/m}^2$

Temperature, $t = 15°\text{C}$

$\therefore$ $T = 273 + 15 = 288 \text{ K}, R = 287 \text{ J/kg°K}, k = 1.4$

For adiabatic process, the velocity of sound is given by equation (15.18), as

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 288} = \mathbf{340.17 \text{ m/s. Ans.}}$$

Problem 15.6 *Calculate the Mach number at a point on a jet propelled aircraft, which is flying at 1100 km/hour at sea-level where air temperature is 20°C. Take k = 1.4 and R = 287 J/kg°K.*

Solution. Given :

Speed of aircraft, $V = 1100 \text{ km/hour} = \dfrac{1100 \times 1000}{60 \times 60} = 305.55 \text{ m/s}$

Temperature, $t = 20°\text{C}$

$\therefore$ $T = 273 + 20 = 293°\text{K}, k = 1.4, R = 287 \text{ J/kg°K}$

Using equation (15.18), the velocity of sound is

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 293} = 343.11 \text{ m/s}$$

Mach number is given by equation (15.19) as

$$M = \frac{V}{C} = \frac{305.55}{343.11} = \mathbf{0.89. \text{ Ans.}}$$

Problem 15.7 *An aeroplane is flying at an height of 15 km where the temperature is – 50°C. The speed of the plane is corresponding to M = 2.0. Assuming k = 1.4 and R = 287 J/kg°K, find the speed of the plane.*

Solution. Given :

Height of the plane, $Z = 15 \text{ km}$ (Extra Data)

Temperature, $t = -50°\text{C}$

$\therefore$ $T = -50 + 273 = 223°\text{K}$

Mach number, $M = 2.0, k = 1.4, R = 287 \text{ J/kg°K}.$

Using equation (15.18), we get the velocity of sound as

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 223} = 299.33 \text{ m/s}$$

Using equation (15.19), we have $M = \dfrac{V}{C}$ or $2.0 = \dfrac{V}{299.33}$

$\therefore$ $V = 2.0 \times 299.33 = 598.66 \text{ m/s}$

$$= \frac{598.66 \times 60 \times 60}{1000} = \mathbf{2155.17 \text{ km/hour. Ans.}}$$

▶15.6 PROPAGATION OF PRESSURE WAVES (OR DISTURBANCES) IN A COMPRESSIBLE FLUID

Whenever any disturbance is produced in a compressible fluid, the disturbance is propagated in all directions with a velocity of sound (*i.e.*, equal to C). The nature of propagation of the disturbance depends upon the Mach number. Let us consider a small projectile moving from left to right in a straight line in a stationary fluid. Due to the movement of the projectile, the disturbances will be created in the fluid. This disturbance will be moving in all directions with a velocity C.

Hence let V = Velocity of the projectile,

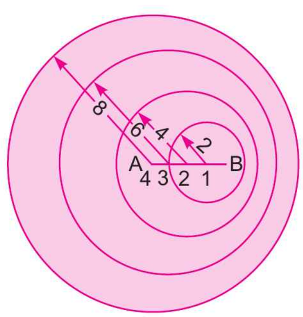
C = Velocity of pressure wave or disturbance created in the fluid.

Let us find the nature of propagation of the disturbance for different Mach numbers.

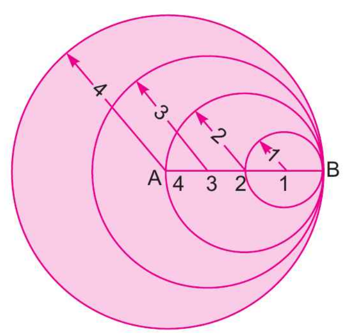
1st Case : When M < 1. When Mach number is less than 1.0, the flow is called sub-sonic flow. For $M < 1$ means $\frac{V}{C} < 1$ or $V < C$. To find the nature of propagation for this case, let $V = 1$ unit and $C = 2$ unit, so that $\frac{V}{C} = \frac{1}{2}$ which is less than 1.0. Let the projectile is at A and is moving towards right. Let in 4 seconds the projectile reaches to the position B. At A, the point 4 is also marked. The position of the projectile after 1 sec, 2 sec, 3 sec and 4 sec along the lines are shown by the points 3, 2, 1 and B respectively. The projectile moves from A to B in 4 seconds and hence the distance $AB = 4 \times V = 4 \times 1 = 4$ units. The disturbance created at A in 4 seconds will move a distance $= 4C = 4 \times 2 = 8$ units in all directions. Hence taking A as centre and radius equal to 8 units, a circle is drawn. This circle gives the position of disturbance after 4 seconds. When the projectile is at point 3, it will reach B in three seconds and distance $3B = 3 \times V = 3 \times 1 = 3$ units. But the disturbance created at point 3 in three seconds will move a distance having a radius $= 3 \times C = 3 \times 2 = 6$ units. Similarly at point 2, the disturbance will have a radius $= 2 \times C = 4$ units and at point 1, the disturbance will have a radius $= 1 \times C = 1 \times 2 = 2$ units. This is shown in Fig. 15.3 (*a*).

As in this case $V < C$, the pressure wave is always ahead of the projectile and point B is inside the sphere of radius 8 units.

2nd Case : When M = 1. When $M = 1$, the flow is known as sonic flow. In this case, the disturbance always travels with the projectile as shown in Fig. 15.3 (*b*). Let $V = 1$ unit, and $C = 1$ unit so that $M = \frac{V}{C} = \frac{1}{1} = 1.0$. Let the projectile moves from A to B in 4 seconds. The disturbance created at A in 4 seconds will move a distance having radius $= 4 \times C = 4 \times 1 = 4$ units in all directions. The projectile



(a) M<1



(b) M=1

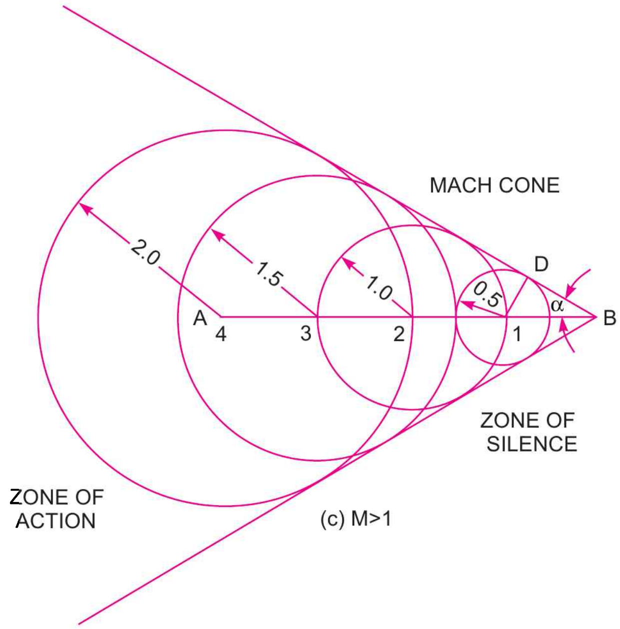


Fig. 15.3 *Propagation of disturbance for different Mach numbers.*

from point 3 will move to position *B* in three seconds. The disturbance created at point 3, will move a distance having radius = 3 × *C* = 3 × 1 = 3 units in all directions in three seconds. Similarly at the point 2 and point 1, the disturbance created at these points will move a distance having radius 2 and 1 in all directions respectively.

3rd Case : When M > 1. When $M > 1$, the flow is known as supersonic flow. Let $V = 1$ unit and $C = 0.5$ unit so that $M = \frac{V}{C} = \frac{1}{0.5} = 2.0$, which is greater than unity. Let the projectile moves from *A* to *B* in 4 seconds. The distance travelled by the projectile in 4 seconds = $4 \times V = 4 \times 1 = 4$ units. Hence, take *AB* = 4 units. The disturbance created at *A* will move in all directions and in 4 seconds, the radius of disturbance will be equal to $4 \times C = 4 \times 0.5 = 2$ units. Hence taking *A* as centre, draw a circle with radius equal to 2 units. After one second from *A*, the projectile will be at point 3 and distance *A*3 = $V \times 1 = 1 \times 1 = 1$ unit. The projectile from point 3 will reach point *B* in three seconds. Hence the disturbance created at point 3 will move in all directions and in three seconds, the radius of disturbance from point 3 will be equal to $3 \times C = 3 \times 0.5 = 1.5$ units. Similarly the radius of disturbance at point 2 and 1 will be $2 \times C = 2 \times 0.5 = 1$ unit and $1 \times C = 1 \times 0.5 = 0.5$ unit respectively as shown in Fig. 15.3 (*c*). In this case the sphere of propagation of disturbance always lags behind the projectile. If we draw a tangent to the different circles which represent the propagated spherical waves on both sides, we shall get a cone with vertex at *B*. This cone is known as **Mach Cone.**

15.6.1 Mach Angle. This is defined as the half of the angle of the Mach cone. In Fig. 15.3 (*c*), angle α is known as Mach angle. In the Δ1*BD* of Fig. 15.3 (*c*), the distance 1*B* = Velocity of projectile = *V*, the distance 1*D* = Velocity of sound wave = *C*. Hence we have

$$\sin \alpha = \frac{1D}{1B} = \frac{C}{V} = \frac{1}{V/C} = \frac{1}{M} \quad ...(15.20)$$

15.6.2 Zone of Action. When $M > 1$, the effect of the disturbance is felt only in the region inside the Mach cone. This region is called the zone of action.

15.6.3 Zone of Silence. When $M > 1$, there is no effect of disturbance in the region outside the Mach cone. The region which is outside the Mach cone is called zone of silence.

Problem 15.8 *A projectile is travelling in air having pressure and temperature as 8.829 N/cm² and – 2°C. If the Mach angle is 40°, find the velocity of the projectile. Take k = 1.4 and R = 287 J/kg°K.*

Solution. Given :

Pressure of air, $p = 8.829 \text{ N/cm}^2 = 8.829 \times 10^4 \text{ N/m}^2$

Temperature of air, $t = -2°\text{C}$

$\therefore$ $T = -2 + 273 = 271°\text{K}$

Mach angle, $\alpha = 40°$, $k = 1.4$, $R = 287$ J/kg°K

Let the velocity of projectile $= V$

Using equation (15.20), we have $\sin \alpha = \dfrac{C}{V}$ or $\sin 40° = 0.6427 = \dfrac{C}{V}$

The velocity of sound, C is given by equation (15.18) as

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 271} = 329.98 \text{ m/s} \simeq 330 \text{ m/s}$$

$$\therefore \quad \sin 40° = 0.6427 = \frac{C}{V} = \frac{330}{V}$$

$$\therefore \quad V = \frac{330}{0.6427} = \textbf{513 m/s. Ans.}$$

Problem 15.9 *A projectile travels in air of pressure 10.1043 N/cm² at 10°C at a speed of 1500 km/hour. Find the Mach number and the Mach angle. Take k = 1.4 and R = 287 J/kg°K.*

Solution. Given :

Pressure, $p = 10.1043 \text{ N/cm}^2 = 10.1043 \times 10^4 \text{ N/cm}^2$

Temperature, $t = 10°\text{C}$

$\therefore$ $T = 10 + 273 = 283°\text{K}$

Speed of projectile, $V = 1500 \text{ km/hour} = \dfrac{1500 \times 1000}{60 \times 60} \text{ m/s} = 416.67 \text{ m/s}$

$k = 1.4$, $R = 287$ J/kg°K

For adiabatic process, the velocity of sound is given by

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 283} = 337.20 \text{ m/s}$$

$\therefore$ Mach number, $M = \dfrac{V}{C} = \dfrac{416.67}{337.20} = \textbf{1.235. Ans.}$

∴ Mach angle is obtained from equation (15.20) as

$$\sin\alpha = \frac{C}{V} = \frac{1}{M} = \frac{1}{1.235} = 0.8097$$

∴ Mach angle, $\alpha = \sin^{-1} 0.8097 =$ **54.06°. Ans.**

Problem 15.10 *Find the velocity of bullet fired in standard air if the Mach angle is 30°. Take R = 287.14 J/kg°K and k = 1.4 for air. Assume temperature as 15°C.*

Solution. Given :

Mach angle $\alpha = 30°$

$R = 287.14$ J/kg°K

$k = 1.4$

Temperature, $t = 15°C$

∴ $T = 15 + 273 = 288°K$

Velocity of sound is given by equation (15.18) as

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287.14 \times 288} = 340.25 \text{ m/s}$$

Using the relation, $\sin\alpha = \frac{C}{V}$ given by equation (15.20)

$$\sin 30° = \frac{340.25}{V}$$

∴ $$V = \frac{340.25}{\sin 20} = \textbf{680.50 m/s. Ans.}$$

▶ 15.7 STAGNATION PROPERTIES

When a fluid is flowing past an immersed body, and at a point on the body if the resultant velocity becomes zero, the values of pressure, temperature and density at that point are called stagnation properties. The point is called the **stagnation point.** The values of pressure, density and temperature are called stagnation pressure, stagnation density and stagnation temperature respectively. They are denoted by p_s, ρ_s and T_s respectively.

15.7.1 Expression for Stagnation Pressure (p_s).

Consider a compressible fluid flowing past an immersed body under frictionless adiabatic conditions as in Fig. 15.4. Consider two points 1 and 2 on a stream-line as shown in Fig. 15.4.

Let p_1 = Pressure of compressible fluid at point 1,

V_1 = Velocity of fluid at 1, and

ρ_1 = Density of fluid at 1,

p_2, V_2, ρ_2 = Corresponding values of pressure, velocity and density at point 2.

Applying Bernoulli's equation for adiabatic flow given by equation (15.13) at point 1 and 2, we get

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} + Z_1 = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} + Z_2$$

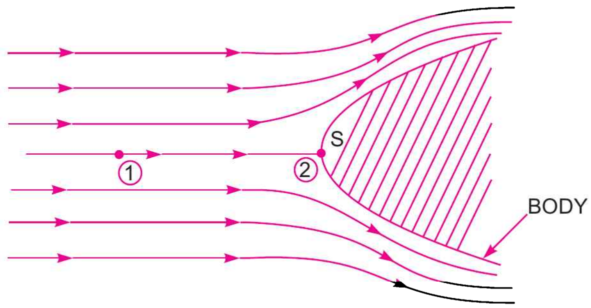


Fig. 15.4 *Stagnation properties.*

But $$Z_1 = Z_2$$

$\therefore$ $$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g}$$

or $$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1} + \frac{V_1^2}{2} = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2} + \frac{V_2^2}{2} \qquad \left(\text{Cancelling } \frac{1}{g}\right)$$

Point 2 is a stagnation point. Hence velocity will become zero at stagnation point and pressure and density will be denoted by p_s and ρ_s.

$\therefore$ $$V_2 = 0,\ p_2 = p_s \text{ and } \rho_2 = \rho_s$$

Substituting these values in the above Bernoulli's equation,

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1} + \frac{V_1^2}{2} = \left(\frac{k}{k-1}\right)\frac{p_s}{\rho_s} + 0$$

or $$\left(\frac{k}{k-1}\right)\left[\frac{p_1}{\rho_1} - \frac{p_s}{\rho_s}\right] = -\frac{V_1^2}{2} \text{ or } \left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1}\left[1 - \frac{p_s}{\rho_s} \times \frac{\rho_1}{p_1}\right] = -\frac{V_1^2}{2}$$

or $$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1}\left[1 - \frac{p_s}{p_1} \times \frac{\rho_1}{\rho_s}\right] = -\frac{V_1^2}{2} \qquad \text{...}(i)$$

But for adiabatic process from equation (15.4), we have

$$\frac{p}{\rho^k} = \text{constant or } \frac{p_1}{\rho_1^k} = \frac{p_s}{\rho_s^k} \text{ or } \frac{p_1}{p_s} = \frac{\rho_1^k}{\rho_s^k} \text{ or } \left(\frac{\rho_1}{\rho_s}\right) = \left(\frac{p_1}{p_s}\right)^{\frac{1}{k}} \qquad \text{...}(ii)$$

Substituting the value of $\frac{\rho_1}{\rho_s}$ in equation (*i*), we get

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1}\left[1 - \frac{p_s}{p_1} \times \left(\frac{p_1}{p_s}\right)^{\frac{1}{k}}\right] = -\frac{V_1^2}{2}$$

or $$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1}\left[1-\frac{p_s}{p_1}\times\left(\frac{p_s}{p_1}\right)^{-\frac{1}{k}}\right] = -\frac{V_1^2}{2}$$

or $$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_s}{p_1}\right)^{1-\frac{1}{k}}\right] = -\frac{V_1^2}{2}$$

or $$\left[1-\left(\frac{p_s}{p_1}\right)^{\frac{k-1}{k}}\right] = -\frac{V_1^2}{2}\times\left(\frac{k-1}{k}\right)\frac{\rho_1}{p_1}$$

or $$1+\frac{V_1^2}{2}\left(\frac{k-1}{k}\right)\frac{\rho_1}{p_1} = \left(\frac{p_s}{p_1}\right)^{\frac{k-1}{k}} \quad \text{...(iii)}$$

Now for adiabatic process, the velocity of sound is given by equation (15.18) as

$$C = \sqrt{kRT} = \sqrt{k\frac{p}{\rho}} \qquad \left(\because \frac{p}{\rho} = RT\right)$$

For the point 1, $C_1 = \sqrt{k\frac{p_1}{\rho_1}}$ or $C_1^2 = k\frac{p_1}{\rho_1}$

Substituting the value of $\frac{k\,p_1}{\rho_1} = C_1^2$ in equation (*iii*)

$$1+\frac{V_1^2}{2}(k-1)\times\frac{1}{C_1^2} = \left(\frac{p_s}{p_1}\right)^{\left(\frac{k-1}{k}\right)} \text{ or } 1+\frac{V_1^2}{2C_1^2}\times(k-1) = \left(\frac{p_s}{p_1}\right)^{\left(\frac{k-1}{k}\right)}$$

or $$1+\frac{M_1^2}{2}(k-1) = \left(\frac{p_s}{p_1}\right)^{\left(\frac{k-1}{k}\right)} \qquad \left(\because \frac{V_1^2}{C_1^2} = M_1^2\right)$$

or $$\left(\frac{p_s}{p_1}\right)^{\frac{k-1}{k}} = \left[1+\frac{(k-1)}{2}M_1^2\right] \text{ or } \frac{p_s}{p_1} = \left[1+\frac{k-1}{2}M_1^2\right]^{\left(\frac{k}{k-1}\right)} \quad \text{...(iv)}$$

∴ $$p_s = p_1\left[1+\frac{k-1}{2}M_1^2\right]^{\left(\frac{k}{k-1}\right)} \quad \text{...(15.21)}$$

Equation (15.21) gives the value of stagnation pressure.

In equation (15.21), for $M < 1$, the term $\frac{k-1}{2}M_1^2$ will be less than 1 and hence the R.H.S. of this equation can be expressed by Binomial theorem as

$$p_s = p_1\left[1+\left(\frac{k}{k-1}\right)\times\left(\frac{k-1}{2}.M_1^2\right)+\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1}-1\right)\left(\frac{k-1}{2}.M_1^2\right)^2\Big/2!\right.$$

$$\left.+\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1}-1\right)\left(\frac{k}{k-1}-2\right)\left(\frac{k-1}{2}.M_1^2\right)^3\Big/\ 3!+\ldots\ldots\right]$$

$$= p_1\left[1+\frac{k}{2}M_1^2+\frac{k}{8}M_1^4+\frac{k(2-k)}{48}M_1^6+\ldots\ldots\right]$$

$$= p_1 + p_1\left[\frac{k}{2}M_1^2+\frac{k}{8}M_1^4+\frac{k(2-k)}{48}M_1^6+\ldots\ldots\right]$$

or
$$\frac{p_s-p_1}{p_1} = \left[\frac{k}{2}M_1^2+\frac{k}{8}M_1^4+\frac{k(2-k)}{48}M_1^6+\ldots\ldots\right]$$

$$= \frac{k}{2}M_1^2\left[1+\frac{1}{4}M_1^2+\frac{(2-k)}{24}M_1^4+\ldots\ldots\right]$$

$$= \frac{k}{2}\frac{V_1^2}{C_1^2}\left[1+\frac{1}{4}M_1^2+\frac{(2-k)}{24}M_1^4+\ldots\ldots\right]$$

$$= \frac{k}{2}\frac{V_1^2}{\left(k\dfrac{p_1}{\rho_1}\right)}\left[1+\frac{1}{4}M_1^2+\frac{(2-k)}{24}M_1^4+\ldots\ldots\right] \qquad \left(\because\ C_1=\sqrt{\frac{kp_1}{\rho_1}}\right)$$

$$= \frac{1}{2}.\frac{\rho_1 V_1^2}{p_1}\left[1+\frac{1}{4}M_1^2+\frac{(2-k)}{24}M_1^4+\ldots\ldots\right]$$

or
$$(p_s-p_1) = \frac{1}{2}\ .\ \rho_1\ .\ V_1^{\ 2}\left[1+\frac{1}{4}M_1^2+\frac{(2-k)}{24}M_1^4+\ldots\ldots\right] \qquad \ldots(15.21A)$$

From the above equation, it is clear that when the approaching velocity V_1 is small compared with the velocity of sound wave C_1, then M_1 will be very small and the term $1+\frac{1}{4}M_1^{\ 2}+\frac{(2-k)}{24}M_1^{\ 4}+\ldots..$ will be nearly equal to 1.

Hence equation (15.21*A*) becomes as

∴
$$p_s - p_1 = \frac{1}{2}\ .\ \rho_1 \times V_1^{\ 2} \text{ or } p_s = p_1 + \frac{1}{2}\ \rho_1\ V_1^{\ 2}$$

But when approaching velocity becomes high then M_1 is not small and equation (15.21*A*) is expressed as

$$\frac{(p_s-p_1)}{\frac{1}{2}\times\rho_1\times V_1^2} = 1+\frac{1}{4}M_1^2+\frac{(2-k)}{24}M_1^{\ 4}+\ldots.. \qquad \ldots(15.21B)$$

15.7.2 Expression for Stagnation Density (ρ_s). From equation (*ii*), we have

$$\left(\frac{\rho_1}{\rho_s}\right) = \left(\frac{p_1}{p_s}\right)^{\frac{1}{k}} \text{ or } \left(\frac{\rho_s}{\rho_1}\right) = \left(\frac{p_s}{p_1}\right)^{\frac{1}{k}} \qquad \text{(Taking reciprocal)}$$

$$\therefore \qquad \rho_s = \rho_1 \left[\frac{p_s}{p_1}\right]^{\frac{1}{k}}$$

Substituting the value of $\left(\frac{p_s}{p_1}\right)$ from equation (*iv*),

$$\rho_s = \rho_1 \left[\left(1 + \frac{k-1}{2} M_1^2\right)^{\frac{k}{k-1}}\right]^{\frac{1}{k}} \text{ or } \rho_s = \rho_1 \left[1 + \frac{k-1}{2} M_1^2\right]^{\frac{1}{k-1}} \qquad ...(15.22)$$

15.7.3 Expression for Stagnation Temperature (T_s). Equation of state is given by equation (15.2) as $\frac{p}{\rho} = RT$

For the stagnation point, we have equation of state as $\frac{p_s}{\rho_s} = RT_s$ ...(15.22*A*)

$$\therefore \qquad T_s = \frac{1}{R}\frac{p_s}{\rho_s}$$

Substituting the value of p_s and ρ_s from equations (15.21) and (15.22), we have

$$T_s = \frac{1}{R}\frac{p_1\left[1 + \left(\frac{k-1}{2}\right) M_1^2\right]^{\left(\frac{k}{k-1}\right)}}{\rho_1\left[1 + \left(\frac{k-1}{2}\right) M_1^2\right]^{\frac{1}{k-1}}}$$

$$= \frac{1}{R}\frac{p_1}{\rho_1}\left[1 + \left(\frac{k-1}{2}\right) M_1^2\right]^{\left(\frac{k}{k-1}\right) - \left(\frac{1}{k-1}\right)}$$

$$= \frac{1}{R}\frac{p_1}{\rho_1}\left[1 + \left(\frac{k-1}{2}\right) M_1^2\right]^{\left(\frac{k-1}{k-1}\right)} = \frac{p_1}{\rho_1 R}\left[1 + \left(\frac{k-1}{2}\right) M_1^2\right]$$

$$= T_1\left[1 + \left(\frac{k-1}{2}\right) M_1^2\right] \qquad \left(\because \frac{p_1}{\rho_1} = RT_1\right) \quad ...(15.23)$$

Problem 15.11 *Find the Mach number when an aeroplane is flying at 1100 km/hour through still air having a pressure of 7 N/cm² and temperature – 5°C. Wind velocity may be taken as zero. Take R = 287.14 J/kg K. Calculate the pressure, temperature and density of air at stagnation point on the nose of the plane. Take k = 1.4.*

Solution. Given :

Speed of aeroplane, $V = 1100 \text{ km/hr} = \dfrac{1100 \times 1000}{60 \times 60} = 305.55 \text{ m/s}$

Pressure of air, $p_1 = 7 \text{ N/cm}^2 = 7 \times 10^4 \text{ N/m}^2$

Temperature, $t_1 = -5°\text{C}$

∴ $T_1 = -5 + 273 = 268°\text{K}$

$R = 287.14 \text{ J/kg K}$

$k = 1.4$

Using relation $C = \sqrt{kRT}$ for velocity of sound for adiabatic process, we have

$$C_1 = \sqrt{1.4 \times 287.14 \times 268} = 328.2 \text{ m/s}$$

∴ Mach number, $M_1 = \dfrac{V_1}{C_1} = \dfrac{305.55}{328.20} = 0.9309 \simeq$ **0.931. Ans.**

Stagnation Pressure, p_s. Using equation (15.21) for stagnation pressure,

$$p_s = p_1 \left[1 + \frac{k-1}{2} M_1^2\right]^{\frac{k}{k-1}}$$

$$= 7.0 \times 10^4 \left[1 + \frac{1.4 - 1.0}{2.0} \times (.931)^2\right]^{\frac{1.4}{1.4-1.0}}$$

$$= 7.0 \times 10^4 \,[1 + .1733]^{\frac{1.4}{.4}}$$

$= 7.0 \times 10^4 \,[1.1733]^{3.5} = 12.24 \times 10^4 \text{ N/m}^2 =$ **12.24 N/cm². Ans.**

Stagnation Temperature, T_s. Using equation (15.23) for stagnation temperature,

$$T_s = T_1 \left[1 + \frac{k-1}{2} M_1^2\right]$$

$$= 268 \left[1 + \frac{1.4 - 1.0}{2.0} \times (.931)^2\right] = 268\,[1.1733] = 314.44°\text{K}$$

∴ $t_s = T_s - 273 = 314.43 - 273 =$ **41.44°C. Ans.**

Stagnation Density, ρ_s. Using equation of state (15.22 A) for stagnation density, $\dfrac{p_s}{\rho_s} = RT_s$

∴ $$\rho_s = \frac{p_s}{RT_s} \qquad \text{...}(i)$$

In equation (*i*) given above, if R is taken as 287.14 J/kg K, then pressure should be taken in N/m² so that the value of ρ is in kg/m³. Hence $p_s = 12.24 \times 10^4$ N/m² and $T_s = 314.44$°K.

∴ $\rho_s = \dfrac{12.24 \times 10^4}{287.14 \times 314.44} =$ **1.355 kg/m³. Ans.**

Problem 15.12 *Calculate the stagnation pressure, temperature and density at the stagnation point on the nose of a plane, which is flying at 800 km/hour through still air having a pressure 8.0 N/cm² (abs.) and temperature – 10°C. Take R = 287 J/kg K and k = 1.4.*

Solution. Given :

Speed of plane, $V = 800 \text{ km/hour} = \dfrac{800 \times 1000}{60 \times 60} = 222.22 \text{ m/s}$

Pressure of air, $p_1 = 8.0 \text{ N/cm}^2 = 8.0 \times 10^4 \text{ N/m}^2$

Temperature, $t_1 = -10°C$

∴ $T_1 = -10 + 273 = 263°K$

$R = 287 \text{ J/kg°K}$

$k = 1.4$

For adiabatic flow, the velocity of sound is given by

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 263} = 325.07 \text{ m/s}$$

∴ Mach number, $M = \dfrac{V}{C} = \dfrac{222.22}{325.07} = 0.683.$

This Mach number is the local Mach number and hence equal to M_1.

∴ $M_1 = 0.683$

Using equation (15.21) for stagnation pressure,

$$p_s = p_1 \left[1 + \frac{k-1}{2} M_1^2\right]^{\left(\frac{k}{k-1}\right)} = 8.0 \times 10^4 \left[1 + \frac{1.4 - 1.0}{2.0} \times (.683)^2\right]^{\left(\frac{1.4}{1.4-1.0}\right)}$$

$$= 8.0 \times 10^4 \, [1.0933]^{3.5} = 10.93 \times 10^4 \text{ N/m}^2 = \mathbf{10.93 \text{ N/cm}^2.\ Ans.}$$

Using equation (15.23) for stagnation temperature,

$$T_s = T_1 \left(1 + \frac{k-1}{2} M_1^2\right) = 263 \left(1 + \frac{1.4 - 1.0}{2.0} \times (.683)^2\right)$$

$$= 263 \, [1.0933] = 287.5 \text{ K}$$

∴ $t_s = T_s - 273 = 287.5 - 723 = \mathbf{14.5°C.\ Ans.}$

Using equation (15.2), $\dfrac{p}{\rho} = RT$

For stagnation point, $\dfrac{p_s}{\rho_s} = RT_s$ ∴ $\rho_s = \dfrac{p_s}{RT_s}$

As $R = 287$ J/kg K, the value of p_s should be taken in N/m² so that the value of ρ_s is obtained in kg/m³.

∴ $p_s = 10.93 \times 10^4 \text{ N/m}^2$

∴ Stagnation density, $\rho_s = \dfrac{10.93 \times 10^4}{287 \times 287.5} = \mathbf{1.324 \text{ kg/m}^3.\ Ans.}$

▶15.8 AREA VELOCITY RELATIONSHIP FOR COMPRESSIBLE FLOW

The area velocity relationship for incompressible fluid is given by the continuity equation as

$$A \times V = \text{Constant.}$$

From the above equation, it is clear that with the increase of area, velocity decreases. But in case of compressible fluid, the continuity equation is given by, $\rho AV = \text{Constant.}$ ...(*i*)

From this relation, it is clear that with the change of area, both the velocity and density are affected. Hence to find the relation between area and velocity for compressible fluid we proceed as given below.

Differentiating equation (*i*), we get

$$\rho d\,(AV) + AVd\rho = 0 \quad \text{or} \quad \rho\,[AdV + VdA] + AVd\rho = 0$$

or $$\rho AdV + \rho VdA + AVd\rho = 0.$$

Dividing by ρAV, we get $$\frac{dV}{V} + \frac{dA}{A} + \frac{d\rho}{\rho} = 0 \qquad ...(ii)$$

The Euler's equation for compressible fluid is given by equation (15.8), as

$$\frac{dp}{\rho} + VdV + gdz = 0$$

Neglecting the z term, the above equation is written as $\frac{dp}{\rho} + VdV = 0.$

This equation can also be written as $\frac{dp}{\rho} \times \frac{d\rho}{d\rho} + VdV = 0$ (Dividing and multiplying by $d\rho$)

or $$\frac{dp}{d\rho} \times \frac{d\rho}{\rho} + VdV = 0$$

But $$\frac{dp}{d\rho} = C^2 \text{ from equation (15.15)}$$

Hence above equation becomes as

$$C^2 \frac{d\rho}{\rho} + VdV = 0 \quad \text{or} \quad C^2 \frac{d\rho}{\rho} = -\,VdV \quad \text{or} \quad \frac{d\rho}{\rho} = -\frac{VdV}{C^2}$$

Substituting the value of $\frac{d\rho}{\rho}$ in equation (*ii*), we get

$$\frac{dV}{V} + \frac{dA}{A} - \frac{VdV}{C^2} = 0 \quad \text{or} \quad \frac{dA}{A} = \frac{VdV}{C^2} - \frac{dV}{V} = \frac{dV}{V}\left[\frac{V^2}{C^2} - 1\right]$$

$$\therefore \quad \frac{dA}{A} = \frac{dV}{V}\left[M^2 - 1\right] \qquad ...(15.24)$$

Equation (15.24) gives the relationship between change of area with change of velocity for different Mach numbers. The following are the important conclusions :

(*i*) For $M < 1$, the flow is sub-sonic and the right-hand side of equation (15.24) is negative as $(M^2 - 1)$ is negative for the values of $M < 1$. Hence $\frac{dA}{A} > 0$, $\frac{dV}{V} < 0$. This means that with the increase of area, the velocity decreases and *vice versa.*

(*ii*) For $M > 1$, the flow is super-sonic. The value of $(M^2 - 1)$ will be positive and hence right-hand side of equation (15.24) will be positive. Hence $\frac{dA}{A} > 0$ and also $\frac{dV}{V} > 0$. This means that with the increase of area, velocity also increases.

(*iii*) For $M = 1$, the flow is called sonic flow. The value of $(M^2 - 1)$ is zero. Hence right-hand side of equation (15.24) will be zero. Hence $\frac{dA}{A} = 0$. This means area is constant.

▶15.9 FLOW OF COMPRESSIBLE FLUID THROUGH ORIFICES AND NOZZLES FITTED TO A LARGE TANK

Consider a compressible fluid filled in a large reservoir or vessel to which a short nozzle is fitted as shown in Fig. 15.5. If the pressure drop of the compressible fluid, flowing through the nozzle from reservoir is small, the process is considered to be isothermal. But if the pressure drop is large, the process is considered to be adiabatic.

Consider two points 1 and 2 inside the tank and at the exit of the nozzle respectively.

Let V_1 = Velocity of fluid in the tank,

p_1 = Pressure of fluid at point 1,

ρ_1 = Density of fluid at point 1,

T_1 = Temperature of fluid at point 1, and

V_2, p_2, ρ_2, T_2 = Corresponding values of velocity, pressure, density and temperature at point 2.

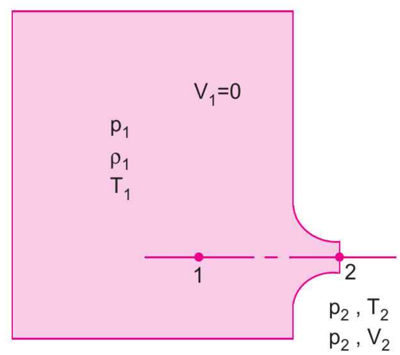


Fig. 15.5 *Pressure tank fitted with a nozzle.*

Considering the process to be adiabatic. Then from Bernoulli's equation for adiabatic flow from equation (15.13), we have

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} + Z_1 = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} + Z_2.$$

But $Z_1 = Z_2$ and also V_1 = Velocity of fluid in tank = 0

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g} + 0 = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g}$$

or
$$\left(\frac{k}{k-1}\right)\left[\frac{p_1}{\rho_1 g} - \frac{p_2}{\rho_2 g}\right] = \frac{V_2^2}{2g}$$

or
$$\left(\frac{k}{k-1}\right)\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) = \frac{V_2^2}{2} \qquad \left(\text{Cancelling } \frac{1}{g}\right)$$

or
$$V_2 = \sqrt{\frac{2k}{(k-1)}\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right]} = \sqrt{\frac{2k}{(k-1)}\frac{p_1}{\rho_1}\left(1 - \frac{p_2}{\rho_2} \times \frac{\rho_1}{p_1}\right)} \qquad \text{...}(i)$$

But for adiabatic flow from equation (15.4), we have

$$\frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} \text{ or } \frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^k$$

$\therefore$
$$\frac{\rho_1}{\rho_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{k}} \qquad ...(ii)$$

Substituting the value of $\frac{\rho_1}{\rho_2}$ in equation (i), we get

$$V_2 = \sqrt{\frac{2k}{(k-1)}\frac{p_1}{\rho_1}\left[1 - \frac{p_2}{p_1}\times\left(\frac{p_1}{p_2}\right)^{\frac{1}{k}}\right]} = \sqrt{\frac{2k}{(k-1)}\frac{p_1}{\rho_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{k}}\right]}$$

$$= \sqrt{\frac{2k}{(k-1)}\frac{p_1}{\rho_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]} \qquad ...(15.25)$$

Let $\frac{p_2}{p_1} = n$. Then above equation becomes as $V_2 = \sqrt{\frac{2k}{(k-1)}\frac{p_1}{\rho_1}\left[1 - n^{\frac{k-1}{k}}\right]}$ ...(iii)

The mass rate of flow of the compressible fluid is given as

$$m = \rho_2 A_2 V_2, \text{ where } A_2 = \text{Area at the exit of nozzle}$$

$$= \rho_2 A_2 \sqrt{\frac{2k}{(k-1)}\frac{p_1}{\rho_1}\left[1 - n^{\frac{k-1}{k}}\right]} \qquad \text{[Substitute } V_2 \text{ from } (iii)]$$

$\Rightarrow$
$$= A_2 \sqrt{\frac{2k}{(k-1)}\frac{p_1}{\rho_1}\times\rho_2^2\left[1 - n^{\frac{k-1}{k}}\right]} \qquad \text{(taking } \rho_2 \text{ inside)}$$

But from equation (ii), we have $\rho_2 = \frac{\rho_1}{\left(\frac{p_1}{p_2}\right)^{(1/k)}} = \rho_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{k}} = \rho_1 n^{\frac{1}{k}}$ $\left(\because \frac{p_2}{p_1} = n\right)$

$\therefore$
$$\rho_2^2 = \rho_1^2\, n^{\frac{2}{k}}$$

Substituting this value of ρ_2^2 in the above equation, we get

$$m = A_2\sqrt{\frac{2k}{k-1}\frac{p_1}{\rho_1}\times\rho_1^2 n^{2/k}\left[1 - n^{\frac{k-1}{k}}\right]} = A_2\sqrt{\frac{2k}{(k-1)}p_1\rho_1\left[n^{\frac{2}{k}} - n^{\frac{k-1}{k}+\frac{2}{k}}\right]}$$

$$= A_2\sqrt{\frac{2k}{k-1}p_1\rho_1\left[n^{\frac{2}{k}} - n^{\frac{k+1}{k}}\right]} \qquad ...(15.26)$$

The mass rate of flow (m) depends on the value of n for the given values of p_1 and ρ_1 at 1.

15.9.1 Value of n or $\frac{p_2}{p_1}$ for Maximum Value of Mass Rate of Flow. For maximum value of m, we have

$$\frac{\partial m}{\partial n} = 0$$

or
$$\frac{\partial}{\partial n}\left[n^{\frac{2}{k}} - n^{\frac{k+1}{k}}\right] = 0 \qquad \left(\because \frac{2k}{k-1} p_1\rho_1 = \text{Constant}\right)$$

or
$$\frac{2}{k} n^{\frac{2}{k}-1} - \frac{k+1}{k} n^{\frac{k+1}{k}-1} = 0 \quad \text{or} \quad \frac{2}{k} n^{\frac{2-k}{k}} = \frac{k+1}{k} n^{\frac{k+1-k}{k}} = \frac{k+1}{k} n^{\frac{1}{k}}$$

or
$$n^{\frac{2-k}{k}} = \left(\frac{k+1}{k}\right) \times \frac{k}{2} \times n^{\frac{1}{k}} = \frac{k+1}{2} n^{\frac{1}{k}} \quad \text{or} \quad \frac{n^{\frac{2-k}{k}}}{n^{\frac{1}{k}}} = \frac{k+1}{2}$$

$\therefore$
$$n^{\frac{2-k}{k}-\frac{1}{k}} = \frac{k+1}{2} \quad \text{or} \quad n^{\frac{1-k}{k}} = \frac{k+1}{2}$$

$\therefore$
$$n^{\frac{-(k-1)}{k}} = \frac{k+1}{2} \quad \text{or} \quad \frac{1}{n^{\frac{k-1}{k}}} = \frac{k+1}{2} = \frac{1}{\left(\frac{2}{k+1}\right)}$$

$\therefore$
$$n^{\frac{k-1}{k}} = \frac{2}{k+1} \qquad \text{...(15.27)}$$

Equation (15.27) is the condition for maximum value of mass rate of flow through the nozzle.

For $k = 1.4$, the value of n can be obtained from equation (15.27) as

$$n^{\frac{1.4-1.0}{1.4}} = \frac{2}{1.4+1} = \frac{2}{2.4} \quad \text{or} \quad n^{2/7} = \frac{2}{2.4}$$

$\therefore$
$$n = \left(\frac{2}{2.4}\right)^{7/2} = 0.528 \quad \text{or} \quad \frac{p_2}{p_1} = n = 0.528. \qquad \text{...(15.28)}$$

15.9.2 Value of V_2 for Maximum Rate of Flow of Fluid. From equation (15.27), the value of n is given as

$$n = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$$

Substituting this value of n in equation (*iii*), we get

$$V_2 = \sqrt{\frac{2k}{k-1}\frac{p_1}{\rho_1}\left(1 - n^{\frac{k-1}{k}}\right)}$$

$$= \sqrt{\frac{2k}{k-1}\frac{p_1}{\rho_1}\left[1-\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}\times\frac{(k-1)}{k}}\right]}$$

$$= \sqrt{\frac{2k}{k-1}\frac{p_1}{\rho_1}\left(1-\frac{2}{k+1}\right)} = \sqrt{\frac{2k}{k-1}\frac{p_1}{\rho_1}\left[\frac{k+1-2}{k+1}\right]}$$

$$= \sqrt{\frac{2k}{k-1}\frac{p_1}{\rho_1}\left[\frac{k-1}{k+1}\right]} = \sqrt{\frac{2k}{k+1}\frac{p_1}{\rho_1}} \qquad \text{...(15.29)}$$

15.9.3 Maximum Rate of Flow of Fluid Through Nozzle. Mass rate of flow of fluid through nozzle is given by equation (15.26) as

$$m = A_2\sqrt{\frac{2k}{(k-1)}p_1\rho_1\left[n^{2/k} - n^{\frac{k+1}{k}}\right]}$$

From maximum rate of flow, from equation (15.27), we have

$$n^{\frac{k-1}{k}} = \frac{2}{k+1}$$

$$\therefore \qquad n = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$$

Substituting the value of n in the above equation, we have

$$m_{max} = A_2\sqrt{\left(\frac{2k}{k-1}\right)p_1\rho_1\left[\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}\times\frac{2}{k}} - \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}\times\frac{k+1}{k}}\right]}$$

$$= A_2\sqrt{\frac{2k}{k-1}p_1\rho_1\left[\left(\frac{2}{k+1}\right)^{\frac{2}{k-1}} - \left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\right]}$$

For air, $k = 1.4$,

$$\therefore \qquad m_{max} = A_2\sqrt{\frac{2\times1.4}{1.4-1.0}p_1\rho_1\left[\left(\frac{2}{1.4+1}\right)^{\frac{2}{1.4-1.0}} - \left(\frac{2}{1.4+1.0}\right)^{\frac{1.4+1.0}{1.4-1.0}}\right]}$$

$$= A_2\sqrt{\frac{2.8}{0.4}p_1\rho_1\left[\left(\frac{2}{2.4}\right)^{2/.4} - \left(\frac{2}{2.4}\right)^{2.4/.4}\right]}$$

$$= A_2\sqrt{7\times p_1\rho_1[.4018-.3348]} = A_2\times0.685\times\sqrt{p_1\rho_1}$$

$$= 0.685\,A_2\sqrt{p_1\rho_1}\,. \qquad \text{...(15.30)}$$

15.9.4 Variation of Mass Rate of Flow of Compressible Fluid with Pressure Ratio $\left(\frac{p_2}{p_1}\right)$.

Fig. 15.6 shows the variation of mass rate of flow of compressible fluid with different pressure ratio $\left(\frac{p_2}{p_1}\right)$. It is seen that when $\frac{p_2}{p_1}$ is less than critical pressure ratio of 0.528, the mass rate of flow is constant and is equal to the mass rate of flow corresponding to pressure ratio = 0.528. But if the pressure ratio is more than 0.528, mass rate of flow decreases as shown by dotted line.

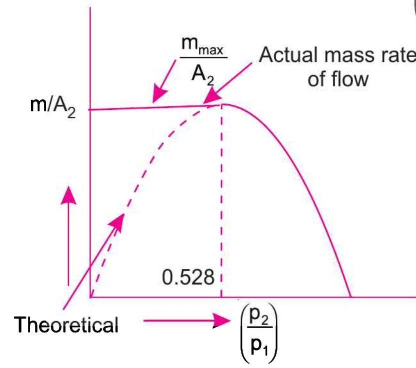


Fig. 15.6

15.9.5 Velocity at Outlet of Nozzle for Maximum Rate of Flow is Equal to Sonic Velocity.

This is proved as given below.

The velocity at the outlet of nozzle for maximum rate of flow is given by equation (15.29) as

$$V_2 = \sqrt{\left(\frac{2k}{k+1}\right)\frac{p_1}{\rho_1}} \qquad \text{...}(i)$$

Now pressure ratio $\qquad \frac{p_2}{p_1} = n \quad \therefore \quad p_1 = \frac{p_2}{n}$

Also for adiabatic flow, $\qquad \frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k}$ or $\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^k$

or $$\frac{\rho_1}{\rho_2} = \left(\frac{p_1}{p_2}\right)^{1/k} = \left(\frac{p_2}{p_1}\right)^{-1/k} = n^{-1/k} \qquad \left(\because \frac{p_2}{p_1} = n\right)$$

$\therefore$ $$\rho_1 = \rho_2 n^{-1/k}$$

Substituting the value of p_1 and ρ_1 in equation (*i*), we get

$$V_2 = \sqrt{\left(\frac{2k}{k+1}\right)\frac{\frac{p_2}{n}}{\rho_2 n^{-1/k}}} = \sqrt{\frac{2k}{k+1} \times \frac{p_2}{\rho_2} \times \frac{1}{n} \times \frac{1}{n^{-1/k}}}$$

$$= \sqrt{\frac{2k}{k+1} \times \frac{p_2}{\rho_2} \times \frac{1}{n^{(1-1/k)}}} = \sqrt{\frac{2k}{k+1} \times \frac{p_2}{\rho_2} \times \frac{1}{n^{(k-1)/k}}}$$

But $n^{\frac{k-1}{k}} = \frac{2}{k+1}$ from equation (15.27)

$\therefore$ $$V_2 = \sqrt{\frac{2k}{k+1} \times \frac{p_2}{\rho_2} \times \frac{(k+1)}{2}} = \sqrt{\frac{kp_2}{\rho_2}}$$

$$= C_2. \qquad \left(\because \sqrt{\frac{kp_2}{\rho_2}} = C_2\right) \qquad \text{...(15.31)}$$

Problem 15.13 *Find the velocity of air flowing at the outlet of a nozzle, fitted to a large vessel which contains air at a pressure of 294.3 N/cm² (abs.) and at a temperature of 20°C. The pressure at the outlet of the nozzle is 206 N/cm² (abs). Take k = 1.4 and K = 287 J/kg°K.*

Solution. Given :

Pressure inside vessel, $p_1 = 294.3 \text{ N/cm}^2 = 294.3 \times 10^4 \text{ N/m}^2$

Temperature inside vessel, $t_1 = 20°C$

$\therefore$ $T_1 = 20 + 273 = 293°K$

Pressure at the nozzle, $p_2 = 206 \text{ N/cm}^2 = 206 \times 10^4 \text{ N/m}^2$

$R = 287 \text{ J/kg°K}$

$k = 1.4$

Using equation (15.25) for the velocity,

$$V_2 = \sqrt{\left(\frac{2k}{k-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]}$$

$$= \sqrt{\left(\frac{2\times 1.4}{1.4\times 1.0}\right)\frac{294.3\times 10^4}{\rho_1}\left[1-\left(\frac{206\times 10^4}{294.3\times 10^4}\right)^{\frac{1.4-1.0}{1.4}}\right]}$$

$$= \sqrt{\frac{2.8}{0.4}\times\frac{294.3\times 10^4}{\rho_1}\left[1-0.7^{0.4/1.4}\right]} = \sqrt{\frac{7\times 294.3\times 10^4}{\rho_1}[1-.903]} \quad ...(i)$$

The value of ρ_1 is calculated from equation of state as

$$\frac{p_1}{\rho_1} = RT_1 \qquad \therefore \quad \rho_1 = \frac{p_1}{RT_1}$$

In this equation if R is taken in J/kg K, p_1 should be in N/m². Then ρ_1 will be in kg/m³.

$\therefore$ $p_1 = 294.3 \times 10^4 \text{ N/m}^2$

$$\therefore \quad \rho_1 = \frac{294.3\times 10^4}{287\times 293} = 34.99 \text{ kg/m}^3.$$

Substituting the value of $\rho_1 = 34.99$ kg/m³ in equation (*i*),

$$V_2 = \sqrt{7\times\frac{294.3\times 10^4}{34.99}[1-.903]} = \mathbf{239.2 \text{ m/s. Ans.}}$$

Problem 15.14 *A tank contains air at a temperature of 30°C. Air flows from the tank into atmosphere through a convergent nozzle. The diameter at the outlet of the nozzle is 25 mm. Assuming adiabatic flow, find the mass rate of flow of air through the nozzle when the pressure of air in tank is (i) 3.924 N/cm² (gauge), (ii) 33.354 N/cm² (gauge). Take k = 1.4, R = 287 J/kg°K and atmospheric pressure = 10.104 N/cm² (abs).*

Solution. Given :

Temperature in tank, $t_1 = 30°C$

$\therefore$ $T_1 = 30 + 273 = 303°K$

Diameter at the nozzle, $D_2 = 25$ mm $= 0.025$ m.

$\therefore$ Area, $A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} (.025)^2 = 0.0004908 \text{ m}^2$

$R = 287$ J/kg°K

$k = 1.4$.

(*i*) Mass rate of flow of air when pressure in tank is 3.924 N/cm^2 (gauge).

$p_1 = 3.924$ N/cm^2 (gauge)

$= 3.924 + 10.104 = 14.028$ N/cm^2 (abs.)

$= 14.028 \times 10^4$ N/cm^2 (abs.)

Pressure at the nozzle, p_2 = Atmospheric pressure

$= 10.104$ N/cm^2 $= 10.104 \times 10^4$ N/m^2

$\therefore$ Pressure ratio, $n = \frac{p_2}{p_1} = \frac{10.104 \times 10^4}{14.028 \times 10^4} = 0.7203.$

This pressure ratio is more than the pressure ratio 0.528 given by equation (15.28), hence mass rate of flow of air is given by equation (15.26), as

$$m = A_2 \sqrt{\frac{2k}{(k-1)} p_1 \rho_1 \left[n^{\frac{2}{k}} - n^{\frac{k+1}{k}} \right]} \quad ...(i)$$

In this equation if p_1 is taken into N/m^2, then ρ_1 will be in kg/m^3. The value of ρ_1 is obtained from equation of state as

$$\frac{p_1}{\rho_1} = RT_1 \text{ or } \rho_1 = \frac{p_1}{RT_1}, \quad \text{where } p_1 = 14.028 \text{ N/m}^2$$

$$\therefore \quad \rho_1 = \frac{14.028 \times 10^4}{287 \times 303} = 1.613 \text{ kg/m}^3.$$

Substituting this value of $\rho_1 = 1.613$ kg/m^3 and $p_1 = 14.028 \times 10^4$ N/m^2 in equation (*i*), we get

$$m = .0004908 \sqrt{\frac{2 \times 1.4}{1.4 - 1.0} \times 14.028 \times 10^4 \times 1.613 \left[.7203^{\frac{2}{1.4}} - .7203^{\frac{1.4+1.0}{1.4}} \right]}$$

$$= .0004908 \sqrt{1583935 \left[.7203^{1.4285} - .7203^{1.7142} \right]}$$

$$= .0004908 \sqrt{1583935 [.6258 - .5698]} = \mathbf{0.146 \text{ kg/s. Ans.}}$$

(*ii*) Mass rate of flow of air when pressure in the tank is 33.354 N/cm^2 (gauge).

$\therefore$ $p_1 = 33.354 + 10.104 = 43.458$ N/cm^2 (abs.) $= 43.458 \times 10^4$ N/m^2 (abs).

$$p_2 = \text{Atmospheric pressure} = 10.104 \times 10^4 \text{ N/m}^2$$

∴ Pressure ratio, $$n = \frac{p_2}{p_1} = \frac{10.104 \times 10^4}{43.458 \times 10^4} = 0.2325.$$

This pressure ratio is less than the pressure ratio of 0.528. Hence as mentioned in Art. 15.9.4, the mass rate of flow will be corresponding to the pressure ratio of 0.528. Hence

$$m = A_2 \sqrt{\frac{2k}{k-1} p_1 \rho_1 \left[n^{\frac{2}{k}} - n^{\frac{(k+1)}{k}} \right]}$$

where $n = 0.528$, $p_1 = 43.458 \times 10^4$ N/m^2.

The value of $$\rho_1 = \frac{p_1}{RT_1} = \frac{43.458 \times 10^4}{287 \times 303} = 4.99 \text{ kg/m}^3$$

∴ $$m = .0004908 \sqrt{\frac{2 \times 1.4}{(1.4 - 1.0)} \times 43.458 \times 10^4 \times 4.99 \left[.528^{\frac{2}{1.4}} - .528^{\frac{1.4+1.0}{1.4}} \right]}$$

$$= .004908 \sqrt{4906875 [.4015 - .3346]} = \mathbf{0.494 \text{ kg/s. Ans.}}$$

Problem 15.15 *A large tank contains air at 28.449 N/cm² gauge pressure and 24°C temperature. The air flows from the tank to the atmosphere through an orifice. If the diameter of the orifice is 20 mm, find the maximum rate of flow of air. Tank R = 287 J/kg°K, k = 1.4, atmospheric pressure = 10.104 N/cm².*

Solution. Given :

Pressure in tank, $p_1 = 28.449$ N/cm^2 (gauge)

$= 28.449 + 10.104 = 38.553$ N/cm^2 (abs.) $= 38.553 \times 10^4$ N/m^2

Temperature in tank, $t_1 = 24°\text{C}$

∴ $T_1 = 273 + 24 = 297°\text{K}$

$R = 287$ J/kg K

$k = 1.4$

Diameter of orifice, $D = 20$ mm $= 0.02$ m

∴ Area, $$A = \frac{\pi}{4} (.02)^2 = .0003141 \text{ m}^2$$

Using equation of state, we get $\frac{p_1}{\rho_1} = RT_1$ or $\rho_1 = \frac{p_1}{RT_1} = \frac{38.553 \times 10^4}{287 \times 297} = 4.522$ kg/m^3

Maximum rate of flow of air is given by equation (15.30) as

$$m_{max} = 0.685 \, A_2 \sqrt{p_1 \rho_1} \qquad \text{(Here } A_2 = A = 0.0003141\text{)}$$

$$= 0.685 \times .0003141 \sqrt{38.553 \times 10^4 \times 4.522} = \mathbf{0.284 \text{ kg/s. Ans.}}$$

▶ 15.10 MASS RATE OF FLOW OF COMPRESSIBLE FLUID THROUGH VENTURIMETER

Consider a compressible fluid flowing through the horizontal venturimeter. Let the conditions of flow is represented by suffix 1 at the inlet of venturimeter and by suffix 2 at the throat of the venturimeter. Considering the flow as adiabatic, we have from Bernoulli's equation at sections 1 and 2 from equation (15.13), as

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g}+\frac{V_1^2}{2g}=\left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g}+\frac{V_2^2}{2g} \qquad (\because \; z_1 = z_2)$$

or
$$\frac{k}{(k-1)}\frac{p_1}{\rho_1}+\frac{V_1^2}{2}=\left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2}+\frac{V_2^2}{2} \qquad \left(\text{Cancelling } \frac{1}{g}\right)$$

or
$$\frac{k}{(k-1)}\left[\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2}$$

$$\frac{k}{(k-1)}\frac{p_1}{\rho_1}\left[1-\frac{p_2}{\rho_2}\times\frac{\rho_1}{p_1}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2} \qquad \ldots(i)$$

For adiabatic flow,
$$\frac{p_1}{\rho_1^k}=\frac{p_2}{\rho_2^k} \quad \therefore \quad \frac{p_1}{p_2}=\left(\frac{\rho_1}{\rho_2}\right)^k$$

or
$$\frac{\rho_1}{\rho_2}=\left(\frac{p_1}{p_2}\right)^{1/k}=\left(\frac{p_2}{p_1}\right)^{-1/k} \qquad \ldots(ii)$$

Substituting this value of $\frac{\rho_1}{\rho_2}$ in equation (*i*), we get

$$\frac{k}{k-1}\frac{p_1}{\rho_1}\left[1-\frac{p_2}{p_1}\times\left(\frac{p_2}{p_1}\right)^{-1/k}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2}$$

or
$$\frac{k}{k-1}\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{1-1/k}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2}$$

or
$$\frac{k}{k-1}\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{k-1/k}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2} \qquad \ldots(iii)$$

Applying continuity for sections 1 and 2, we have

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad \therefore \quad V_1 = \frac{\rho_2 A_2 V_2}{\rho_1 A_1}$$

Substituting the value of V_1 in equation (*iii*), we get

$$\frac{k}{(k-1)}\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{k-1/k}\right]=\frac{V_2^2}{2}-\left(\frac{\rho_2 A_2 V_2}{\rho_1 A_1}\right)^2\times\frac{1}{2}=\frac{V_2^2}{2}\left[1-\frac{\rho_2^2 A_2^2}{\rho_1^2 A_1^2}\right] \quad ...(iv)$$

But from equation (*ii*), we have

$$\frac{\rho_1}{\rho_2}=\left(\frac{p_1}{p_2}\right)^{1/k} \quad \text{or} \quad \frac{\rho_2}{\rho_1}=\left(\frac{p_2}{p_1}\right)^{1/k} \quad \therefore \left(\frac{\rho_2}{\rho_1}\right)^2=\left(\frac{p_2}{p_1}\right)^{2/k}$$

Substituting this value in equation (*iv*), we get

$$\frac{k}{(k-1)}\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{k-1/k}\right]=\frac{V_2^2}{2}\left[1-\left(\frac{p_2}{p_1}\right)^{2/k}\times\frac{A_2^2}{A_1^2}\right]$$

$$V_2^2=\frac{\frac{2k}{(k-1)}\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{k-1/k}\right]}{\left[1-\left(\frac{p_2}{p_1}\right)^{2/k}\times\frac{A_2^2}{A_1^2}\right]}$$

$$\therefore \quad V_2=\sqrt{\frac{\frac{2k}{k-1}\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]}{1-\left(\frac{p_2}{p_1}\right)^{2/k}\times\frac{A_2^2}{A_1^2}}}$$

$\therefore$ Mass rate of flow through venturimeter,

$$m=\rho_2 A_2 V_2=\rho_2 A_2\sqrt{\frac{\frac{2k}{k-1}\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{k-1/k}\right]}{\left[1-\left(\frac{A_2^2}{A_1^2}\right)\left(\frac{p_2}{p_1}\right)^{2/k}\right]}} \quad ...(15.32)$$

The only condition for equation (15.32) is that the pressure ratio $\left(\frac{p_2}{p_1}\right)$ should be more than the pressure ratio 0.528

Problem 15.16 *Find the mass rate of flow of air though a venturimeter having inlet diameter as 300 mm and throat diameter 150 mm. The pressure at the inlet of venturimeter is 1.4 kgf/cm^2 (1.4 $\times$ 9.81 N/cm^2) absolute and temperature of air at inlet is 15°C. The pressure at the throat is given as 1.3 kgf/cm^2 (1.3 $\times$ 9.81 N/cm^2) absolute. Take R = 287 J/kg°K and k = 1.4.*

Solution. Given :

Diameter at inlet, D_1 = 300 mm = 0.30 m

$\therefore$ Area, $A_1=\frac{\pi}{4}D_1^2=\frac{\pi}{4}(.30)^2=.07068\text{ m}^2$

Diameter at throat, $D_2 = 150 \text{ mm} = 0.15 \text{ m}$

$\therefore$ Area, $A_2 = \frac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$

Pressure of air at inlet, $p_1 = 1.4 \text{ kgf/cm}^2 = 1.4 \times 10^4 \text{ kgf/m}^2 = 1.4 \times 10^4 \times 9.81 \text{ N/m}^2$

Throat pressure, $p_2 = 1.3 \text{ kgf/cm}^2 = 1.3 \times 10^4 \times 9.81 \text{ N/m}^2$

$R = 287 \text{ J/kg°K}$

$k = 1.4$

Temperature at inlet, $t_1 = 15°\text{C}$

The pressure ratio, $\frac{p_2}{p_1} = \frac{1.3 \times 10^4 \times 9.81}{1.4 \times 10^4 \times 9.81} = 0.9285$

Density of gas at inlet is obtained from equation of state,

$$\frac{p_1}{\rho_1} = RT_1 \text{ or } \rho_1 = \frac{p_1}{RT_1} = \frac{1.4 \times 10^4 \times 9.81}{287 \times (273 + 15)}$$

where $T_1 = t_1 + 273 = 15 + 273 = 288°\text{K}$

$$\therefore \qquad \rho_1 = \frac{1.4 \times 10^4 \times 9.81}{287 \times 288} = 1.66 \text{ kg/cm}^3$$

For adiabatic process, we have $\frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k}$ or $\left(\frac{\rho_2}{\rho_1}\right)^k = \frac{p_2}{p_1}$ or $\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{1/k}$

$$\therefore \qquad \rho_2 = \rho_1 \left(\frac{p_2}{p_1}\right)^{1/k} = 1.66 \times (.9285)^{1/1.4} \qquad \left(\because \frac{p_2}{p_1} = .9285\right)$$

$$= 1.574 \text{ kg/m}^3.$$

Using equation (15.32) for the mass rate of flow through venturimeter, we get

$$m = \rho_2 A_2 \sqrt{\frac{\frac{2k}{k-1}\frac{p_1}{\rho_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{k-1/k}\right]}{\left[1 - \left(\frac{A_2^2}{A_1^2}\right)\left(\frac{p_2}{p_1}\right)^{2/k}\right]}}$$

$$= 1.574 \times .01767 \sqrt{\frac{\frac{2 \times 1.4}{1.4 - 1} \times \frac{1.4 \times 10^4 \times 9.81}{1.66}\left[1 - .9285^{\frac{1.4-1.0}{1.4}}\right]}{\left[1 - \left(\frac{.01767^2}{.07068^2}\right)\left(.9285^{2/1.4}\right)\right]}}$$

$$= .0278 \sqrt{\frac{579144\left[1 - .9285^{.2857}\right]}{1 - .0625 \times .899}} = .0278 \times \sqrt{\frac{12145.57}{.9438}}$$

= 315 kg/s. Ans.

▶ 15.11 PITOT-STATIC TUBE IN A COMPRESSIBLE FLOW

The pitot-static tube, when used for determining the velocity at any point in a compressible fluid, gives only the difference between the stagnation head and static head. From this difference, the velocity of the incompressible fluid at that point is obtained from the relation

$$V = \sqrt{2gh}\text{, where } h = \text{Difference in two heads.}$$

But when the pitot-static tube is used for finding velocity at any point in a compressible fluid, the actual pressure difference shown by the gauges of the pitot-tube should be multiplied by a factor, for obtaining correct velocity at that point. The value of the factor depends upon the Mach number of the flow. Let us find an expression for the correction factor for *sub-sonic* flow.

At a point in pitot-static tube, the pressure becomes stagnation pressure, denoted by p_s. The expression for stagnation pressure, p_s is given by equation (15.21), as

$$p_s = p_1\left[1 + \frac{k-1}{2}M_1^2\right]^{\frac{k}{k-1}} \qquad ...(i)$$

where p_1 = Pressure of fluid far away from stagnation point,
M_1 = Mach number at point 1 far away from stagnation point.

For $M < 1$, the term $\frac{k-1}{2}M_1^2$ will be less than 1 and hence the right-hand side of equation (*i*) can be expanded by Binomial theorem* as

$$p_s = \left[1 + \frac{k-1}{2}M_1^2 \times \frac{k}{k-1} \times \frac{\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1}-1\right)}{2!} \times \left(\frac{k-1}{2}M_1^2\right)^2 + \frac{\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1}-1\right)\left(\frac{k}{k-1}-2\right)\left(\frac{k-1}{2}M_1^2\right)^3}{3!} + \dots\right]$$

$$= p_1\left[1 + \frac{k}{2}M_1^2 + \frac{k}{8}M_1^4 + \frac{k(2-k)}{48}M_1^6 + \dots\right]$$

$$= p_1 + p_1\left[\frac{k}{2}M_1^2 + \frac{k}{8}M_1^4 + \frac{k(2-k)}{48}M_1^6 + \dots\right]$$

$$p_s - p_1 = p_1 \times \frac{k}{2}M_1^2\left[1 + \frac{M_1^2}{4} + \frac{(2-k)}{24}M_1^4 + \dots\right] \qquad ...(ii)$$

But $$M_1^2 = \frac{V_1^2}{C_1^2}, \text{ where } C_1^2 = \frac{kp_1}{\rho_1}$$

---

* Binomial Theorem $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \frac{n(n-1)(n-2)(n-3)x^4}{4!}$ .....

$$= \frac{V_1^2}{\frac{kp_1}{\rho_1}} = \frac{V_1^2 \rho_1}{kp_1}$$

Substituting the value of M_1^2 in equation (*ii*), we get

$$p_s - p_1 = p_1 \times \frac{k}{2} \times \frac{V_1^2 \rho_1}{kp_1} \left[1 + \frac{M_1^2}{4} + \frac{(2-k)}{24} M_1^4 + \ldots\right]$$

$$= \frac{\rho_1 V_1^2}{2} \left[1 + \frac{M_1^2}{4} + \frac{(2-k)}{24} M_1^4 + \ldots\right]$$

The term $\left[1 + \frac{M_1^2}{4} + \frac{(2-k)}{24} M_1^4 + \ldots\right]$ is known as **Compressibility Correction Factor.** And $\frac{\rho_1 V_1^2}{2}$ is the reading of the pitot-static tube. Thus the reading of the pitot- tube must be multiplied by a correction factor given below for correct value of velocity measured by the pitot-tube.

Compressibility Correction Factor, C.C.F. = $\left[1 + \frac{M_1^2}{4} + \frac{2-k}{24} M_1^4 + \ldots\right]$ ...(15.33)

Problem 15.17 *Calculate the numerical factor by which the actual pressure difference shown by the gauge of a pitot-static tube must be multiplied to allow for compressibility when the value of the Mach number is 0.9. Take k = 1.4.*

Solution. Given :

Mach number, $M_1 = 0.9$

$k = 1.4$

Using equation (15.33), Compressibility Correction Factor is

$$\text{C.C.F.} = \left[1 + \frac{M_1^2}{4} + \frac{2-k}{24} M_1^4 + \ldots\right] = 1 + \frac{.9^2}{4} + \frac{2-1.4}{24} (.9)^4 + \ldots$$

$= 1.0 + .2025 + .0164 + \ldots.. =$ **1.2189. Ans.**

∴ Numerical factor by which the actual pressure difference is to be multiplied = 1.2189.

HIGHLIGHTS

1. A flow in which density does not remain constant during flow, is called compressible flow.
2. Equation of state is given by, $\frac{p}{\rho} = RT$

   where p = Absolute pressure in kgf/m^2 or N/m^2
   T = Absolute temperature = 273 + t°C
   R = Gas constant in J/kg K or m/kg
   ρ = Density of gas.

   If the value of p is taken in N/m^2, R is J/kg °K and T in °K, the value of density is given in kg/m^3

3. The pressure density of a gas are related as

$$\frac{p}{\rho} = \text{Constant.... for isothermal process}$$

$$\frac{p}{\rho^k} = \text{Constant....for adiabatic process}$$

where k = Ratio of specific heats = 1.4 for air.

4. Continuity equation for compressible flow is given as ρAV = Constant.

And in differential form, continuity equation is $\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0.$

5. Bernoulli's equation for compressible fluids is given as

$$\frac{p}{\rho g}\log_e p + \frac{V^2}{2g} + Z = \text{Constant......for isothermal process.}$$

$$\frac{k}{(k-1)}\frac{p}{\rho g} + \frac{V^2}{2g} + Z = \text{Constant......for adiabatic process.}$$

6. Velocity of sound wave is given by

$$C = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{K}{\rho}} \qquad \text{......in term of Bulk modulus}$$

$$C = \sqrt{\frac{p}{\rho}} = \sqrt{RT} \qquad \text{......for isothermal process}$$

$$= \sqrt{\frac{kp}{\rho}} = \sqrt{kRT} \qquad \text{......for adiabatic process.}$$

7. Mach number, M is given as $M = \frac{V}{C}$

If $M < 1$ ......flow is sub-sonic flow,

$M > 1$ ......flow is super-sonic flow,

$M = 1$ ......flow is sonic flow.

8. In sub-sonic flow, the disturbance always moves ahead of the projectile. In sonic flow, the disturbance moves along the projectile while in super-sonic flow, the projectile always moves ahead of the disturbance .

9. Mach angle is given by $\sin\alpha = \frac{C}{V} = \frac{1}{M}$.

10. The pressure, temperature and density at a point where velocity is zero are called stagnation pressure, temperature and stagnation density. Their values are given as

$$p_s = p_1\left[1 + \frac{k-1}{2}M_1^2\right]^{\frac{k}{k-1}} ; \rho_s = \rho_1\left[1 + \frac{k-1}{2}M_1^2\right]^{\frac{1}{k-1}}$$

$$T_s = T_1\left[1 + \frac{k-1}{2}M_1^2\right] \text{ also } = \frac{1}{R}\frac{p_s}{\rho_s}.$$

11. Area velocity relationship for compressible fluid is given as $\frac{dA}{A} = \frac{dV}{V}\,[M^2 - 1]$

If $M < 1, \frac{dV}{V} < 0$ and $\frac{dA}{A} > 0$ which means with the increase of area, velocity decreases.

If $M > 1, \frac{dV}{V} > 0$ and $\frac{dA}{A} > 0$ which means with the increase of area, velocity also increases.

If $M = 1$, $\frac{dA}{A} = 0$ means area is constant.

12. Velocity through a nozzle or orifice fitted to a large tank is

$$V_2 = \sqrt{\frac{2k}{k-1}\frac{p_1}{\rho_1}\left[1 - \left(\frac{p_2}{\rho_1}\right)^{\frac{k-1}{k}}\right]}$$

where p_2 = Pressure at the outlet of nozzle or orifice, p_1 = Pressure in the tank.

13. The mass rate of flow of compressible fluid though orifice or nozzle fitted to the tank is

$$m = A_2\sqrt{\frac{2k}{k-1}\,p_1\rho_1\left[n^{2/k} - n^{\frac{k+1}{k}}\right]}$$

where n = Pressure ratio = $\frac{p_2}{p_1}$.

14. For maximum flow through orifice or nozzle fitted to the tank, pressure ratio = $\frac{p_2}{p_1} = n = 0.528$

Also $$n^{\frac{k-1}{k}} = \frac{2}{k+1}.$$

And velocity at the outlet of the orifice or nozzle is $V_2 = \sqrt{\frac{2k}{k+1}\frac{p_1}{\rho_1}} = C_2$.

And mass rate of flow of fluid is given by $m_{max} = 0.685\,A_2\sqrt{p_1\rho_1}$.

15. If the pressure ratio for the nozzle or orifice fitted to the large tank is less than 0.528, the mass rate of flow of the fluid is always corresponding to the pressure ratio of 0.528. But if the pressure ratio is more than 0.528, the mass rate of flow of fluid is corresponding to the given pressure ratio.

16. Mass rate of flow of compressible fluid through venturimeter is given by

$$m = \rho_2 A_2 \sqrt{\frac{\frac{2k}{k-1}\frac{p_1}{\rho_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{k-1/k}\right]}{\left[1 - \left(\frac{A_2}{A_1}\right)^2\left(\frac{p_2}{p_1}\right)^{2/k}\right]}}$$

where A_2 = Area at the throat, A_1 = Area at inlet.

17. The compressibility correction factor is given by C.C.F = $\left[1 + \frac{M_1^2}{4} + \frac{2-k}{24}M_1^4 + \ldots\ldots\right]$

EXERCISE

(A) THEORETICAL PROBLEMS

1. Define compressible and incompressible flow.
2. What is the relation between pressure and density of a compressible fluid for (*a*) isothermal process and (*b*) adiabatic process ?
3. Derive the continuity equation for one-dimensional compressible flow in differential form.
4. State the Bernoulli's theorem for compressible flow. Derive an expression for Bernoulli's equation when the process is (*i*) isothermal and (*ii*) adiabatic.
5. Write an expression for momentum equation for compressible fluid. (*a*) isothermal process and (*b*) adiabatic process ?
6. (*a*) Obtain an expression for velocity of the sound wave in a compressible fluid in terms of change of pressure and change of density.
   (*b*) Show that the velocity of propagation of the pressure wave in a compressible fluid is given by $C = \sqrt{E/\rho}$, where E is volume modulus of elasticity of fluid.
7. Prove that the velocity of sound wave in a compressible fluid is given by $C = \sqrt{\dfrac{K}{\rho}}$
   where K = Bulk modulus of fluid, ρ = Density of fluid.
8. Derive an expression for the velocity of sound wave for compressible fluid when the process is assumed as (*i*) isothermal and(*ii*) adiabatic.
9. Define Mach number. What is the significance of Mach number in compressible fluid flows ?
10. Define the terms: Sub-sonic flow, super-sonic flow, sonic flow, Mach angle and Mach cone.
11. Show by means of diagrams the nature of propagation of disturbance in compressible flow when Mach number is less than one, is equal to one and is more than one. *(Delhi University, Dec. 2002)*
12. What do you understand by stagnation pressure? Obtain an expression for stagnation pressure of a compressible fluid in terms of approaching Mach number and pressure.
13. Prove that stagnation temperature and stagnation density are given as
$$T_s = T_1\left[1 + \frac{k-1}{2} M_1^2\right] \text{ and } \rho_s = \rho_1\left[1 + \frac{k-1}{2} M_1^2\right]^{\frac{1}{k-1}}$$
14. Derive an expression for area velocity relationship for a compressible fluid in the form.
$$\frac{dA}{A} = \frac{dV}{V}\,[M^2 - 1]$$ *(Delhi University, Dec. 2002)*
15. Find an expression for mass rate of flow of compressible fluid through an orifice or nozzle fitted to a large tank. What is condition for maximum rate of flow?
16. What do you mean by compressibility correction factor ? Find an expression for compressibility factor.
17. Derive an expression for velocity of sound for an adiabatic process.
18. What do you mean by sub-sonic, sonic and super-sonic flows?
19. For frictionless adiabatic flow(*i.e.*, isentropic flow) show that the stagnation pressure at a given point is given by
$$\frac{p_s}{p_1} = 1 + \frac{1}{4} M_0^2 + \frac{(2-k)}{24} M_0^4 + \ldots$$
   where p_s = stagnation pressure, p_0 = pressure in ambient flow and $M_0 = U_0/\sqrt{E/\rho}$.

20. Differentiate between isentropic and adiabatic processes.
21. (*a*) Derive an expression for the velocity of sound waves moving in a compressible fluid.
(*b*) Define and explain the terms :
Mach number, Froude number, Reynolds number, Mach cone and Mach angle
22. State the Bernoulli's theorem for compressible flow. Derive an expression for Bernoulli's equation when the process is adiabatic. *(R.G.P.V., Summer, 2002)*

(B) NUMERICAL PROBLEMS

1. A gas is flowing through a horizontal pipe of cross-sectional area of 30 cm^2. At a point the pressure is 30 N per cm^2 (gauge) and temperature 20°C. At another section the area of cross-section is 15 cm^2 and pressure is 25 N/cm^2 gauge. If the mass rate of flow of gas is 0.15 kg/s, find the velocities of the gas at these two sections, assuming an isothermal change. Take R = 287 J/kg K, and atmospheric pressure 10 N/cm^2. [**Ans.** V_1 = 10.71 m/s ; V_2 = 24.5 m/s]
2. A gas with a velocity of 350 m/s is flowing through a horizontal pipe at a section where pressure is 8 N/cm^2 (absolute) and temperature is 30°C. The pipe changes in diameter and at this section the pressure is 12 N/cm^2 (absolute). Find the velocity of the gas at this section if the flow of the gas is adiabatic. Take R = 287 J/kg K and k = 1.4. [**Ans.** 218.63 m/s]
3. Find the speed of the sound wave in air at sea-level where the pressure and temperature are 9.81 N/cm^2 (abs.) and 20°C respectively. Take R = 287 J/kg K and k = 1.4. [**Ans.** 343.11 m/s]
4. Calculate the Mach number at a point on a jet propelled aircraft which is flying at 900 km/hour at sea-level where air temperature is 15°C. Take k = 1.4 and R = 287 J/kg K. [**Ans.** 0.735]
5. An aeroplane is flying at an height of 20 km, where the temperature is – 40°C. The speed of the plane is corresponding to M = 1.8. Assuming k = 1.4 and R = 287 J/kg K, find the speed of the plane. [**Ans.** 1982.66 m/hr]
6. A projectile is travelling in air having pressure and temperature as 8.829 N/cm^2 and – 5°C. If the Mach angle is 30°, find the velocity of the projectile. Take k = 1.4 and R = 287 J/kg K. [**Ans.** 656.30 m/s]
7. A projectile travels in air of pressure 8.829 N/cm^2 at –10°C at a speed of 1200 km/hour. Find the Mach number and the Mach angle. Take k = 1.4 and R = 287 J/kg K. [**Ans.** 1.025, θ = 77.2°]
8. Find the Mach number when an aeroplane is flying at 900 km/hour through still air having a pressure of 8.0 N/cm^2 and temperature –15°C. Take k = 1.4 and R = 287 J/kg K. Calculate the pressure, temperature and density of air at the stagnation point on the nose of the plane. [**Ans.** 0.776, 11.9 N/cm^2 , 16.06°C, 1.434 kg/m^3]
9. Find the velocity of air flowing at the outlet of a nozzle, fitted to a large vessel which contains air at a pressure of 294.3 N/cm^2 (abs.) and at a temperature of 30°C. The pressure at the outlet of the nozzle is 137.34 N/cm^2 (abs.) Take k = 1.4 and R = 287 J/kg K. [**Ans.** 242.98 m/s]
10. A nozzle of diameter 20 mm is fitted to a large tank which contains air at 20°C. The air flows from the tank into atmosphere. For adiabatic flow, find the mass rate of flow of air through the nozzle when pressure of air in tank is (*i*) 5.886 N/cm^2 (gauge) and (*ii*) 29.43 N/cm^2 (gauge). Take k = 1.4 and R = 287 J/kg K and atmospheric pressure = 9.81 N/cm^2. [**Ans.** (*i*) 0.114 kg/s, (*ii*) 0.291 kg/s]
11. Find the mass rate of flow of air through a venturimeter having inlet diameter as 400 mm and throat diameter 200 mm. The pressure at the inlet of the venturimeter is 27.468 N/cm^2 (abs.) and temperature of air at inlet is 20°C. The pressure at the throat is given as 25.506 N/cm^2 absolute. Take k = 1.4 and R = 287 J/kg K. [**Ans.** 11.13 kg/s]
12. Calculate the numerical factor by which the actual pressure difference shown by the gauge of a pitot-tube should be multiplied to allow for compressibility when the value of the Mach number is 0.7. Take k = 1.4. [**Ans.** 1.1285]

13. Find the Mach number when an aeroplane is flying at 1000 km/hour through still air having pressure of 7 N/cm^2 and temperature of –5°C. Take R = 287.14 J/kg K. Calculate the pressure and temperature of air at stagnation point. Take k = 1.4. *(Delhi University, Dec. 2002)*

[**Hint.** $V = 1000 \text{ km/hour} = \frac{1000 \times 1000}{60 \times 60} = 277.77 \text{ m/s}$; $p_1 = 7 \text{ N/m}^2 = 7 \times 10^4 \text{ N/m}^2$, $t_1 = -5°\text{C}$

$\therefore \quad T_1 = -5 + 273 = 268° \text{ K}$; $R = 287.14$ J/kg K, $k = 1.4$

Now $C_1 = \sqrt{kRT_1} = \sqrt{1.4 \times 287.14 \times 268} = 328.2 \text{ m/s}$ $\quad \therefore M = M_1 = \frac{V}{C} = \frac{V_1}{C_1} = \frac{277.77}{328.2}$

$$\therefore \quad M_1 = 0.846 \qquad \therefore p_s = p_1 \left[1 + \frac{k-1}{2} M_1^2\right]^{\frac{k}{k-1}} = 7 \times 10^4 \left[1 + \frac{1.4-1}{2} \times (0.846)^2\right]^{\frac{1.4}{1.4-1}}$$

$$= 7 \times 10^4 \left[1 + 0.2 \times 0.846^2\right]^{\frac{1.4}{0.4}} = 11.18 \times 10^4 \text{ N/m}^2 = \mathbf{11.18 \text{ N/cm}^2}.$$

$$T_s = T_1 \left[1 + \frac{k-1}{2} M_1^2\right] = 268 \left[1 + \frac{1.4-1}{2} \times (0.846)^2\right] = 306.38° \text{ K} = 306.38 - 273 = \mathbf{33.38°C.]}$$