13

CHAPTER

BOUNDARY LAYER FLOW

▶ 13.1 INTRODUCTION

When a real fluid flows past a solid body or a solid wall, the fluid particles adhere to the boundary and condition of no slip occurs. This means that the velocity of fluid close to the boundary will be same as that of the boundary. If the boundary is stationary, the velocity of fluid at the boundary will be zero. Farther away from the boundary, the velocity will be higher and as a result of this variation of velocity, the velocity gradient $\frac{du}{dy}$ will exist. The velocity of fluid increases from zero velocity on the stationary boundary to free-stream velocity (U) of the fluid in the direction normal to the boundary. This variation of velocity from zero to free-stream velocity in the direction normal to the boundary takes place in a narrow region in the vicinity of solid boundary. This narrow region of the fluid is called boundary layer. The theory dealing with boundary layer flows is called boundary layer theory.

According to boundary layer theory, the flow of fluid in the neighbourhood of the solid boundary may be divided into two regions as shown in Fig. 13.1.

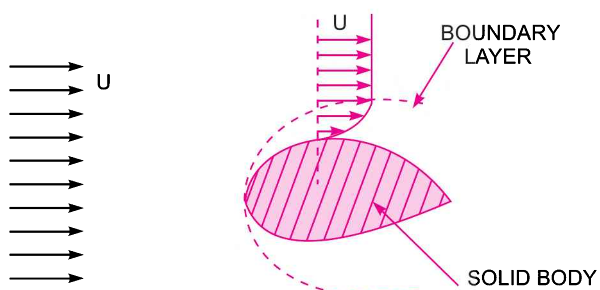

Fig. 13.1 *Flow over solid body.*

1. A very thin layer of the fluid, called the boundary layer, in the immediate neighbourhood of the solid boundary, where the variation of velocity from zero at the solid boundary to free-stream velocity in the direction normal to the boundary takes place. In this region, the velocity gradient $\frac{du}{dy}$ exists and hence the fluid exerts a shear stress on the wall in the direction of motion. The value of shear stress is given by

$$\tau = \mu \frac{du}{dy}.$$

2. The remaining fluid, which is outside the boundary layer. The velocity outside the boundary layer is constant and equal to free-stream velocity. As there is no variation of velocity in this region, the velocity gradient $\frac{du}{dy}$ becomes zero. As a result of this the shear stress is zero.

▶ 13.2 DEFINITIONS

13.2.1 Laminar Boundary Layer. For defining the boundary layer (*i.e.*, laminar boundary layer or turbulent boundary layer) consider the flow of a fluid, having free-stream velocity (U), over a smooth thin plate which is flat and placed parallel to the direction for free stream of fluid as shown in Fig. 13.2. Let us consider the flow with zero pressure gradient on one side of the plate, which is stationary.

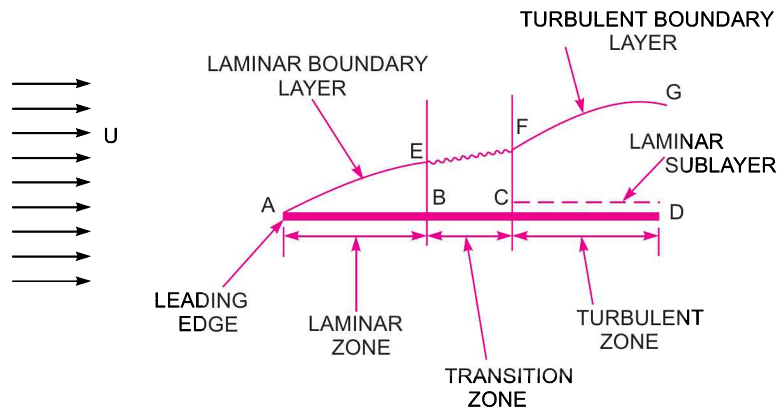

Fig. 13.2 *Flow over a plate.*

The velocity of fluid on the surface of the plate should be equal to the velocity of the plate. But plate is stationary and hence velocity of fluid on the surface of the plate is zero. But at a distance away from the plate, the fluid is having certain velocity. Thus a velocity gradient is set up in the fluid near the surface of the plate. This velocity gradient develops shear resistance, which retards the fluid. Thus the fluid with a uniform free stream velocity (U) is retarded in the vicinity of the solid surface of the plate and the boundary layer region begins at the sharp leading edge. At subsequent points downstream the leading edge, the boundary layer region increases because the retarded fluid is further retarded. This is also referred as the growth of boundary layer. Near the leading edge of the surface of the plate, where the thickness is small, the flow in the boundary layer is laminar though the main flow is turbulent. This layer of the fluid is said to be laminar boundary layer. This is shown by AE in Fig. 13.2. The length of the plate from the leading edge, upto which laminar boundary layer exists, is called laminar zone. This is shown by distance AB. The distance of B from leading edge is obtained from Reynold number equal to 5×10^5 for a plate. Because upto this Reynold number the boundary layer is laminar. The Reynold number is given by $(R_e)_x = \frac{U \times x}{\nu}$

where x = Distance from leading edge,
U = Free-stream velocity of fluid,
ν = Kinematic viscosity of fluid,

Hence for laminar boundary layer, we have $5 \times 10^5 = \dfrac{U \times x}{\nu}$ ...(13.1)

If the values of U and ν are known, x or the distance from the leading edge upto which laminar boundary layer exists can be calculated.

13.2.2 Turbulent Boundary Layer. If the length of the plate is more than the distance x, calculated from equation (13.1), the thickness of boundary layer will go on increasing in the down-stream direction. Then the laminar boundary layer becomes unstable and motion of fluid within it, is disturbed and irregular which leads to a transition from laminar to turbulent boundary layer. This short length over which the boundary layer flow changes from laminar to turbulent is called transition zone. This is shown by distance BC in Fig. 13.2. Further downstream the transition zone, the boundary layer is turbulent and continues to grow in thickness. This layer of boundary is called turbulent boundary layer, which is shown by the portion FG in Fig. 13.2.

13.2.3 Laminar Sub-layer. This is the region in the turbulent boundary layer zone, adjacent to the solid surface of the plate as shown in Fig. 13.2. In this zone, the velocity variation is influenced only by viscous effects. Though the velocity distribution would be a parabolic curve in the laminar sub-layer zone, but in view of the very small thickness we can reasonably assume that velocity variation is linear and so the velocity gradient can be considered constant. Therefore, the shear stress in the laminar sub-layer would be constant and equal to the boundary shear stress τ_0. Thus the shear stress in the sub-layer is

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \frac{u}{y} \qquad \left\{\because \text{ For linear variation, } \frac{\partial u}{\partial y} = \frac{u}{y}\right\}$$

13.2.4 Boundary Layer Thickness (δ). It is defined as the distance from the boundary of the solid body measured in the y-direction to the point, where the velocity of the fluid is approximately equal to 0.99 times the free stream velocity (U) of the fluid. It is denoted by the symbol δ. For laminar and turbulent zone it is denoted as :

1. δ_{lam} = Thickness of laminar boundary layer,
2. δ_{tur} = Thickness of turbulent boundary layer, and
3. δ' = Thickness of laminar sub-layer.

13.2.5 Displacement Thickness (δ*). It is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in flow rate on account of boundary layer formation. It is denoted by δ^*. It is also defined as :

"The distance perpendicular to the boundary, by which the free-stream is displaced due to the formation of boundary layer".

Expression for δ*.

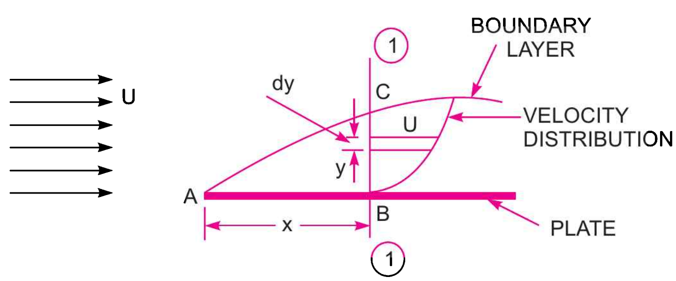

Fig. 13.3 *Displacement thickness.*

Consider the flow of a fluid having free-stream velocity equal to U over a thin smooth plate as shown in Fig. 13.3. At a distance x from the leading edge consider a section 1-1. The velocity of fluid at B is zero and at C, which lies on the boundary layer, is U. Thus velocity varies from zero at B to U at C, where BC is equal to the thickness of boundary layer *i.e.*,

Distance $BC = \delta$

At the section 1-1, consider an elemental strip.

Let y = distance of elemental strip from the plate,

dy = thickness of the elemental strip,

u = velocity of fluid at the elemental strip,

b = width of plate.

Then area of elemental strip, $dA = b \times dy$

Mass of fluid per second flowing through elemental strip

$$= \rho \times \text{Velocity} \times \text{Area of elemental strip}$$

$$= \rho u \times dA = \rho u \times b \times dy \qquad \text{...}(i)$$

If there had been no plate, then the fluid would have been flowing with a constant velocity equal to free-stream velocity (U) at the section 1-1. Then mass of fluid per second flowing through elemental strip would have been

$$= \rho \times \text{Velocity} \times \text{Area} = \rho \times U \times b \times dy \qquad \text{...}(ii)$$

As U is more than u, hence due to the presence of the plate and consequently due to the formation of the boundary layer, there will be a reduction in mass flowing per second through the elemental strip.

This reduction in mass/sec flowing through elemental strip

$$= \text{mass/sec given by equation } (ii) - \text{mass/sec given by equation } (i)$$

$$= \rho U b dy - \rho u b dy = \rho b(U - u)dy$$

$\therefore$ Total reduction in mass of fluid/s flowing through BC due to plate

$$= \int_0^\delta \rho b(U - u)dy = \rho b \int_0^\delta (U - u)dy \qquad \text{...}(iii)$$

{if fluid is incompressible}

Let the plate is displaced by a distance δ^* and velocity of flow for the distance δ^* is equal to the free-stream velocity (*i.e.*, U). Loss of the mass of the fluid/sec flowing through the distance δ^*

$$= \rho \times \text{Velocity} \times \text{Area}$$

$$= \rho \times U \times \delta^* \times b \qquad \{\because \text{Area} = \delta^* \times b\} \text{ ...}(iv)$$

Equating equation (*iii*) and (*iv*), we get

$$\rho b \int_0^\delta (U - u)dy = \rho \times U \times \delta^* b$$

Cancelling ρb from both sides, we have

$$\int_0^\delta (U - u)dy = U \times \delta^*$$

or

$$\delta^* = \frac{1}{U}\int_0^\delta (U - u)dy = \int_0^\delta \frac{(U - u)dy}{U} \qquad \left\{\begin{array}{l}\because U \text{ is constant and can} \\ \text{be taken inside the integral}\end{array}\right\}$$

$$\therefore \qquad \delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy. \qquad \text{...(13.2)}$$

13.2.6 Momentum Thickness (θ). Momentum thickness is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in **momentum** of the flowing fluid on account of boundary layer formation. It is denoted by θ.

Consider the flow over a plate as shown in Fig. 13.3. Consider the section 1-1 at a distance x from leading edge. Take an elemental strip at a distance y from the plate having thickness (dy). The mass of fluid flowing per second through this elemental strip is given by equation (i) and is equal to $\rho ubdy$.

Momentum of this fluid = Mass × Velocity = $(\rho ubdy)u$

Momentum of this fluid in the absence of boundary layer = $(\rho ubdy)U$

$\therefore$ Loss of momentum through elemental strip = $(\rho ubdy)U - (\rho ubdy) \times u = \rho bu(U - u)dy$

$\therefore$ Total loss of momentum/sec through $BC = \int_0^\delta \rho bu(U - u)dy$ ...(13.3)

Let θ = distance by which plate is displaced when the fluid is flowing with a constant velocity U

$\therefore$ Loss of momentum/sec of fluid flowing through distance θ with a velocity U

$$= \text{Mass of fluid through } \theta \times \text{velocity}$$
$$= (\rho \times \text{area} \times \text{velocity}) \times \text{velocity}$$
$$= [\rho \times \theta \times b \times U] \times U \qquad \{\because \text{Area} = \theta \times b\}$$
$$= \rho\theta bU^2 \qquad \text{...(13.4)}$$

Equating equations (13.4) and (13.3), we have

$$\rho\theta bU^2 = \int_0^\delta \rho bu(U - u)dy = \rho b \int_0^\delta u(U - u)dy \qquad \{\text{If fluid is assumed incompressible}\}$$

or
$$\theta U^2 = \int_0^\delta u(U - u)dy \qquad \{\text{cancelling } \rho b \text{ from both sides}\}$$

or
$$\theta = \frac{1}{U^2}\int_0^\delta u(U - u)dy = \int_0^\delta \frac{u(U - u)dy}{U^2}$$

$\therefore$
$$\theta = \int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right] dy. \qquad \text{...(13.5)}$$

13.2.7 Energy Thickness (δ^{**}). It is defined as the distance measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in kinetic energy of the flowing fluid on account of boundary layer formation. It is denoted by δ^{**}.

Consider the flow over the plate as shown in Fig. 13.3 having section 1-1 at a distance x from leading edge. The mass of fluid flowing per second through the elemental strip of thickness 'dy' at a distance y from the plate as given by equation (i) = $\rho ubdy$

Kinetic energy of this fluid $= \frac{1}{2} m \times \text{velocity}^2 = \frac{1}{2} (\rho ubdy) u^2$

Kinetic energy of this fluid in the absence of boundary layer

$$= \frac{1}{2} (\rho ubdy)U^2$$

$\therefore$ Loss of K.E. through elemental strip

$$= \frac{1}{2} (\rho ubdy)U^2 - \frac{1}{2} (\rho ubdy) u^2 = \frac{1}{2} \rho ub [U^2 - u^2] dy$$

∴ Total loss of K.E. of fluid passing through BC

$$= \int_0^\delta \frac{1}{2} \rho u b\,[U^2 - u^2]\,dy = \frac{1}{2}\rho b \int_0^\delta u\,(U^2 - u^2)\,dy$$

{If fluid is considered incompressible}

Let δ^{**} = distance by which the plate is displaced to compensate for the reduction in K.E.

∴ Loss of K.E. through δ^{**} of fluid flowing with velocity U

$$= \frac{1}{2}\,(\text{mass}) \times \text{velocity}^2 = \frac{1}{2}\,(\rho \times \text{area} \times \text{velocity}) \times \text{velocity}^2$$

$$= \frac{1}{2}\,(\rho \times b \times \delta^{**} \times U)U^2 \qquad \{\because \quad \text{Area} = b \times \delta^{**}\}$$

$$= \frac{1}{2}\,\rho b \delta^{**} U^3$$

Equating the two losses of K.E., we get

$$\frac{1}{2}\,\rho b \delta^{**} U^3 = \frac{1}{2}\rho b \int_0^\delta u\,(U^2 - u^2)\,dy$$

or

$$\delta^{**} = \frac{1}{U^3}\int_0^\delta u\,(U^2 - u^2)\,dy$$

∴

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1 - \frac{u^2}{U^2}\right] dy. \qquad \text{...(13.6)}$$

Problem 13.1 *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $\frac{u}{U} = \frac{y}{\delta}$, *where u is the velocity at a distance y from the plate and u = U at y = δ, where δ = boundary layer thickness. Also calculate the value of* δ^*/θ.

Solution. Given :

Velocity distribution $\quad \frac{u}{U} = \frac{y}{\delta}$

(*i*) Displacement thickness δ^* is given by equation (13.2),

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy \qquad \left\{\because \quad \frac{u}{U} = \frac{y}{\delta}\right\}$$

$$= \left[y - \frac{y^2}{2\delta}\right]_0^\delta \qquad \{\delta \text{ is constant across a section}\}$$

$$= \delta - \frac{\delta^2}{2\delta} = \delta - \frac{\delta}{2} = \frac{\delta}{2}.\ \textbf{Ans.}$$

(*ii*) Momentum thickness, θ is given by equation (13.5),

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = \frac{y}{\delta}$,

$$\theta = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy$$

$$= \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2} = \frac{\delta}{2} - \frac{\delta}{3} = \frac{3\delta - 2\delta}{6} = \frac{\delta}{6}. \text{ **Ans.**}$$

(*iii*) Energy thickness δ^{**} is given by equation (13.6), as

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1 - \frac{u^2}{U^2}\right] dy = \int_0^\delta \frac{y}{\delta}\left[1 - \frac{y^2}{\delta^2}\right] dy \qquad \left\{\because \frac{u}{U} = \frac{y}{\delta}\right\}$$

$$= \int_0^\delta \left[\frac{y}{\delta} - \frac{y^3}{\delta^3}\right] dy = \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3}$$

$$= \frac{\delta}{2} - \frac{\delta}{4} = \frac{2\delta - \delta}{4} = \frac{\delta}{4}. \text{ **Ans.**}$$

(*iv*)
$$\frac{\delta^*}{\theta} = \frac{\left(\frac{\delta}{2}\right)}{\left(\frac{\delta}{6}\right)} = \frac{\delta}{2} \times \frac{6}{\delta} = 3. \text{ **Ans.**}$$

Problem 13.2 *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$.

Solution. Given :

Velocity distribution $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

(*i*) Displacement thickness δ^* is given by equation (13.2),

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$, we have

$$\delta^* = \int_0^\delta \left\{1 - \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right\} dy$$

$$= \int_0^\delta \left\{1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right\} dy = \left[y - \frac{2y^2}{2\delta} + \frac{y^3}{3\delta^2}\right]_0^\delta$$

$$= \delta - \frac{\delta^2}{\delta} + \frac{\delta^3}{3\delta^2} = \delta - \delta + \frac{\delta}{3} = \frac{\delta}{3}. \quad \text{**Ans.**}$$

(*ii*) Momentum thickness θ, is given by equation (13.5),

$$\theta = \int_0^\delta \frac{u}{U}\left\{1-\frac{u}{U}\right\}dy = \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left[1-\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\right]dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]\left[1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right]dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{4y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy = \left[\frac{2y^2}{2\delta}-\frac{5y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \left[\frac{\delta^2}{\delta}-\frac{5\delta^3}{3\delta^2}+\frac{\delta^4}{\delta^3}-\frac{\delta^5}{5\delta^4}\right] = \delta-\frac{5\delta}{3}+\delta-\frac{\delta}{5}$$

$$= \frac{15\delta-25\delta+15\delta-3\delta}{15} = \frac{30\delta-28\delta}{15} = \mathbf{\frac{2\delta}{15}}. \quad \textbf{Ans.}$$

(*iii*) Energy thickness δ** is given by equation (13.6),

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1-\frac{u^2}{U^2}\right]dy = \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]^2\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\left[\frac{4y^2}{\delta^2}+\frac{y^4}{\delta^4}-\frac{4y^3}{\delta^3}\right]\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\frac{4y^2}{\delta^2}-\frac{y^4}{\delta^4}+\frac{4y^3}{\delta^3}\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{8y^3}{\delta^3}-\frac{2y^5}{\delta^5}+\frac{8y^4}{\delta^4}-\frac{y^2}{\delta^2}+\frac{4y^4}{\delta^4}+\frac{y^6}{\delta^6}-\frac{4y^5}{\delta^5}\right)dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}-\frac{8y^3}{\delta^3}+\frac{12y^4}{\delta^4}-\frac{6y^5}{\delta^5}+\frac{y^6}{\delta^6}\right]dy$$

$$= \left[\frac{2y^2}{2\delta}-\frac{y^3}{3\delta^2}-\frac{8y^4}{4\delta^3}+\frac{12y^5}{5\delta^4}-\frac{6y^6}{6\delta^5}+\frac{y^7}{7\delta^6}\right]_0^\delta$$

$$= \frac{\delta^2}{\delta}-\frac{\delta^3}{3\delta^2}-\frac{2\delta^4}{\delta^3}+\frac{12\delta^5}{5\delta^4}-\frac{\delta^6}{\delta^5}+\frac{\delta^7}{7\delta^6} = \delta-\frac{\delta}{3}-2\delta+\frac{12}{5}\delta-\delta+\frac{\delta}{7}$$

$$= -2\delta-\frac{\delta}{3}+\frac{12}{5}\delta+\frac{\delta}{7} = \frac{-210\delta-35\delta+252\delta+15\delta}{105}$$

$$= \frac{-245\delta+267\delta}{105} = \mathbf{\frac{22\delta}{105}}. \quad \textbf{Ans.}$$

▶ 13.3 DRAG FORCE ON A FLAT PLATE DUE TO BOUNDARY LAYER

Consider the flow of a fluid having free-stream velocity equal to U, over a thin plate as shown in Fig. 13.4. The drag force on the plate can be determined if the velocity profile near the plate is known. Consider a small length Δx of the plate at a distance of x from the leading edge as shown in Fig. 13.4 (a). The enlarged view of the small length of the plate is shown in Fig. 13.4 (b).

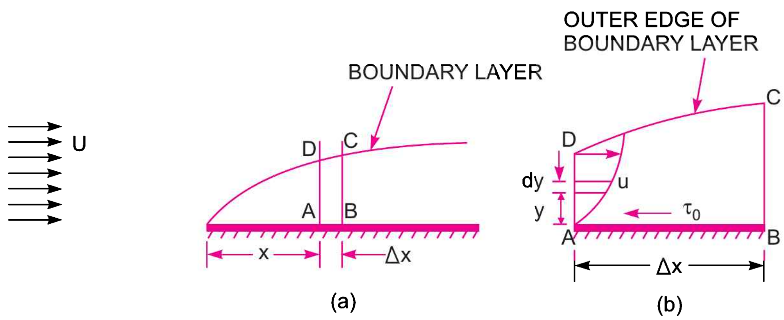

Fig. 13.4 *Drag force on a plate due to boundary layer.*

The shear stress τ_0 is given by $\tau_0 = \mu \left(\frac{du}{dy}\right)_{y=0}$, where $\left(\frac{du}{dy}\right)_{y=0}$ is the velocity distribution near the plate at $y = 0$.

Then drag force or shear force on a small distance Δx is given by

$$\Delta F_D = \text{shear stress} \times \text{area}$$

$$= \tau_0 \times \Delta x \times b \quad \text{...(13.7)} \quad \{\text{Taking width of plate} = b\}$$

where ΔF_D = drag force on distance Δx

The drag force ΔF_D must also be equal to the rate of change of momentum over the distance Δx.

Consider the flow over the small distance Δx. Let $ABCD$ is the control volume of the fluid over the distance Δx as shown in Fig. 13.4 (b). The edge DC represents the outer edge of the boundary layer.

Let u = velocity at any point within the boundary layer

b = width of plate

Then mass rate of flow entering through the side AD

$$= \int_0^\delta \rho \times \text{velocity} \times \text{area of strip of thickness } dy$$

$$= \int_0^\delta \rho \times u \times b \times dy \quad \{\because \text{ Area of strip} = b \times dy\}$$

$$= \int_0^\delta \rho u b dy$$

Mass rate of flow leaving the side BC

$$= \text{mass through } AD + \frac{\partial}{\partial x}(\text{mass through } AD) \times \Delta x$$

$$= \int_0^\delta \rho u b dy \, \frac{\partial}{\partial x}\left[\int_0^\delta (\rho u b dy)\right] \times \Delta x$$

From continuity equation for a steady incompressible fluid flow, we have
Mass rate of flow entering AD + mass rate of flow entering DC
= mass rate of flow leaving BC

$\because$ Mass rate of flow entering DC = mass rate of flow through BC – mass rate of flow through AD

$$= \int_0^\delta \rho u b dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u b dy\right] \times \Delta x - \int_0^\delta \rho u b dy$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u b dy\right] \times \Delta x$$

The fluid is entering through side DC with a uniform velocity U.
Now let us calculate momentum flux through control volume.
Momentum flux entering through AD

$$= \int_0^\delta \text{momentum flux through strip of thickness } dy$$

$$= \int_0^\delta \text{mass through strip} \times \text{velocity} = \int_0^\delta (\rho u b dy) \times u = \int_0^\delta \rho u^2 b dy$$

Momentum flux leaving the side $BC = \int_0^\delta \rho u^2 b dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy\right] \times \Delta x$

Momentum flux entering the side DC = mass rate through DC × velocity

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u b dy\right] \times \Delta x \times U \qquad (\because \text{ Velocity} = U\}$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u U b dy\right] \times \Delta x$$

As U is constant and so it can be taken inside the differential and integral.

$\therefore$ Rate of change of momentum of the control volume
= Momentum flux through BC – Momentum flux through AD
– momentum flux through DC

$$= \int_0^\delta \rho u^2 b dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy\right] \times \Delta x - \int_0^\delta \rho u^2 b dy - \frac{\partial}{\partial x}\left[\int_0^\delta \rho u U b dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy\right] \times \Delta x - \frac{\partial}{\partial x}\left[\int_0^\delta \rho u U b dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy - \int_0^\delta \rho u U b dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\rho u^2 b - \rho u U b\right) dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\rho b \int_0^\delta \left(u^2 - uU\right) dy\right] \times \Delta x$$

{For incompressible fluid ρ is constant}

$$= \rho b \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right)dy\right] \times \Delta x \qquad ...(13.8)$$

Now the rate of change of momentum on the control volume *ABCD* must be equal to the total force on the control volume in the same direction according to the momentum principle. But for a flat plate $\frac{\partial p}{\partial x} = 0$, which means there is no external pressure force on the control volume. Also the force on the side *DC* is negligible as the velocity is constant and velocity gradient is zero approximately. The only external force acting on the control volume is the shear force acting on the side *AB* in the direction from *B* to *A* as shown in Fig. 13.4 (*b*). The value of this force is given by equation (13.7) as

$$\Delta F_D = \tau_0 \times \Delta x \times b$$

∴ Total external force in the direction of rate of change of momentum

$$= - \tau_0 \times \Delta x \times b \qquad ...(13.9)$$

According to momentum principle, the two values given by equations (13.9) and (13.8) should be the same.

$$\therefore \qquad - \tau_0 \times \Delta x \times b = \rho b \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right)dy\right] \times \Delta x$$

Cancelling $\Delta x \times b$, to both sides, we have

$$- \tau_0 = \rho \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right)dy\right]$$

or

$$\tau_0 = - \rho \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right)dy\right] = \rho \frac{\partial}{\partial x}\left[\int_0^\delta \left(uU - u^2\right)dy\right]$$

$$= \rho \frac{\partial}{\partial x}\left[\int_0^\delta U^2\left(\frac{u}{U} - \frac{u^2}{U^2}\right)dy\right] = \rho U^2 \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right]$$

or

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right] \qquad ...(13.10)$$

In equation (13.10), the expression $\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]dy$ is equal to momentum thickness θ. Hence equation (13.10) is also written as

$$\frac{\tau_0}{\rho U^2} = \frac{\partial \theta}{\partial x} \qquad ...(13.11)$$

Equation (13.11) is known as **Von Karman momentum integral equation** for boundary layer flows.

This is applied to :

1. Laminar boundary layers,
2. Transition boundary layers, and
3. Turbulent boundary layer flows.

For a given velocity profile in laminar zone, transition zone or turbulent zone of a boundary layer, the shear stress τ_0 is obtained from equation (13.10) or (13.11). Then drag force on a small distance Δx of the plate is obtained from equation (13.7) as

$$\Delta F_D = \tau_0 \times \Delta x \times b$$

Then total drag on the plate of length L on one side is

$$F_D = \int \Delta F_D = \int_0^L \tau_0 \times b \times dx \quad \{\text{change } \Delta x = dx\}. \qquad ...(13.12)$$

13.3.1 Local Co-efficient of Drag [C_D*]. It is defined as the ratio of the shear stress τ_0 to the quantity $\frac{1}{2}\rho U^2$. It is denoted by C_D*

Hence
$$C_D^* = \frac{\tau_0}{\frac{1}{2}\rho U^2}. \qquad ...(13.13)$$

13.3.2 Average Co-efficient of Drag [C_D]. It is defined as the ratio of the total drag force to the quantity $\frac{1}{2}\rho A U^2$. It is also called co-efficient of drag and is denoted by C_D.

Hence
$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2} \qquad ...(13.14)$$

where A = Area of the surface (or plate)
U = Free-stream velocity
ρ = Mass density of fluid.

13.3.3 Boundary Conditions for the Velocity Profiles. The followings are the boundary conditions which must be satisfied by any velocity profile, whether it is in laminar boundary layer zone, or in turbulent boundary layer zone :

1. At $y = 0$, $u = 0$ and $\frac{du}{dy}$ has some finite value
2. At $y = \delta$, $u = U$
3. At $y = \delta$, $\frac{du}{dy} = 0$.

Problem 13.3 *For the velocity profile for laminar boundary layer flows given as*

$$\frac{u}{U} = 2(y/\delta) - (y/\delta)^2$$

find an expression for boundary layer thickness (δ), shear stress (τ_0) and co-efficient of drag (C_D) in terms of Reynold number.

Solution. Given :

(*i*) The velocity distribution $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ ...(*i*)

Substituting this value of $\frac{u}{U}$ in equation (13.10), we get

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] = \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]\left[1-\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\right]dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]\left[1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right]dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta}-\frac{4y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy\right]$$

$$= \frac{\partial}{\partial x}\int_0^\delta \left[\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy = \frac{\partial}{\partial x}\left[\frac{2y^2}{2\delta}-\frac{5\times y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{\delta^2}{\delta}-\frac{5}{3}\frac{\delta^3}{\delta^2}+\frac{\delta^4}{\delta^3}-\frac{\delta^5}{5\delta^4}\right] = \frac{\partial}{\partial x}\left[\delta-\frac{5}{3}\delta+\delta-\frac{\delta}{5}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{15\delta-25\delta+15\delta-3\delta}{15}\right] = \frac{\partial}{\partial x}\left[\frac{30\delta-28\delta}{15}\right] = \frac{\partial}{\delta x}\left[\frac{2\delta}{15}\right] = \frac{2}{15}\frac{\partial}{\partial x}\,[\delta]$$

$\therefore$
$$\tau_0 = \rho U^2 \times \frac{2}{15}\frac{\partial}{\partial x}[\delta] = \frac{2}{15}\rho U^2\frac{\partial[\delta]}{\partial x} \qquad ...(13.15)$$

The shear stress at the boundary in laminar flow is also given by Newton's law of viscosity as

$$\tau_0 = \mu\left(\frac{du}{dy}\right)_{y=0} \qquad ...(ii)$$

But from equation (*i*), $u = U\left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]$

$\therefore$ $\frac{du}{dy} = U\left[\frac{2}{\delta}-\frac{2y}{\delta^2}\right]$ $\{\because$ U is constant$\}$

$\therefore$ $\left(\frac{du}{dy}\right)_{y=0} = U\left[\frac{2}{\delta}-\frac{2\times(0)}{\delta^2}\right] = \frac{2U}{\delta}$

Substituting this value in (*ii*), we get

$$\tau_0 = \mu \times \frac{2U}{\delta} = \frac{2\mu U}{\delta} \qquad ...(iii)$$

Equating the two values of τ_0 given by equation (13.15) and (*iii*)

$$\frac{2}{15}\rho U^2 \frac{\partial}{\partial x}[\delta] = \frac{2\mu U}{\delta}$$

or
$$\frac{\delta\partial}{\partial x}[\delta] = \frac{15\mu U}{\rho U^2} = \frac{15\mu}{\rho U} \quad \text{or} \quad \delta\partial[\delta] = \frac{15\mu}{\rho U}\partial x$$

As the boundary layer thickness (δ) is a function of x only.
Hence partial derivative can be changed to total derivative

$\therefore$ $$\delta d[\delta] = \frac{15\mu}{\rho U}\,dx$$

On integration, we get $$\frac{\delta^2}{2} = \frac{15\mu}{\rho U}\,x + C \qquad \left\{\frac{\mu}{\rho U} \text{ is constant}\right\}$$

$x = 0$, $\delta = 0$ and hence $C = 0$

$\therefore$ $$\frac{\delta^2}{2} = \frac{15\mu x}{\rho U}$$

$\therefore$ $$\delta = \sqrt{\frac{2 \times 15\mu x}{\rho U}} = \sqrt{\frac{30\mu x}{\rho U}} = 5.48\sqrt{\frac{\mu x}{\rho U}} \qquad \text{...(13.16)}$$

$$= 5.48\sqrt{\frac{\mu x \times x}{\rho U \times x}} = 5.48\sqrt{\frac{x^2}{R_{e_x}}} \qquad \left\{\because \; R_{e_x} = \frac{\rho U x}{\mu}\right\}$$

$$= 5.48\,\frac{x}{\sqrt{R_{e_x}}} \qquad \text{...(13.17)}$$

In equation (13.16), μ, ρ and U are constant and hence it is clear from this equation that thickness of laminar boundary layer is proportional to the square root of the distance from the leading edge. Equation (13.17) gives the thickness of laminar boundary layer in terms of Reynolds number.

(*ii*) **Shear stress (τ_0) in terms of Reynolds number**

From equation (*iii*), we have $\tau_0 = \dfrac{2\mu U}{\delta}$

Substituting the value of δ from equation (13.17), in the above equation, we get

$$\tau_0 = \frac{2\mu U}{5.48\dfrac{x}{\sqrt{R_{e_x}}}} = \frac{2\mu U\sqrt{R_{e_x}}}{5.48x} = 0.365\,\frac{\mu U}{x}\sqrt{R_{e_x}}$$

(*iii*) **Co-efficient of Drag (C_D)**

From equation (13.14), we have $C_D = \dfrac{F_D}{\frac{1}{2}\rho A U^2}$

where F_D is given by equation (13.12) as

$$F_D = \int_0^L \tau_0 \times b \times dx = \int_0^L 0.365\frac{\mu U}{x}\sqrt{R_{e_x}} \times b \times dx$$

$$= 0.365\int_0^L \frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}} \times b \times dx \qquad \left\{\because \; R_{e_x} = \frac{\rho U x}{\mu}\right\}$$

$$= 0.365\int_0^L \mu U\sqrt{\frac{\rho U}{\mu}} \times \frac{1}{\sqrt{x}} \times b \times dx$$

$$= 0.365\ \mu U \sqrt{\frac{\rho U}{\mu}} \times b \int_0^L x^{-1/2}\, dx = 0.365\ \mu U \sqrt{\frac{\rho U}{\mu}} \times b \times \left[\frac{x^{1/2}}{\frac{1}{2}}\right]_0^L$$

$$= 0.365 \times 2\mu U \sqrt{\frac{\rho U}{\mu}} \times b \times \sqrt{L}$$

$$= 0.73\ b\mu U \sqrt{\frac{\rho UL}{\mu}} \qquad \text{...(13.18)}$$

$$\therefore \qquad C_D = \frac{0.73\ b\mu U \sqrt{\frac{\rho UL}{\mu}}}{\frac{1}{2}\rho A U^2}$$

where A = Area of plate = Length of plate × width = $L \times b$

$$\therefore \qquad C_D = \frac{0.73\ b\mu U}{\frac{1}{2}\rho \times L \times b \times U^2} \sqrt{\frac{\rho UL}{\mu}} = \frac{1.46\ \mu}{\rho LU} \sqrt{\frac{\rho UL}{\mu}}$$

$$= \frac{1.46\sqrt{\mu}}{\sqrt{\rho UL}} = 1.46 \sqrt{\frac{\mu}{\rho UL}} = \frac{1.46}{\sqrt{R_{e_L}}} \text{ ...(13.19)} \qquad \left\{\because \sqrt{\frac{\mu}{\rho UL}} = \frac{1}{\sqrt{R_{e_L}}}\right\}$$

Problem 13.4 *For the velocity profile given in problem 13.3, find the thickness of boundary layer at the end of the plate and the drag force on one side of a plate 1 m long and 0.8 m wide when placed in water flowing with a velocity of 150 mm per second. Calculate the value of co-efficient of drag also. Take* μ *for water = 0.01 poise.*

Solution. Given :

Length of plate, $L = 1$ m

Width of plate, $b = 0.8$ m

Velocity of fluid (water), $U = 150$ mm/s = 0.15 m/s

μ for water $= 0.01 \text{ poise} = \frac{0.01}{10} \frac{\text{Ns}}{\text{m}^2} = 0.001 \frac{\text{Ns}}{\text{m}^2}$

Reynold number at the end of the plate *i.e.,* at a distance of 1 m from leading edge is given by

$$R_{e_L} = \frac{\rho UL}{\mu} = 1000 \times \frac{0.15 \times 1.0}{.001} \qquad (\because \quad \rho = 1000)$$

$$= \frac{1000 \times .15 \times 1.0}{0.001} = 150000$$

(*i*) As laminar boundary layer exists upto Reynold number = 2×10^5. Hence this is the case of laminar boundary layer. Thickness of boundary layer at $x = 1.0$ m is given by equation (13.17) as

$$\delta = 5.48 \frac{x}{\sqrt{R_{e_x}}} = \frac{5.48 \times 1.0}{\sqrt{150000}} = 0.01415 \text{ m} = \mathbf{14.15\ mm.\ Ans.}$$

(*ii*) Drag force on one side of the plate is given by equation (13.18)

$$F_D = 0.73\, b\mu U \sqrt{\frac{\rho U L}{\mu}}$$

$$= 0.73 \times 0.8 \times 0.001 \times 0.15 \times \sqrt{150000} \qquad \left\{\because \frac{\rho U L}{\mu} = R_{e_L}\right\}$$

$$= \mathbf{0.0338\ N.\ Ans.}$$

(*iii*) Co-efficient of drag, C_D is given by equation (13.19) as

$$C_D = \frac{1.46}{\sqrt{R_{e_L}}} = \frac{1.46}{\sqrt{150000}} = \mathbf{.00376.\ Ans.}$$

Problem 13.5 *For the velocity profile for laminar boundary layer* $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$. *Determine the boundary layer thickness, shear stress, drag force and co-efficient of drag in terms of Reynold number.*

Solution. Given :

Velocity distribution, $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$

Using equation (13.10), we have $\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right]$

Substituting the value of $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ in the above equation

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1 - \left\{\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}\right] dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{3y}{2\delta} - \frac{y^3}{2\delta^3}\right)\left(1 - \frac{3y}{2\delta} + \frac{y^3}{2\delta^3}\right) dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{3y}{2\delta} - \frac{9y^2}{4\delta^2} + \frac{3y^4}{4\delta^4} - \frac{y^3}{2\delta^3} + \frac{3y^4}{4\delta^4} - \frac{y^6}{4\delta^6}\right) dy\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{3y^2}{2\times 2\delta} - \frac{9y^3}{3\times 4\delta^2} + \frac{3y^5}{5\times 4\delta^4} - \frac{y^4}{4\times 2\delta^3} + \frac{3y^5}{5\times 4\delta^4} - \frac{y^7}{7\times 4\delta^6}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{3\delta^2}{4\delta} - \frac{3\delta^3}{4\delta^2} + \frac{3}{20}\frac{\delta^5}{\delta^4} - \frac{1}{8}\frac{\delta^4}{\delta^3} + \frac{3}{20}\frac{\delta^5}{\delta^4} - \frac{1}{28}\frac{\delta^7}{\delta^6}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{3}{4}\delta - \frac{3}{4}\delta + \frac{3}{20}\delta - \frac{1}{8}\delta + \frac{3}{20}\delta - \frac{1}{28}\delta\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{6}{20}\delta - \frac{1}{8}\delta - \frac{1}{28}\delta\right] = \frac{\partial \delta}{\partial x}\left[\frac{84-35-10}{280}\right] = \frac{39}{280}\frac{\partial \delta}{\partial x}$$

$$\tau_0 = \rho U^2 \times \frac{39}{280}\frac{\partial \delta}{\partial x} = \frac{39}{280}\rho U^2 \frac{\partial \delta}{\partial x} \qquad ...(13.20)$$

Also the shear stress τ_0 is given by $\tau_0 = \mu \left(\frac{du}{dy}\right)_{y=0}$, where $u = U\left[\frac{3}{2}\frac{y}{\delta} - \frac{y^3}{2\delta^3}\right]$

$$\therefore \qquad \frac{du}{dy} = U\left[\frac{3}{2\delta} - \frac{3y^2}{2\delta^3}\right]$$

Hence
$$\left(\frac{du}{dy}\right)_{y=0} = U\left[\frac{3}{2\delta} - \frac{3}{2\delta^3}\times 0\right] = \frac{3U}{2\delta}$$

$$\therefore \qquad \tau_0 = \mu\left(\frac{du}{dy}\right)_{y=0} = \mu\frac{3U}{2\delta} = \frac{3}{2}\frac{\mu U}{\delta} \qquad ...(13.21)$$

Equating the two values of τ_0 given by equations (13.20) and (13.21)

$$\frac{39}{280}\rho U^2 \frac{\partial \delta}{\partial x} = \frac{3}{2}\frac{\mu U}{\delta}$$

$$\therefore \qquad \delta \partial \delta = \frac{3}{2}\mu U \times \frac{280}{39} \times \frac{1}{\rho U^2}\partial x = \frac{420}{39}\frac{\mu}{\rho U}\partial x$$

Integrating, we get
$$\frac{\delta^2}{2} = \frac{420}{39}\frac{\mu}{\rho U}x + C$$

where $x = 0$, $\delta = 0$, $\therefore$ $C = 0$

$$\therefore \qquad \frac{\delta^2}{2} = \frac{420}{39}\cdot\frac{\mu}{\rho U}x$$

or
$$\delta = \sqrt{\frac{420\times 2}{39}\frac{\mu}{\rho U}x} = 4.64\sqrt{\frac{\mu x}{\rho U}} = 4.64\sqrt{\frac{\mu x \times x}{\rho U x}}$$

$$= 4.64\sqrt{\frac{\mu}{\rho U x}}\,x = \frac{4.64\,x}{\sqrt{R_{e_x}}} \qquad ...(13.22)$$

(*i*) **Shear Stress τ_0.** Substituting the value of δ from equation (13.22) into equation (13.21), we get

$$\tau_0 = \frac{3}{2}\frac{\mu U}{\frac{4.64\,x}{\sqrt{R_{e_x}}}} = \frac{3}{9.28}\frac{\mu U\sqrt{R_{e_x}}}{x} = 0.323\frac{\mu U}{x}\sqrt{R_{e_x}}$$

(*ii*) **Drag force (F_D)**

Using equation (13.12), we get the drag force as

$$F_D = \int_0^L \tau_0 \times b \times dx = \int_0^L 0.323\frac{\mu U}{x}\sqrt{R_{e_x}} \times b \times dx$$

$$= 0.323 \int_0^L \frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}} \times b \times dx = 0.323\ \mu U \sqrt{\frac{\rho U}{\mu}} \times b \int_0^L \frac{1}{\sqrt{x}} dx$$

$$= 0.323\ \mu U \sqrt{\frac{\rho U}{\mu}} \times b \int_0^L x^{-1/2}\, dx$$

$$= 0.323\ \mu U \sqrt{\frac{\rho U}{\mu}} \times b \left[\frac{x^{1/2}}{\frac{1}{2}}\right]_0^L = 0.323 \times 2\ \mu U \sqrt{\frac{\rho U}{\mu}} \times b\left[\sqrt{L}\right]$$

$$= 0.646\ \mu U \sqrt{\frac{\rho U L}{\mu}} \times b \qquad \text{...(13.23)}$$

(*iii*) **Drag Co-efficient (C_D).** Using equation (13.14), we get the value of C_D as

$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2}, \text{ where } A = b \times L$$

$$= \frac{0.646\ \mu U \sqrt{\frac{\rho U L}{\mu}} \times b}{\frac{1}{2}\rho \times b \times L \times U^2} = 0.646 \times 2 \times \frac{\mu}{\rho U L} \times \sqrt{\frac{\rho U L}{\mu}} = \frac{1.292}{\sqrt{\frac{\rho U L}{\mu}}}$$

$$= \frac{1.292}{\sqrt{R_{e_L}}}. \qquad \left\{\because \sqrt{\frac{\rho U L}{\mu}} = \sqrt{R_{e_L}}\right\} \qquad \text{...(13.24)}$$

Problem 13.6 *For the velocity profile for laminar boundary layer*

$$\frac{u}{U} = 2\,(y/\delta) - 2(y/\delta)^3 + (y/\delta)^4$$

obtain an expression for boundary layer thickness, shear stress, drag force on one side of the plate and co-efficient of drag in term of Reynold number.

Solution. Given :

(*i*) The velocity profile, $\frac{u}{U} = \frac{2y}{\delta} - \frac{2y^3}{\delta^3} + \frac{y^4}{\delta^4}$

Using equation (13.10), we have

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right]$$

Substituting the given velocity profile in the above equation

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{2y}{\delta} - \frac{2y^3}{\delta^3} + \frac{y^4}{\delta^4}\right)\left(1 - \left\{\frac{2y}{\delta} - \frac{2y^3}{\delta^3} + \frac{y^4}{\delta^4}\right\}\right) dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{2y}{\delta} - \frac{2y^3}{\delta^3} + \frac{y^4}{\delta^4}\right)\left(1 - \frac{2y}{\delta} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4}\right) dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{4y^4}{\delta^4} - \frac{2y^5}{\delta^5} - \frac{2y^3}{\delta^3} + \frac{4y^4}{\delta^4} - \frac{4y^6}{\delta^6} + \frac{2y^7}{\delta^7} + \frac{y^4}{\delta^4} - \frac{2y^5}{\delta^5} + \frac{2y^7}{\delta^7} - \frac{y^8}{\delta^8}\right) dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{2y}{\delta} - \frac{4y^2}{\delta^2} - \frac{2y^3}{\delta^3} + \frac{9y^4}{\delta^4} - \frac{4y^5}{\delta^5} - \frac{4y^6}{\delta^6} + \frac{4y^7}{\delta^7} - \frac{y^8}{\delta^8}\right) dy\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{2y^2}{2\delta} - \frac{4y^3}{3\delta^2} - \frac{2y^4}{4\delta^3} + \frac{9y^5}{5\delta^4} - \frac{4y^6}{6\delta^5} - \frac{4y^7}{7\delta^6} + \frac{4y^8}{8\delta^7} - \frac{y^9}{9\delta^8}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\delta - \frac{4}{3}\delta - \frac{1}{2}\delta + \frac{9}{5}\delta - \frac{2}{3}\delta - \frac{4}{7}\delta + \frac{1}{2}\delta - \frac{1}{9}\delta\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{315 - 420 + 63 \times 9 - 210 - 45 \times 4 - 35}{315}\right]\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{315 - 420 + 567 - 210 - 180 - 35}{315}\right]\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{882 - 845}{815}\right]\delta = \frac{\partial}{\partial x}\left[\frac{37}{315}\right]\delta = \frac{37}{315}\frac{\partial\delta}{\partial x}$$

$$\therefore \quad \tau_0 = \frac{37}{315}\rho U^2 \frac{\partial\delta}{\partial x} \qquad \text{...(13.25)}$$

Also shear stress is given by Newton's law of viscosity as

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

where $u = U\left[\frac{2y}{\delta} - \frac{2y^2}{\delta^2} + \frac{y^4}{\delta^4}\right]$

$$\therefore \qquad \left(\frac{du}{dy}\right) = U\left[\frac{2}{\delta} - \frac{4y}{\delta^2} - \frac{4y^3}{\delta^4}\right]$$

$$\therefore \qquad \left(\frac{\partial u}{\partial y}\right)_{y=0} = U\left[\frac{2}{\delta} - \frac{4}{\delta^2}(0) - \frac{4}{\delta^4}(0)\right] = \frac{2U}{\delta}$$

$$\therefore \qquad \tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \times \frac{2U}{\delta} = \frac{2U\mu}{\delta} \qquad \text{...(13.26)}$$

Equating the two values of τ_0 given by equations (13.25) and (13.26)

$$\frac{37}{315}\rho U^2 \frac{\partial\delta}{\partial x} = \frac{2U\mu}{\delta} \quad \text{or} \quad \delta\partial\delta = \frac{315}{37} \times \frac{2U\mu}{\rho U^2}\partial x = \frac{630}{37}\frac{\mu}{\rho U}\partial x$$

On integration, we get $\frac{\delta^2}{2} = \frac{630}{37}\frac{\mu}{\rho U}x + C$, where C = Constant of integration

At $x = 0$, $\delta = 0$ and hence $C = 0$

$$\therefore \quad \frac{\delta^2}{2} = \frac{630}{37}\frac{\mu}{\rho U}x$$

$$\therefore \quad \delta = \sqrt{\frac{630 \times 2}{37}\frac{\mu}{\rho U}x} = 5.84\sqrt{\frac{\mu x}{\rho U}}$$

$$= 5.84\sqrt{\frac{\mu x \times x}{\rho U x}} = 5.84\sqrt{\frac{\mu}{\rho U x}} \times x = \frac{5.84x}{\sqrt{R_{e_x}}} \quad \text{...(13.27)}$$

(*ii*) **Shear Stress (τ_0).** Substituting the value of δ from (13.27) into (13.26)

$$\tau_0 = \frac{2U\mu}{\delta} = \frac{2U\mu}{\frac{5.84x}{\sqrt{R_{e_x}}}} = \frac{2U\mu}{5.84x}\sqrt{R_{e_x}} = 0.34\frac{U\mu}{x}\sqrt{R_{e_x}}.$$

(*iii*) **Drag Force (F_D)** on one side of the plate :
Using equation (13.12), we get

$$F_D = \int_0^L \tau_0 \times b \times dx = \int_0^L 0.34\frac{U\mu}{x}\sqrt{R_{e_x}} \times b \times dx = \int_0^L 0.34\frac{U\mu}{\mu}\frac{\sqrt{\rho U \mu}}{\mu} bdx$$

$$= 0.34\, U\mu\sqrt{\frac{\rho U}{\mu}} \times b\int_0^L x^{-1/2}\,dx = 0.34\, U\mu\sqrt{\frac{\rho U}{\mu}} \times b \times \left[\frac{x^{1/2}}{1/2}\right]_0^L$$

$$= 0.34 \times 2U\mu\sqrt{\frac{\rho U}{\mu}}\, b\sqrt{L} = 0.68\, b\mu U\sqrt{\frac{\rho U L}{\mu}} \quad \text{...(13.28)}$$

(*iv*) **Drag Co-efficient (C_D)**

Using equation (13.14), $C_D = \dfrac{F_D}{\frac{1}{2}\rho A U^2}$, where $A = b \times L$

$$= \frac{0.68 \times b \times \mu U \times \sqrt{\frac{\rho U L}{\mu}}}{\frac{1}{2}\rho \times b \times L \times U^2} = 0.68 \times 2\frac{\mu}{\rho U L} \times \sqrt{\frac{\rho U L}{\mu}} = 1.36 \times \frac{1}{\sqrt{\frac{\rho U L}{\mu}}}$$

$$= 1.36 \times \frac{1}{\sqrt{R_{e_L}}}. \quad \text{...(13.29)}$$

Problem 13.7 *For the velocity profile for laminar boundary flow* $\dfrac{u}{U} = \sin\left(\dfrac{\pi}{2}\dfrac{y}{\delta}\right)$.
Obtain an expression for boundary layer thickness, shear stress, drag force on one side of the plate and co-efficient of drag in terms of Reynold number.

Solution. (*i*) The velocity profile is $\dfrac{u}{U} = \sin\left(\dfrac{\pi}{2}\dfrac{y}{\delta}\right)$.

Substituting this value in equation (13.10), we have

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] = \frac{\partial}{\partial x}\left[\int_0^\delta \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\left[1-\sin\left(\frac{\pi}{2}.\frac{y}{\delta}\right)\right]dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left[\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]dy\right]$$

$$= \frac{\partial}{\partial x}\left[\left[\frac{-\cos\frac{\pi y}{2\delta}}{\frac{\pi}{2\delta}}\right] - \left[\frac{\frac{\pi y}{2\delta}\times\frac{1}{2}}{\frac{\pi}{2\delta}} - \frac{\sin 2\left(\frac{\pi}{2}\frac{y}{\delta}\right)}{4\times\frac{\pi}{2\delta}}\right]\right]_0^\delta$$

$$\left\{\because \int \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)dy = \frac{\frac{\pi y}{2\delta}\times\frac{1}{2}}{\frac{\pi}{2\delta}} - \frac{\sin 2\left(\frac{\pi}{2}\frac{y}{\delta}\right)}{4\times\frac{\pi}{2\delta}}\right\}$$

$$\because \quad \frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\left(\frac{-\cos\frac{\pi}{2}\frac{\delta}{\delta}}{\frac{\pi}{2\delta}} + \frac{\cos\frac{\pi}{2}\times\frac{0}{\delta}}{\frac{\pi}{2\delta}}\right) - \left[\frac{\frac{\pi}{2}\frac{\delta}{\delta}\times\frac{1}{2}}{\frac{\pi}{2\delta}} - 0\right]\right]$$

$$= \frac{\partial}{\partial x}\left[\left(0+\frac{1}{\frac{\pi}{2\delta}}\right) - \frac{\left(\frac{\pi}{4}\right)}{\frac{\pi}{2\delta}}\right] = \frac{\partial}{\partial x}\left[\frac{2\delta}{\pi} - \frac{\pi}{4}\times\frac{2\delta}{\pi}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{2\delta}{\pi} - \frac{\delta}{2}\right] = \frac{\partial}{\partial x}\left[\frac{4-\pi}{2\pi}\right]\delta = \left(\frac{4-\pi}{2\pi}\right)\frac{\partial\delta}{\partial x}$$

$$\therefore \quad \tau_0 = \left(\frac{4-\pi}{2\pi}\right)\rho U^2 \frac{\partial\delta}{\partial x} \qquad ...(13.30)$$

τ_0 is also equal $= \mu\left(\frac{du}{dy}\right)_{\text{at } y=0}$

But $$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

$$\therefore \quad \left(\frac{du}{dy}\right) = U\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\times\frac{\pi}{2\delta}$$

$$\left(\frac{du}{dy}\right)_{y=0} = U\times\frac{\pi}{2\delta}\cos\left(\frac{\pi}{2}\times\frac{0}{\delta}\right) = \frac{U\pi}{2\delta}$$

$$\therefore \qquad \tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U \pi}{2\delta} \qquad ...(13.31)$$

Equating the two values τ_0 given by equations (13.30) and (13.31)

$$\left(\frac{4-\pi}{2\pi}\right)\rho U^2 \frac{\partial \delta}{\partial x} = \frac{\mu U \pi}{2\delta} \quad \text{or} \quad \delta \partial \delta = \frac{\mu U \pi}{2} \times \frac{2\pi}{4-\pi} \times \frac{1}{\rho U^2} \partial x$$

$$\therefore \qquad \delta \partial \delta = \frac{\pi^2}{(4-\pi)} \frac{\mu U}{\rho U^2} \,.\, \partial x = 11.4975 \frac{\mu}{\rho U} \partial x$$

Integrating, we get $\dfrac{\delta^2}{2} = 11.4975 \dfrac{\mu}{\rho U} x + C$

At $x = 0$, $\delta = 0$ and hence $C = 0$

$$\therefore \qquad \frac{\delta^2}{2} = 11.4975 \frac{\mu}{\rho U} x$$

$$\therefore \qquad \delta = \sqrt{2 \times 11.4975 \frac{\mu}{\rho U} x} = 4.795 \sqrt{\frac{\mu}{\rho U}\, x}$$

$$= 4.795 \sqrt{\frac{\mu}{\rho U x}} = 4.795 \sqrt{\frac{\mu}{\rho U x}} \times x$$

$$= \frac{4.795\, x}{\sqrt{R_{e_x}}} \qquad ...(13.32)$$

(*ii*) **Shear Stress (τ_0)**

From equation (13.31), $\tau_0 = \dfrac{\mu U \pi}{2\delta} = \dfrac{\mu U \pi}{\dfrac{2 \times 4.795\, x}{\sqrt{R_{e_x}}}} = \dfrac{\mu U \pi \sqrt{R_{e_x}}}{2 \times 4.795\, x}$

$$= \frac{\pi}{2 \times 4.795} \frac{\mu U}{x} \sqrt{R_{e_x}} = 0.327 \frac{\mu U}{x} \sqrt{R_{e_x}} \,.$$

(*iii*) **Drag force (F_D)** on one side of the plate is given by equation (13.12)

$$F_D = \int_0^L \tau_0 \times b \times dx = \int_0^L 0.327 \frac{\mu U}{x} \sqrt{R_{e_x}} \times b \times dx = 0.327\, \mu U \times b \int_0^L \frac{1}{x} \sqrt{\frac{\rho U x}{\mu}}\, dx$$

$$= 0.327\, \mu U \times b \times \sqrt{\frac{\rho U}{\mu}} \int_0^L x^{-1/2}\, dx = 0.327\, \mu U \times b \times \sqrt{\frac{\rho U}{\mu}} \left[\frac{x^{1/2}}{\frac{1}{2}}\right]_0^L$$

$$= 0.327 \times 2 \times \mu U \times b \sqrt{\frac{\rho U}{\mu}} \times \sqrt{L}$$

$$= 0.655 \times \mu U \times b \times \sqrt{\frac{\rho U L}{\mu}} \qquad ...(13.33)$$

(*iv*) **Co-efficient of drag, C_D** is given by equation (13.14),

$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2}, \text{ where } A = b \times L$$

$$\therefore \quad C_D = \frac{0.655 \times \mu U \times b \times \sqrt{\frac{\rho U L}{\mu}}}{\frac{1}{2}\rho U^2 \times b \times L} = 0.655 \times 2 \times \frac{\mu}{\rho U L} \times \sqrt{\frac{\rho U L}{\mu}}$$

$$= 1.31 \times \frac{1}{\sqrt{\frac{\rho U L}{\mu}}} = \frac{1.31}{\sqrt{R_{e_L}}} \quad \text{...(13.34)}$$

Note. $\int \sin^2 x ds = \left(\frac{x}{2} - \frac{\sin 2x}{4}\right)$ is used.

Table 13.1 shows the values of boundary layer thickness and co-efficients of drag in terms of Reynold number for various velocity distributions

Table 13.1

	Velocity Distribution	δ	C_D
1.	$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$	$5.48\, x/\sqrt{R_{e_x}}$	$1.46/\sqrt{R_{e_L}}$
2.	$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$	$4.64\, x/\sqrt{R_{e_x}}$	$1.292/\sqrt{R_{e_L}}$
3.	$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$	$5.84\, x/\sqrt{R_{e_x}}$	$1.36/\sqrt{R_{e_L}}$
4.	$\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$	$4.79\, x/\sqrt{R_{e_x}}$	$1.31/\sqrt{R_{e_L}}$
5.	Blasius's Solution	$4.91\, x/\sqrt{R_{e_x}}$	$1.328/\sqrt{R_{e_L}}$

Problem 13.8 *For the velocity profile in laminar boundary layer as,*

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

find the thickness of the boundary layer and the shear stress 1.5 m from the leading edge of a plate. The plate is 2 m long and 1.4 m wide and is placed in water which is moving with a velocity of 200 mm per second. Find the total drag force on the plate if μ for water = .01 poise.

Solution. Given :

Velocity profile is $\quad \frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$

Distance of x from leading edge, $x = 1.5$ m
Length of plate, $L = 2$ m
Width of plate, $b = 1.4$ m
Velocity of plate, $U = 200$ mm/s $= 0.2$ m/s

Viscosity of water, $\mu = 0.01 \text{ poise} = \frac{0.01}{10} = 0.001 \text{ Ns/m}^2$

For the given velocity profile, thickness of boundary layer is given by equation (13.22) as

$$\delta = \frac{4.64\, x}{\sqrt{R_{e_x}}}$$

$$\left[\text{Here } R_{e_x} = \frac{\rho U x}{\mu} = 1000 \times \frac{0.2 \times 1.5}{0.001} = 300000\right]$$

$$\delta = \frac{4.64 \times 1.5}{\sqrt{300000}} = 0.0127 \text{ m} = \mathbf{12.7 \text{ mm. Ans.}}$$

Shear stress (τ_0) is given by $\tau_0 = 0.323 \frac{\mu U}{x} \sqrt{R_{e_x}}$

$$= 0.323 \times 0.001 \times \frac{0.2}{1.5} \times \sqrt{300000} = \mathbf{0.0235 \text{ N/m}^2\text{. Ans.}}$$

Drag Force (F_D) on one side of the plate is given by (13.23) as

$$F_D = 0.646\, \mu U \sqrt{\frac{\rho U L}{\mu}} \times b$$

$$= 0.646 \times 0.001 \times 0.2 \times \sqrt{1000 \times \frac{0.2 \times 2.0}{0.001}} \times 1.4$$

$$= .646 \times 0.001 \times 0.2 \times \sqrt{400000} \times 1.4 = 0.1138 \text{ N}$$

$\therefore$ Total drag force = Drag force on both sides of the plate
= 2×0.1138 = **0.2276 N. Ans.**

Problem 13.9 *Air is flowing over a smooth plate with a velocity of 10 m/s. The length of the plate is 1.2 m and width 0.8 m. If laminar boundary layer exists up to a value of $R_e = 2 \times 10^5$, find the maximum distance from the leading edge upto which laminar boundary layer exists. Find the maximum thickness of laminar boundary layer if the velocity profile is given by*

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

Take kinematic viscosity for air = 0.15 stokes.

Solution. Given :
Velocity of air, $U = 10$ m/s
Length of plate, $L = 1.2$ m
Width of plate, $b = 0.8$ m
Reynold number upto which laminar boundary exists $= 2 \times 10^5$
ν for air $= 0.15$ stokes $= 0.15 \times 10^{-4}$ m^2/s

Reynold number $R_{e_x} = \frac{\rho U x}{\mu} = \frac{U x}{\nu}$

If $R_{e_x} = 2 \times 10^5$, then x denotes the distance from leading edge upto which laminar boundary layer exists

$$\therefore \quad 2 \times 10^5 = \frac{10 \times x}{0.15 \times 10^{-4}}$$

$$\therefore \quad x = \frac{2 \times 10^5 \times 0.15 \times 10^{-4}}{10} = 0.30 \text{ m} = \textbf{300 mm. Ans.}$$

Maximum thickness of the laminar boundary for the velocity profile, $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ is given by equation (13.17) as

$$\delta = \frac{5.48 \times x}{\sqrt{R_{e_x}}} = \frac{5.48 \times 0.30}{\sqrt{2 \times 10^5}} = 0.00367 \text{ m} = \textbf{3.67 mm. Ans.}$$

Problem 13.10 *Air is flowing over a flat plate 500 mm long and 600 mm wide with a velocity of 4 m/s. The kinematic viscosity of air is given as 0.15 × 10^{-4} m^2/s. Find (i) the boundary layer thickness at the end of the plate, (ii) Shear stress at 200 mm from the leading edge and (iii) drag force on one side of the plate. Take the velocity profile over the plate as* $\frac{u}{U} = \sin\left(\frac{\pi}{2} \cdot \frac{y}{\delta}\right)$ *and density of air 1.24 kg/m^3.*

Solution. Given :

Length of plate, $L = 500 \text{ mm} = 0.5 \text{ m}$

Width of plate, $b = 600 \text{ mm} = 0.6 \text{ m}$

Velocity of air, $U = 4 \text{ m/s}$

Kinematic viscosity, $\nu = 0.15 \times 10^{-4} \text{ m}^2\text{/s}$

∴ Mass density, $\rho = 1.24 \text{ kg/m}^3$

For the velocity profile $\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$, we have

(*i*) Boundary layer thickness at the end of the plate means value of δ at $x = 0.5$ m. First find Reynold number.

$$R_{e_x} = \frac{\rho U x}{\mu} = \frac{U x}{\nu} = \frac{4 \times 0.5}{0.15 \times 10^{-4}} = 1.33 \times 10^5.$$

Hence boundary layer is laminar over the entire length of the plate as Reynold number at the end of the plate is 1.33×10^5.

∴ δ at $x = 0.5$ m for the given velocity profile is given by equation (13.32) as

$$\delta = \frac{4.795x}{\sqrt{R_{e_x}}} = \frac{4.795 \times 0.5}{\sqrt{1.33 \times 10^5}} = 0.00656 \text{ m} = \textbf{6.56 mm.} \quad \textbf{Ans.}$$

(*ii*) Shear stress at any distance from leading edge is given by $\tau_0 = 0.327 \frac{\mu U}{x}\sqrt{R_{e_x}}$

At x = 200 mm = 0.2 m, $R_{e_x} = \dfrac{U \times x}{\nu} = \dfrac{4 \times 0.2}{0.15 \times 10^{-4}}$ = 53333

$$\therefore \qquad \tau_0 = \frac{0.327 \times \mu \times 4 \times \sqrt{53333}}{0.2}$$

But $\mu = \nu \times \rho$ $\quad \left\{\because \quad \nu = \dfrac{u}{\rho}, \quad \therefore \quad \mu = \nu \times \rho\right\}$

$$= 0.15 \times 10^{-4} \times 1.24 = 0.186 \times 10^{-4}$$

$$\therefore \qquad \tau_0 = \frac{0.327 \times 0.186 \times 10^{-4} \times 4 \times \sqrt{53333}}{0.2} = \mathbf{0.02805\ N/m^2.\ Ans.}$$

(*iii*) Drag force on one side of the plate is given by equation (13.33)

$$F_D = 0.655 \times \mu U \times b \times \sqrt{\frac{\rho UL}{\mu}}$$

$$= 0.655 \times 0.186 \times 10^{-4} \times 4.0 \times 0.6 \times \sqrt{\frac{UL}{\nu}} \qquad \left\{\because \quad \nu = \frac{\mu}{\rho}\right\}$$

$$= 0.29234 \times 10^{-4} \times \sqrt{\frac{4 \times 0.5}{.15 \times 10^{-4}}} = \mathbf{0.01086\ N.\ Ans.}$$

Problem 13.11 *A thin plate is moving in still atmospheric air at a velocity of 5 m/s. The length of the plate is 0.6 m and width 0.5 m. Calculate (i) the thickness of the boundary layer at the end of the plate, and (ii) drag force on one side of the plate. Take density of air as 1.24 kg/m³ and kinematic viscosity 0.15 stokes.*

Solution. Given :

Velocity of plate, U = 5 m/s

Length of plate, L = 0.6 m

Width of plate, b = 0.5 m

Density of air, ρ = 1.24 kg/m³

Kinematic viscosity, ν = 0.15 stokes = 0.15 × 10^{-4} m²/s

Reynold number, $R_e = \dfrac{UL}{\nu} = \dfrac{5 \times 0.6}{0.15 \times 10^{-4}}$ = 200000.

As R_e is less than 5 × 10^5, hence boundary layer is laminar over the entire length of the plate.

(*i*) Thickness of boundary layer at the end of the plate by Blasius's solution is

$$\delta = \frac{4.91x}{\sqrt{R_{e_x}}} = \frac{4.91\,L}{\sqrt{R_{e_1}}} = \frac{4.91 \times 0.6}{\sqrt{200000}} = .00658 \text{ m} = \mathbf{6.58\ mm.\ Ans.}$$

(*ii*) Drag force on one side of the plate is given by equation (13.14) as

$$C_D = \frac{F_D}{\frac{1}{2}\rho AU^2}$$

$$\therefore \qquad F_D = \frac{1}{2}\rho\, AU^2 \times C_D$$

where C_D from Blasius's solution, $C_D = \frac{1.328}{\sqrt{R_{e_L}}} = \frac{1.328}{\sqrt{200000}} = 0.002969 \simeq .00297$

$$\therefore \quad F_D = \frac{1}{2} \times 1.24 \times 0.6 \times 0.5 \times 5^2 \times .002970 \qquad \{\because A = L \times b = 0.6 \times 0.5\}$$

$$= \mathbf{0.01373\ N.\ Ans.}$$

Note. If no velocity profile is given in the numerical problem but boundary layer is laminar, then Blasius's solution is used.

Problem 13.12 *A plate of 600 mm length and 400 mm wide is immersed in a fluid of sp. gr. 0.9 and kinematic viscosity (ν =) 10^{-4} m^2/s. The fluid is moving with a velocity of 6 m/s. Determine (i) boundary layer thickness, (ii) shear stress at the end of the plate, and (iii) drag force on one side of the plate.*

Solution. As no velocity profile is given in the above problem, hence Blasius's solution will be used.

Given : length of plate, $L = 600$ mm $= 0.60$ m
Width of plate, $b = 400$ mm $= 0.40$ m
Sp. gr. of fluid, $S = 0.9$
$\therefore$ Density, $\rho = 0.9 \times 1000 = 900$ kg/m^3
Velocity of fluid, $U = 6$ m/s
Kinematic viscosity, $\nu = 10^{-4}$ m^2/s

Reynold number, $R_{e_L} = \frac{U \times L}{\nu} = \frac{6 \times 0.6}{10^{-4}} = 3.6 \times 10^4.$

As R_{e_L} is less than 5×10^5, hence boundary layer is laminar over the entire length of the plate.

(*i*) Thickness of boundary layer at the end of the plate from Blasius's solution is

$$\delta = \frac{4.91\,x}{\sqrt{R_{e_x}}}, \text{ where } x = 0.6 \text{ m and } R_{e_x} = 3.6 \times 10^4$$

$$= \frac{4.91 \times 0.6}{\sqrt{3.6 \times 10^4}} = 0.0155 \text{ m} = \mathbf{15.5\ mm.\ Ans.}$$

(*ii*) Shear stress at the end of the plate is

$$\tau_0 = 0.332\, \frac{\rho U^2}{\sqrt{R_{e_L}}} = \frac{0.332 \times 900 \times 6^2}{\sqrt{3.6 \times 10^4}} = \mathbf{56.6\ N/m^2.\ Ans.}$$

(*iii*) Drag force (F_D) on one side of the plate is given by

$$F_D = \frac{1}{2} \rho A U^2 \times C_D$$

where C_D from Blasius's solution is $C_D = \frac{1.328}{\sqrt{R_{e_L}}} = \frac{1.328}{\sqrt{3.6 \times 10^4}} = 0.00699$

$$\therefore \quad F_D = \frac{1}{2} \rho A U^2 \times C_D$$

$$= \frac{1}{2} \times 900 \times 0.6 \times 0.4 \times 6^2 \times .00699 \qquad \{\because A = L \times b = 0.6 \times .4\}$$

$$= \mathbf{26.78\ N.\ Ans.}$$

▶ 13.4 TURBULENT BOUNDARY LAYER ON A FLAT PLATE

The thickness of the boundary layer, drag force on one side of the plate and co-efficient of drag due to turbulent boundary layer on a smooth plate at zero pressure gradient are determined as in case of laminar boundary layer provided the velocity profile is known. Blasius on the basis of experiments give the following velocity profile for turbulent boundary layer

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^n \qquad ...(13.35)$$

where $n = \frac{1}{7}$ for $R_e < 10^7$ but more than 5×10^5

$$\therefore \qquad \frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \qquad ...(13.36)$$

Equation (13.36) is not applicable very near the boundary, where the thin laminar sub-layer of thickness δ' exists. Here velocity distribution is influenced only by viscous effects.

The value of τ_0 for flat plate is taken as $\tau_0 = 0.0225\, \rho\, U^2 \left(\frac{\mu}{\rho \delta U}\right)^{1/4}$ ...(13.37)

Problem 13.13 *For the velocity profile for turbulent boundary layer* $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$, *obtain an expression for boundary layer thickness, shear stress, drag force on one side of the plate and co-efficient of drag in terms of Reynold number. Given the shear stress* (τ_0) *for turbulent boundary layer as*

$$\tau_0 = 0.0225\, \rho\, U^2 \left(\frac{\mu}{\rho U \delta}\right)^{1/4}.$$

Solution. Given : $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$

(*i*) Substituting this value in Von Karman momentum integral equation (13.10),

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right] dy\right] = \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{y^{1/7}}{\delta^{1/7}} - \frac{y^{2/7}}{\delta^{2/7}}\right) dy\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{y^{1/7+1}}{\left(\frac{1}{7}+1\right)\delta^{1/7}} - \frac{y^{2/7+1}}{\left(\frac{2}{7}+1\right)\delta^{2/7}}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{7}{8}\frac{y^{8/7}}{\delta^{1/7}} - \frac{7}{9}\frac{y^{9/7}}{\delta^{2/7}}\right]_0^\delta = \frac{\partial}{\partial x}\left[\frac{7}{8}\frac{\delta^{8/7}}{\delta^{1/7}} - \frac{7}{9}\frac{\delta^{9/7}}{\delta^{2/7}}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{7}{8}\delta - \frac{7}{9}\delta\right] = \frac{\partial}{\partial x}\left[\frac{63-56}{72}\right]\delta = \frac{\partial}{\partial x}\left[\frac{7}{72}\right]\delta = \frac{7}{72}\frac{\partial\delta}{\partial x}$$

In the above expression, the integration limits should be from δ' to δ. But as the laminar sub-layer is very thin that is δ' is very small. Hence the limits of integration are taken from 0 to δ.

Now $$\tau_0 = \frac{7}{72}\rho U^2 \frac{\partial\delta}{\partial x} \quad \text{...(13.38)}$$

But the value of τ_0 for turbulent boundary layer is given,

$$\tau_0 = 0.0225\,\rho U^2 \left(\frac{\mu}{\rho U \delta}\right)^{1/4} \quad \text{...(13.39)}$$

Equating the two values of τ_0 given by equations (13.38) and (13.39), we have

$$\frac{7}{72}\rho U^2 \frac{\partial\delta}{\partial x} = .0225\,\rho U^2 \left(\frac{\mu}{\rho U \delta}\right)^{1/4}$$

or $$\frac{7}{72}\frac{\partial\delta}{\partial x} = .0225\left(\frac{\mu}{\rho U}\right)^{1/4} \times \frac{1}{\delta^{1/4}} \qquad \{\text{cancelling } \rho U^2\}$$

or $$\delta^{1/4}\partial\delta = .0225 \times \frac{72}{7} \times \left(\frac{\mu}{\rho U}\right)^{1/4} \partial x = 0.2314\left(\frac{\mu}{\rho U}\right)^{1/4} \partial x.$$

Integrating, we get $$\frac{\delta^{1/4+1}}{\left(\frac{1}{4}+1\right)} = 0.2314\left(\frac{\mu}{\rho U}\right)^{1/4} x + C$$

or $$\frac{4}{5} \times \delta^{5/4} = 0.2314\left(\frac{\mu}{\rho U}\right)^{1/4} x + C$$

where C is constant of integration.

To determine the value of C, assume turbulent boundary layer starts from the leading edge, though in actual practice the turbulent boundary layer starts after the transition from laminar boundary layer. The laminar layer exists for a very short distance and hence this assumption will not affect the subsequent analysis.

Hence at $x = 0$, $\delta = 0$ and so $C = 0$

$\therefore$ $$\frac{4}{5}\delta^{5/4} = 0.2314\left(\frac{\mu}{\rho U}\right)^{1/4} x \text{ or } \delta^{5/4} = \frac{0.2314 \times 5}{4}\left(\frac{\mu}{\rho U}\right)^{1/4} x$$

or $$\delta = \left[\frac{0.2314 \times 5}{4}\left(\frac{\mu}{\rho U}\right)^{1/4} x\right]^{4/5} = \left(\frac{0.2314 \times 5}{4}\right)^{4/5}\left(\frac{\mu}{\rho U}\right)^{1/5} x^{4/5}$$

$$= 0.37\left(\frac{\mu}{\rho U}\right)^{1/5} x^{4/5} \quad \text{...(13.40)}$$

$$= 0.37 \left(\frac{\mu}{\rho U x}\right)^{1/5} x^{1/5} \times x^{4/5} = 0.37 \left(\frac{1}{R_{e_x}}\right)^{1/5} \times x = \frac{0.37x}{\left(R_{e_x}\right)^{1/5}} \quad ...(13.41)$$

From equation (13.40), it is clear that δ varies as $x^{4/5}$ in turbulent boundary layer while in case of laminar boundary layer δ varies as $\sqrt{x}$.

(*ii*) **Shear Stress (τ_0)** at any point from leading edge is given by equation (13.40) as

$$\tau_0 = 0.225\ \rho U^2 \left(\frac{\mu}{\rho U \delta}\right)^{1/4}$$

Substituting the value of δ from equation (13.40), we have

$$\tau_0 = 0.0225\ \rho U^2 \left(\frac{\mu}{\rho U \times 0.37 \times \left(\frac{\mu}{\rho U}\right)^{1/5} \times x^{4/5}}\right)^{1/4}$$

$$= \frac{.0225 \times 2}{2} \rho U^2 \left(\frac{\mu^{4/5}}{0.37 \times (\rho U)^{4/5} \times x^{4/5}}\right)^{1/4}$$

$$= .0225 \times 2 \times \frac{\rho U^2}{2} \times \frac{1}{(0.37)^{1/4}} \left(\frac{\mu}{\rho U x}\right)^{1/5}$$

$$= 0.0577 \times \frac{\rho U^2}{2} \left(\frac{\mu}{\rho U x}\right)^{1/5} \quad ...(13.42)$$

(*iii*) **Drag force (F_D)** on one side of the plate is

$$F_D = \int_0^L \tau_0 \times b \times dx = \int_0^L 0.0577 \times \frac{\rho U^2}{2} \times \left(\frac{\mu}{\rho U}\right)^{1/5} \frac{1}{x^{1/5}} \times b \times dx$$

$$= 0.0577 \times \frac{\rho U^2}{2} \times \left(\frac{\mu}{\rho U}\right)^{1/5} \times b \int_0^L x^{-1/5}\, dx$$

$$= .0577 \times \frac{\rho U^2}{2} \times \left(\frac{\mu}{\rho U}\right)^{1/5} \times b \times \left[\frac{x^{4/5}}{4/5}\right]_0^L$$

$$= .0577 \times \frac{5}{4} \times \frac{\rho U^2}{2} \left(\frac{\mu}{\rho U}\right)^{1/5} \times b \times L^{4/5}$$

$$= 0.072 \times \frac{\rho U^2}{2} \times \left(\frac{\mu}{\rho U}\right)^{1/5} \times b \times L^{4/5}$$

(*iv*) **Drag co-efficient, C_D** is given by

$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2}, \text{ where } A = L \times b$$

$$= \frac{.072 \times \frac{\rho U^2}{2} \times \left(\frac{\mu}{\rho U}\right)^{1/5} \times b \times L^{4/5}}{\frac{\rho U^2}{2} \times b \times L}$$

$$= 0.072 \times \left(\frac{\mu}{\rho U}\right)^{1/5} \cdot \frac{1}{L^{1/5}} = 0.072 \left(\frac{\mu}{\rho UL}\right)^{1/5}$$

$$= \frac{.072}{R_{e_L}^{1/5}} \qquad \text{...(13.43)} \quad \left\{\because \quad R_{e_L} = \frac{\rho UL}{\mu}\right\}$$

This is valid for $R_{e_L} > 5 \times 10^5$ but less than 10^7.

▶ 13.5 ANALYSIS OF TURBULENT BOUNDARY LAYER

(*a*) If Reynold number is more than 5×10^5 and less than 10^7 the thickness of boundary layer and drag co-efficient are given as :

$$\delta = \frac{0.37x}{\left(R_{e_x}\right)^{1/5}} \text{ and } C_D = \frac{0.072}{\left(R_{e_L}\right)^{1/5}} \qquad \text{...(13.44)}$$

where x = Distance from the leading edge

R_{e_x} = Reynold number for length x

R_{e_L} = Reynold number at the end of the plate.

(*b*) If Reynold number is more than 10^7 but less than 10^9, Schlichting gave the empirical equation as

$$C_D = \frac{0.455}{\left(\log_{10} R_{e_L}\right)^{2.58}} \qquad \text{...(13.44}A\text{)}$$

▶ 13.6 TOTAL DRAG ON A FLAT PLATE DUE TO LAMINAR AND TURBULENT BOUNDARY LAYER

Consider the flow over a flat plate as shown in Fig. 13.5.

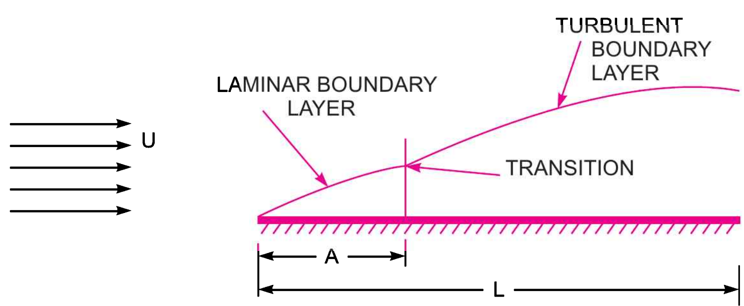

Fig. 13.5 *Drag due to laminar and turbulent boundary layer.*

Let L = Total length of the plate, b = Width of plate,

A = Length of laminar boundary layer

If the length of transition region is assumed negligible, then

$L - A$ = Length of turbulent boundary layer.

We have obtained the drag on a flat plate for the laminar as well as turbulent boundary layer on the assumption that turbulent boundary layer starts from the leading edge. This assumption is valid only when the length of laminar boundary layer is negligible. But if the length of laminar boundary layer is not negligible, then the total drag on the plate due to laminar and turbulent boundary layer is calculated as :

(1) Find the length from the leading edge upto which laminar boundary layer exists. This is done by equating $5 \times 10^5 = \frac{Ux}{\nu}$. The value of x gives the length of laminar boundary layer. Let this length is equal to A.

(2) Find drag using Blasius solution for laminar boundary layer for length A.

(3) Find drag due to turbulent boundary layer for the whole length of the plate.

(4) Find the drag due to turbulent boundary layer for a length A only

Then total drag on the plate

= Drag given by (2) + Drag given by (3) – Drag given by (4)

= Drag due to laminar boundary layer for length A

+ Drag due to turbulent boundary layer for length L

– Drag due to turbulent boundary layer for length A. ...(13.45)

Problem 13.14 (S.I. Units). *Determine the thickness of the boundary layer at the trailing edge of smooth plate of length 4 m and of width 1.5 m, when the plate is moving with a velocity of 4 m/s in stationary air. Take kinematic viscosity of air as 1.5×10^{-5} m²/s.*

Solution. Given :

Length of plate, $L = 4$ m

Width of plate, $b = 1.5$ m

Velocity of plate, $U = 4$ m/s

Kinematic viscosity, $\nu = 1.5 \times 10^{-5}$ m²/s

Reynold number, $R_{e_L} = \frac{U \times L}{\nu} = \frac{4.0 \times .40}{1.5 \times 10^{-5}} = 10.66 \times 10^5$

As the Reynold number is more than 5×10^5 and hence the boundary layer at the trailing edge is turbulent.

The boundary layer thickness for turbulent boundary layer is given by equation (13.44) as

$$\delta = \frac{0.37x}{\left(R_{e_x}\right)^{1/5}} \qquad \text{| Here } x = L \text{ and } R_{e_x} = R_{e_L}$$

$$= \frac{0.37 \times 4.0}{\left(10.66 \times 10^5\right)^{1/5}} = 0.0921 \text{ m} = \mathbf{92.1 \text{ mm. Ans.}}$$

Problem 13.15 *In Problem 13.14, determine the total drag on one side of the plate assuming that (i) the boundary layer is laminar over the entire length of the plate and (ii) the boundary layer is turbulent from the very beginning. Take ρ for air = 1.226 kg/m³.*

Solution. The data of problem 13.14,

$$L = 4 \text{ m}, b = 1.5 \text{ m}, U = 4 \text{ m/s},$$
$$\nu = 1.5 \times 10^{-5} \text{ m}^2/\text{s}$$
$$R_{e_L} = 10.66 \times 10^5 \text{ and } \rho = 1.226 \text{ kg/m}^3.$$

(*i*) **When the boundary layer is laminar over the entire length**, the value of C_D is given by Blasius's solution as

$$C_D = \frac{1.328}{\sqrt{R_{e_L}}} = \frac{1.328}{\sqrt{10.66 \times 10^5}} = .001286$$

∴ Drag force (F_D) on one side of the plate is

$$F_D = \frac{1}{2} \rho A U^2 \times C_D$$

where $A = b \times L = 1.5 \times 4 = 6.0 \text{ m}^2$

$$= \frac{1}{2} \times 1.226 \times 6.0 \times 4^2 \times .001286 = \mathbf{0.0757 \text{ N. Ans.}}$$

(*ii*) **When the boundary layer is turbulent from the very beginning**, the value of co-efficient of drag, C_D is given by equation (13.43) as

$$C_D = \frac{0.072}{\left(R_{e_L}\right)^{1/5}} = \frac{0.072}{\left(10.66 \times 10^5\right)^{1/5}} = .00448$$

∴ Drag force, $$F_D = \frac{1}{2} \rho A U^2 \times C_D$$

$$= \frac{1}{2} \times 1.226 \times 6.0 \times 4^2 \times .00448 \qquad \{\because A = b \times L = 1.5 \times 4 = 6 \text{ m}^2\}$$

$$= \mathbf{0.2637 \text{ N. Ans.}}$$

Problem 13.16 *Water is flowing over a thin smooth plate of length 4 m and width 2 m at a velocity of 1.0 m/s. If the boundary layer flow changes from laminar to turbulent at a Reynold number 5×10^5, find (i) the distance from leading edge upto which boundary layer is laminar, (ii) the thickness of the boundary layer at the transition point, and (iii) the drag force on one side of the plate. Take viscosity of water $\mu = 9.81 \times 10^{-4}$ Ns/m^2.*

Solution. Given :

Length of plate, $L = 4$ m
Width of plate, $b = 2$ m
Velocity of flow, $U = 1.0$ m/s
Reynold number for laminar boundary layer = 5×10^5
Viscosity of water, $\mu = 9.81 \times 10^{-4} \frac{\text{Ns}}{\text{m}^2}$

(*i*) Let the distance from leading edge upto which laminar boundary layer exists = x

∴ $$5 \times 10^5 = \frac{\rho U x}{\mu} = 1000 \times \frac{1.0 \times x}{9.81 \times 10^{-4}} \qquad (\because \rho = 1000)$$

∴ $$x = \frac{5 \times 10^5 \times 9.81 \times 10^{-4}}{1000} = 0.4900 \text{ m} = \mathbf{490 \text{ mm. Ans.}}$$

(*ii*) Thickness of boundary layer at the point where the boundary layer changes from laminar to turbulent *i.e.*, at Reynold number = 5×10^5, is given by Blasius's solution as

$$\delta = \frac{4.91 \times x}{\sqrt{R_{e_x}}} \qquad \text{| Here } x = 49 \text{ cm} = 0.49 \text{ m}, \; R_{e_x} = 5 \times 10^5$$

$$\delta = \frac{4.91 \times 0.49}{\sqrt{5 \times 10^5}} = 0.0034 \text{ m} = \mathbf{3.4 \text{ mm. Ans.}}$$

(*iii*) Drag force on the plate on one side

= Drag due to laminar boundary layer + Drag due to turbulent boundary.

(*a*) Drag due to laminar boundary layer (*i.e.,* from *E* to *F*)

$$F_{EF} = \frac{1}{2} \rho A U^2 \times C_D \qquad \text{...(}i\text{)}$$

where C_D is given by Blasius solution for laminar boundary layer as

$$C_D = \frac{1.328}{\sqrt{R_{e_x}}}, \text{ where for } EF, \; R_{e_x} = 5 \times 10^5$$

$$= \frac{1.328}{\sqrt{5 \times 10^5}} = 0.001878$$

A = Area of plate upto laminar boundary layer
= $0.49 \times b = 0.49 \times 2 = 0.98$ m^2

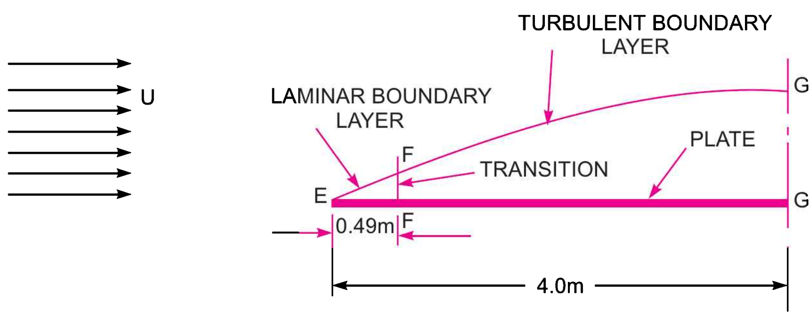

Fig. 13.6 (*a*)

Substituting the value of C_D and A in equation (*i*), we get

$$F_{EF} = \frac{1}{2} \times 1000 \times 0.98 \times 1.0^2 \times .001878 = \mathbf{0.92 \text{ N}}. \qquad \text{...(}ii\text{)}$$

(*b*) Drag force due to turbulent boundary layer from *F* to *G*

= Drag force due to turbulent boundary layer from *E* to *G*

– Drag force due to turbulent flow from *E* to *F*

= $(F_{EG})_{\text{turb.}} - (F_{EF})_{\text{turb.}}$

Now $\quad (F_{FG})_{\text{turb.}} = \frac{1}{2} \rho A U^2 \times C_D$

where C_D from equation (13.44) is $C_D = \frac{0.072}{(R_{e_L})^{1/5}}$

But $$R_{e_L} = \frac{\rho UL}{\mu} = 1000 \times \frac{1.0 \times 4.0}{9.81 \times 10^{-4}} = 40.77 \times 10^5$$

$\therefore$ $$C_D = \frac{0.072}{\left(40.77 \times 10^5\right)^{1/5}} = 0.00343$$

$\therefore$ $$(F_{EG})_{\text{turb.}} = \frac{1}{2}\,\rho AU^2 \times C_D = \frac{1}{2} \times 1000 \times (4 \times 2) \times 1^2 \times .00343 = \mathbf{13.72\ N}$$

Also $$(F_{EF})_{\text{turb.}} = \frac{1}{2}\,\rho A_{EF} \times U^2 \times C_D$$

where A_{EF} = Area of plate upto $EF = EF \times b = 0.49 \times 2 = 0.98$ m^2

and $$C_D = \frac{0.072}{\left(R_{EF}\right)^{1/5}} = \frac{0.072}{\left(5 \times 10^5\right)^{1/5}} = .00522$$

$$(F_{EF})_{\text{turb.}} = \frac{1}{2} \times 1000 \times 0.98 \times 1^2 \times .00522 = \mathbf{2.557\ N}$$

$\therefore$ Drag force due to turbulent boundary layer from F to G

$= (F_{EG})_{\text{turb.}} - (F_{EF})_{\text{turb.}} = 13.72 - 2.557 = 11.163$ N

$\therefore$ Drag force on the plate on one side

= Drag force due to laminar boundary layer upto F

+ Drag due to turbulent boundary layer from F to G

= 0.92 + 11.163 = **12.083 N. Ans.**

Problem 13.16 (A) *Air flows at 10 m/s past a smooth rectangular flat plate 0.3 m wide and 3 m long. Assuming that the turbulence level in the oncoming stream is low and that transition ocurs at* $R_e = 5 \times 10^5$, *calculate ratio of total drag when the flow is parallel to the length of the plate to the value when the flow is parallel to the width.* (R.G.P.V., Bhopal S 2001)

Solution. Given :

$U = 10$ m/s ; $b = 0.3$ m ; $L = 3$ m ;

Reynolds number for laminar B.L. = 5×10^5.

The kinematic viscosity of air and density of air may be assumed as their values are not given in the question. Take $\rho = 1.24$ kg/m^2 and $\nu = 0.15$ stoke

$\therefore$ $\rho = 1.24$ kg/m^3 and $\nu = 0.15$ stoke $= 0.15 \times 10^{-4}$ m^2/s.

(*i*) **Drag when flow is parallel to the length of the plate**

Let x = the distance from leading edge upto which laminar boundary exists

$\therefore$ $$5 \times 10^5 = \frac{\rho \times U \times x}{\mu} = \frac{U \times x}{\nu} = \frac{10 \times x}{0.15 \times 10^{-4}}$$

$\therefore$ $$x = \frac{5 \times 10^5 \times 0.15 \times 10^{-4}}{10} = 0.75 \text{ m}$$

Now the drag force on the plate on one side

= Drag due to laminar boundary layer + Drag due to turbulent boundary layer ...(*i*)

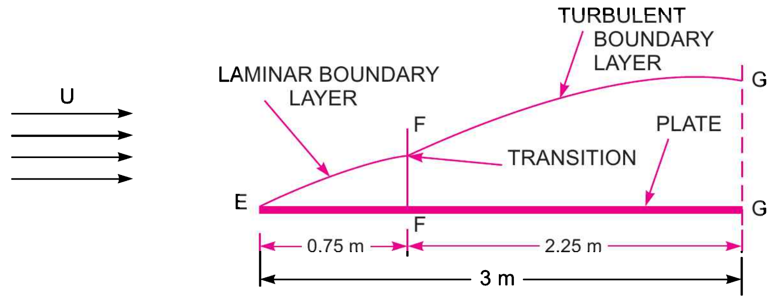

Fig. 13.6 (*b*)

(*a*) **Drag due to laminar boundary layer (*i.e.*, from E to F)**

$$F_{EF} = \frac{1}{2}\,\rho A U^2 \times C_D$$

where C_D is given by Blasius solution for laminar boundary layer as

$$C_D = \frac{1.328}{\sqrt{R_{e_x}}}, \text{ where } R_{ex} = 5 \times 10^5$$

$$= \frac{1.328}{\sqrt{5 \times 10^5}} = 0.001878$$

A = Area of plate upto laminar boundary layer

$= 0.75 \times b = 0.75 \times 0.3 = 0.225$ m^2

$\rho = 1.24$ kg/m^3

$$\therefore \quad F_{EF} = \frac{1}{2} \times 1.24 \times 0.225 \times 10^2 \times 0.001878 = 0.0262 \text{ N}$$

(*b*) **Drag force due to turbulent boundary layer from F to G**

= Drag force due to turbulent boundary layer from E to G

– Drag force due to turbulent B.L. from E to F

$= (F_{EG})_{\text{turb.}} - (F_{EF})_{\text{turb.}}$

Now $$(F_{EG})_{\text{turb.}} = \frac{1}{2}\,\rho A U^2 \times C_D$$

where C_D for turbulent boundary layer is given by equation (13.44) as

$$C_D = \frac{0.072}{\left(R_{e_L}\right)^{1/5}}$$

But $$R_{e_L} = \frac{U \times L}{\nu} = \frac{10 \times 3}{0.15 \times 10^{-4}} = 20 \times 10^5$$

$$\therefore \quad C_D = \frac{0.072}{\left(20 \times 10^5\right)^{1/5}} = 0.00395$$

$$\therefore \quad (F_{EG})_{\text{turb.}} = \frac{1}{2}\,\rho A U^2 \times C_D = \frac{1}{2} \times 1.24 \times (3 \times 0.3) \times 10^2 \times 0.00395$$

$= \mathbf{0.2204\ N}$

Now $$(F_{EF})_{\text{turb.}} = \frac{1}{2}\rho \times A_{EF} \times U^2 \times C_D$$

where A_{EF} = Area of plate upto $EF = EF \times b = 0.75 \times 0.3 = 0.225 \text{ m}^2$

and $$C_D = \frac{0.072}{\left[(R_e)_{EF}\right]^{1/5}} = \frac{0.072}{\left(5 \times 10^5\right)^{1/5}} = 0.00522$$

$\therefore$ $$(F_{EF})_{\text{turb.}} = \frac{1}{2} \times 1.24 \times 0.225 \times 10^2 \times 0.00522 = \mathbf{0.0728\ N}$$

$\therefore$ Drag force due to turbulent boundary layer from F to G

$= (F_{EG})_{\text{turb}} - (F_{EF})_{\text{turb}} = 0.2204 - 0.0728 = $ **0.1476 N**

$\therefore$ Total drag force when flow is parallel to the length of the plate

= Drag due to laminar boundary layer upto F

+ Drag due to turbulent boundary layer from F to G

$= 0.0262 + 0.1476 = $ **0.1738 N** ...(A)

(*ii*) **Drag when flow is parallel to the width of the plate**

We have already calculated that upto the length of 0.75 m from the leading edge, the boundary layer is laminar. As the width of the plate is only 0.3 m, hence when flow is parallel to the width of the plate, only laminar boundary layer will be formed.

$\therefore$ Drag force on the plate

$$= \frac{1}{2}\rho A U^2 \times C_D$$

where C_D from Blasius solution for laminar boundary layer is given as

$$C_D = \frac{1.328}{\sqrt{R_{e_x}}}, \text{ here } \quad x = \text{width of plate} = 0.3 \text{ m hence}$$

$$R_{e_x} = \frac{U \times x}{\nu} = \frac{10 \times 0.3}{0.15 \times 10^{-4}} = 2 \times 10^5$$

$$= \frac{1.328}{\sqrt{2 \times 10^5}} = 0.00297$$

A = Area of plate upto width (0.3 m) $= 3 \times 0.3 = 0.9$

$\rho = 1.24 \text{ kg/m}^3$

$\therefore$ Total drag on the plate $$= \frac{1}{2} \times 1.24 \times 0.9 \times 10^2 \times 0.00297$$

= **0.1657 N** ...(B)

$\therefore$ Ratio of two total drags given by equations (A) and (B) becomes as

$$\frac{\text{Total drag when flow is parallel to the length of the plate}}{\text{Total drag when flow is parallel to the width of the plate}} = \frac{\text{Equation }(A)}{\text{Equation }(B)} = \frac{0.1738}{0.1657} = \mathbf{1.05.\ Ans.}$$

Problem 13.17 *Oil with a free-stream velocity of 2 m/s flows over a thin plate 2 m wide and 2 m long. Calculate the boundary layer thickness and the shear stress at the trailing end point and determine the total surface resistance of the plate. Take specific gravity as 0.86 and kinematic viscosity as* 10^{-5} m^2/s.

Solution. Given :

Free-stream velocity of oil, $U = 2$ m/s

Width of plate, $b = 2$ m

Length of plate, $L = 2$ m

∴ Area of plate, $A = b \times L = 2 \times 2 = 4\ \text{m}^2$

Specific gravity of oil, $S = 0.86$

∴ Density of oil, $\rho = 0.86 \times 1000 = 860\ \text{kg/m}^3$

Kinematic viscosity, $\nu = 10^{-5}\ \text{m}^2/\text{s}$

Now the Reynold number at the trailing end,

$$R_{e_L} = \frac{UL}{\nu} = \frac{2 \times 2}{10^{-5}} = 4 \times 10^5.$$

Since R_{e_L} is less than 5×10^5, the boundary layer is laminar over the entire length of the plate.

∴ Thickness of boundary layer at the end of the plate from Blasius's solution is,

$$\delta = \frac{4.91 \times L}{\sqrt{R_{e_L}}} = \frac{4.91 \times 2.0}{\sqrt{4 \times 10^5}} = 0.0155\ \text{m} = \mathbf{15.5\ mm.\ Ans.}$$

Shear stress at the end of the plate is, $\tau_0 = 0.332 \times$ = **1.805 N/m². Ans.**

Surface resistance on one side of the plate is given by

$$F_D = \frac{1}{2} \rho A U^2 \times C_D$$

where $$C_D = \frac{1.328}{\sqrt{R_{e_L}}} = \frac{1.328}{\sqrt{4 \times 10^5}} = 0.0021$$

∴ $$F_D = \frac{1}{2} \times 860 \times 4.0 \times 2^2 \times .0021 = 14.44\ \text{N}$$

∴ Total resistance $= 2 \times F_D = 2 \times 14.44 =$ **28.88 N. Ans.**

▶ 13.7 SEPARATION OF BOUNDARY LAYER

When a solid body is immersed in a flowing fluid, a thin layer of fluid called the boundary layer is formed adjacent to the solid body. In this thin layer of fluid, the velocity varies from zero to free-stream velocity in the direction normal to the solid body. Along the length of the solid body, the thickness of the boundary layer increases. The fluid layer adjacent to the solid surface has to do work against surface friction at the expense of its kinetic energy. This loss of the kinetic energy is recovered from the immediate fluid layer in contact with the layer adjacent to solid surface through momentum exchange process. Thus the velocity of the layer goes on decreasing. Along the length of the solid body, at a certain point a stage may come when the boundary layer may not be able to keep sticking to the solid body if it cannot provide kinetic energy to overcome the resistance offered by the solid body. In other words, the boundary layer will be separated from the surface. This phenomenon is called the boundary layer separation. The point on the body at which the boundary layer is on the verge of separation from the surface is called point of separation.

13.7.1 Effect of Pressure Gradient on Boundary Layer Separation. The effect of pressure gradient $\left(\frac{dp}{dx}\right)$ on boundary layer separation can be explained by considering the flow over a curved surface *ABCSD* as shown in Fig. 13.7. In the region *ABC* of the curved surface, the area of flow decreases and hence velocity increases. This means that flow gets accelerated in this region. Due to the increase of the velocity, the pressure decreases in the direction of the flow and hence pressure gradient $\frac{dp}{dx}$ is negative in this region. As long as $\frac{dp}{dx} < 0$, the entire boundary layer moves forward as shown in Fig. 13.7.

Region *CSD* of the curved surface. The pressure is minimum at the point *C*. Along the region *CSD* of the curved surface, the area of flow increases and hence velocity of flow along the direction of fluid decreases. Due to decrease of velocity, the pressure increases in the direction of flow and hence pressure gradient $\frac{dp}{dx}$ is positive or $\frac{dp}{dx} > 0$. Thus in the region *CSD*, the pressure gradient is positive and velocity of fluid layer along the direction of flow decreases. As explained in the Art. 13.7, the velocity of the layer adjacent to the solid surface along the length of the solid surface goes on decreasing as the kinetic energy of the layer is used to overcome the frictional resistance of the surface. Thus the combined effect of positive pressure gradient and surface resistance reduce the momentum of the fluid is unable to the surface. A stage comes, when the momentum of the fluid is unable to overcome the surface resistance and the boundary layer starts separating from the surface at the point *S*. Downstream the point *S*, the flow is taking place in reverse direction and the velocity gradient becomes negative.

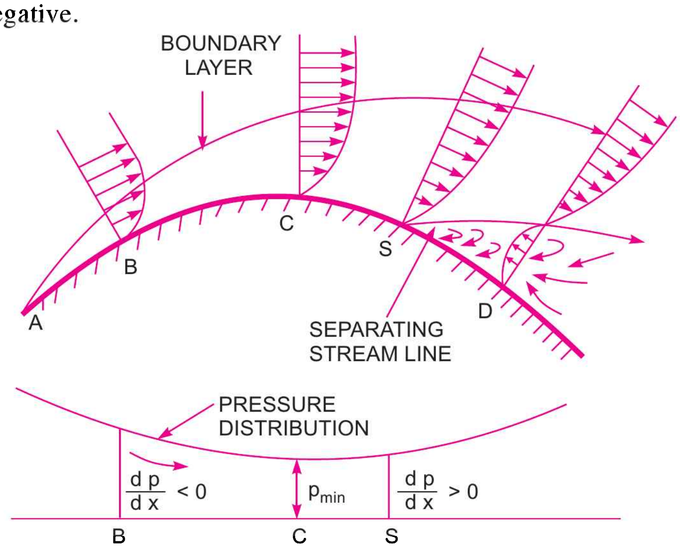

Fig. 13.7 *Effect of pressure gradient on boundary layer separation.*

Thus the positive pressure gradient helps in separating the boundary layer.

13.7.2 Location of Separation Point.

The separation point *S* is determined from the condition,

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \qquad \text{...(13.46)}$$

For a given velocity profile, it can be determined whether the boundary layer has separated, or on the verge of separation or will not separate from the following conditions :

1. If $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is negative ... the flow has separated.

2. If $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ = 0 ... the flow is on the verge of separation.

3. If $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is positive ... the flow will not separate or flow will remain attached with the surface.

Problem 13.18 *For the following velocity profiles, determine whether the flow has separated or on the verge of separation or will attach with the surface :*

(*i*) $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$, (*ii*) $\frac{u}{U} = 2\left(\frac{y}{\delta}\right)^2 - \left(\frac{y}{\delta}\right)^3$,

(*iii*) $\frac{u}{U} = -2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2$.

Solution. Given :

1st Velocity Profile

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \quad \text{or} \quad u = \frac{3U}{2}\left(\frac{y}{\delta}\right) - \frac{U}{2}\left(\frac{y}{\delta}\right)^3$$

Differentiating w.r.t. y, the above equation becomes,

$$\frac{\partial u}{\partial y} = \frac{3U}{2} \times \frac{1}{\delta} - \frac{U}{2} \times 3\left(\frac{y}{\delta}\right)^2 \times \frac{1}{\delta}$$

At $y = 0$, $$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{3U}{2\delta} - \frac{3U}{2}\left(\frac{0}{\delta}\right)^2 \times \frac{1}{\delta} = \frac{3U}{2\delta}.$$

As $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is positive. Hence flow will not separate or flow will remain attached with the surface.

2nd Velocity Profile

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right)^2 - \left(\frac{y}{\delta}\right)^3$$

$$\therefore \quad u = 2U\left(\frac{y}{\delta}\right)^2 - U\left(\frac{y}{\delta}\right)^3$$

$$\therefore \quad \frac{\partial u}{\partial y} = 2U \times 2\left(\frac{y}{\delta}\right) \times \frac{1}{\delta} - U \times 3\left(\frac{y}{\delta}\right)^2 \times \frac{1}{\delta}$$

at $y = 0$, $$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 2U \times 2\left(\frac{0}{\delta}\right) \times \frac{1}{\delta} - U \times 3\left(\frac{0}{\delta}\right)^2 \times \frac{1}{\delta} = 0$$

As $\left(\dfrac{\partial u}{\partial y}\right)_{y=0} = 0$, the flow is on the verge of separation. **Ans.**

3rd Velocity Profile

$$\frac{u}{U} = -2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2$$

$$\therefore \qquad u = -2U\left(\frac{y}{\delta}\right) + U\left(\frac{y}{\delta}\right)^2$$

$$\therefore \qquad \frac{\partial u}{\partial y} = -2U\left(\frac{1}{\delta}\right) + 2U\left(\frac{y}{\delta}\right) \times \frac{1}{\delta}$$

at $y = 0$, $$\left(\frac{\partial u}{\partial y}\right)_{y=0} = -\frac{2U}{\delta} + 2U\left(\frac{0}{\delta}\right) \times \frac{1}{\delta} = -\frac{2U}{\delta}$$

As $\left(\dfrac{\partial u}{\partial y}\right)_{y=0}$ is negative the flow has separated. **Ans.**

13.7.3 Methods of Preventing the Separation of Boundary Layer. When the boundary layer separates from the surface as shown in Fig. 13.7 at point *S*, a certain portion adjacent to the surface has a back flow and eddies are continuously formed in this region and hence continuous loss of energy takes place. Thus separation of boundary layer is undesirable and attempts should be made to avoid separation by various methods. The following are the methods for preventing the separation of boundary layer :

1. Suction of the slow moving fluid by a suction slot.
2. Supplying additional energy from a blower.
3. Providing a bypass in the slotted wing.
4. Rotating boundary in the direction of flow.
5. Providing small divergence in a diffuser.
6. Providing guide-blades in a bend.
7. Providing a trip-wire ring in the laminar region for the flow over a sphere.

HIGHLIGHTS

1. When a solid body is immersed in a flowing fluid, there is a narrow region of the fluid in the neighbourhood of the solid body, where the velocity of fluid varies from zero to free-stream velocity. This narrow region of fluid is called boundary layer.
2. The boundary layer is called laminar boundary layer if the Reynold number of the flow defined as

   $$R_e = \frac{U \times x}{\nu} \text{ is less than } 5 \times 10^5$$

   where U = Free-stream velocity of flow, x = Distance from leading edge,
   and ν = Kinematic viscosity of fluid.
3. If the Reynold number is more than 5×10^5, the boundary layer is called turbulent boundary layer.

4. The distance from the surface of the solid body in the direction perpendicular to flow, where the velocity of fluid is approximately equal to 0.99 times the free-stream velocity is called boundary layer thickness and is denoted by δ. For different zones, δ is represented as

$\delta_{lam.}$ = Thickness of laminar boundary layer
$\delta_{tur.}$ = Thickness of turbulent boundary layer
δ' = Thickness of laminar sub-layer.

5. Displacement thickness (δ^*) is given by $\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$.

6. Momentum thickness (θ) is given by $\theta = \int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right] dy$.

7. Energy thickness (δ^{**}) is given by $\delta^{**} = \int_0^\delta \frac{u}{U}\left[1 - \frac{u^2}{U^2}\right] dy$.

8. Von Karman momentum integral equation is given as $\frac{\tau_0}{\rho U^2} = \frac{\partial\theta}{\partial x}$

where $\theta = \int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right] dy$, τ_0 = shear stress at surface.

This equation is applicable to laminar, transition and turbulent boundary layer flows.

9. Thickness of laminar boundary layer and co-efficient of drag from Blasius's solution is given as

$$\delta = \frac{4.91\, x}{\sqrt{R_{e_x}}}$$

where R_{e_x} = Reynold number, $C_D = \frac{1.328}{\sqrt{R_{e_L}}}$

10. Velocity profile for turbulent boundary layer is $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$

This equation is not valid very near the boundary, where laminar sub-layer exists.

11. The shear stress at the boundary for turbulent boundary layer over a flat plate is given as

$$\tau_0 = 0.0225\, \rho U^2 \left(\frac{\mu}{\rho U \delta}\right)^{1/4}$$

12. Total drag on a flat plate due to laminar and turbulent boundary layer flows = Drag due to laminar boundary layer upto distance x + Drag due to turbulent boundary layer for length L – Drag due to turbulent boundary layer for length x.

$$\left[\text{where } x \text{ is given by } 5 \times 10^5 = \frac{Ux}{\nu}\right]$$

13. If the pressure gradient is positive, the boundary layer separates from the surface and back flow and eddies formation take place due to which a great loss of energy occur.

14. The conditions for separation, attached flow and detached flow are :

(*i*) $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$ ..... condition for separation

(*ii*) $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ = positive ..... condition for attached flow

(*iii*) $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ = negative ..... condition for detached flow.

EXERCISE

(A) THEORETICAL PROBLEMS

1. What do you understand by the terms boundary layer, and boundary layer theory ?

2. Define : laminar boundary layer, turbulent boundary layer, laminar sub-layer and boundary layer thickness.

3. Define displacement thickness. Derive an expression for the displacement thickness.

4. Prove that the momentum thickness and energy thickness for boundary layer flows are given by

$$\theta = \int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right] dy \quad \text{and} \quad \delta^{**} = \int_0^\delta \frac{u}{U}\left[1 - \frac{u^2}{U}\right] dy.$$

5. Obtain an expression for the boundary shear stress in terms of momentum thickness.

6. Obtain Von Karman momentum integral equation.

7. What are the boundary conditions that must be satisfied by a given velocity profile in laminar boundary layer flows ?

8. How will you find the drag on a flat plate due to laminar and turbulent boundary layers ?

9. What do you mean by separation of boundary layer ? What is the effect of pressure gradient on boundary layer separation ?

10. How will you determine whether a boundary layer flow is attached flow, detached flow or on the verge of separation ?

11. What are the different methods of preventing the separation of boundary layers ?

12. What is meant by boundary layer ? Why does it increase with distance from the upstream edge ?

13. Define the terms : boundary layer, boundary layer thickness, drag, lift and momentum thickness.

14. What do you mean by boundary layer separation ? What is the effect of pressure gradient on boundary layer separation ? (*R.G.P.V., Bhopal S, 2001*)

(B) NUMERICAL PROBLEMS

1. (*a*) Find the ratios of displacement thickness to momentum thickness and momentum thickness to energy thickness for the velocity distribution in the boundary layer given by

$$\frac{u}{U} = 2\,(y/\delta) - (y/\delta)^2$$

where u = Velocity in boundary layer at a distance y

U = Free-stream velocity

δ = Boundary layer thickness **[Ans.** 2.5, 7/11]

(*b*) Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2.$$

(*Delhi University, December, 2002*)

2. For the velocity profile in laminar boundary layer given as $\frac{u}{U}=\frac{3}{2}(y/\delta)-\frac{1}{2}(y/\delta)^3$, find the thickness of the boundary layer and shear stress 1.8 m from the leading edge of a plate. The plate is 2.5 m long and 1.5 m wide and is placed in water which is moving with a velocity of 15 cm per second. Find the drag on one side of the plate if the viscosity of water = 0.01 poise. [**Ans.** 1.6 cm, 0.0014 N/m^2, 0.0889 N]
3. Air is flowing over a smooth plate with a velocity of 8 m/s. The length of the plate is 1.5 m and width 1 m. If the laminar boundary exists upto a value of Reynold number = 5×10^5, find the maximum distance from the leading edge upto which laminar boundary layer exists. Find the maximum thickness of laminar boundary layer if the velocity profile is given by

$$\frac{u}{U} = (y/\delta) - (y/\delta)^2.$$ Take ν for air = 0.15 stokes. [**Ans.** 0.9375 m, 7.26 mm]

4. If in Problem 3, the velocity profile over the plate is given as $\frac{u}{U} = \sin\left(\frac{\pi}{2} \times \frac{y}{\delta}\right)$ and density of air as 1.24 kg/m^3, find : (*i*) maximum thickness of the laminar boundary layer, (*ii*) shear stress at 20 cm from the leading edge and (*iii*) drag force on one side of the plate assuming the laminar boundary layer over the entire length of the plate. [**Ans.** (*i*) 0.635 cm, (*ii*) 0.099 N/m^2, (*iii*) 0.0871 N]
5. A thin plate is moving in still atmospheric air at a velocity of 4 m/s. The length of the plate is 0.5 m and width 0.4 m. Calculate the (*i*) thickness of the boundary layer at the end of the plate and (*ii*) drag force on one side of the plate. Take density of air as 1.25 kg/m^3 and kinematic viscosity 0.15 stokes. [**Ans.** (*i*) 0.672 cm (*ii*) 0.00728 N]
6. Find the frictional drag on one side of the plate 200 mm wide and 500 mm long placed longitudinally in a steam of crude oil (specific gravity = 0.925, kinematic viscosity = 0.9 stoke) flowing with undisturbed velocity of 5 m/s. Also find the thickness of boundary layer and the shear stress at the trailing edge of the plate. [**Ans.** 9.34 N, 14.75 mm]
7. A smooth flat plate of length 5 m and width 2 m is moving with a velocity of 4 m/s in stationary air of density as 1.25 kg/m^3 and kinematic viscosity 1.5×10^{-5} m^2/s. Determine thickness of the boundary layer at the trailing edge of the smooth plate. Find the total drag on one side of the plate assuming that the boundary layer is turbulent from the very beginning. [**Ans.** 110 mm, 0.43 N]
8. Water is flowing over a thin smooth plate of length 4.5 m and width 2.5 m at a velocity of 0.9 m/s. If the boundary layer flow changes from laminar to turbulent at a Reynold number 5×10^5, find (*i*) the distance from leading edge upto, which boundary layer is laminar, (*ii*) thickness of the boundary layer at the transition point, and (*iii*) the drag force on-one side of the plate. Take viscosity of water as 0.01 poise. [**Ans.** (*i*) 555 mm (*ii*) 3.85 mm (*iii*) 13.75 N]
9. For the velocity profile given below, state whether the boundary layer has separated or on the verge of separation or will remain attached with the surface :

(*i*) $\frac{u}{U} = 2\,(y/\delta) - (y/\delta)^2$ (*ii*) $\frac{u}{U} = -\,2\,(y/\delta) + \frac{1}{2}\,(y/\delta)^3$ and

(*iii*) $\frac{u}{U} = \frac{3}{2}\,(y/\delta)^2 + \frac{1}{2}(y/\delta)^3$.

[**Ans**. (*i*) Remain attached (*ii*) has separated (*iii*) on the verge of separation]

10. Oil with a free-stream velocity of 1.5 m/s flow over a thin plate 1.4 m wide and 2.2 m long. Calculate the boundary layer thickness and the shear stress at the trailing end point and determine the total surface resistance of the plate. Take specific gravity of oil as 0.80 and kinematic viscosity as 0.1 stoke. [**Ans.** 1.88 cm, 1.04 N/cm^2, 12.8 N]

11. (*a*) For the velocity profile $\frac{u}{U}=\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^2$. Calculate the co-efficient of drag in terms of Reynolds number.

(*b*) A thin smooth plate of 0.3 m width and 1.0 m length moves at 4 m/s viscosity in still atmospheric air of density 1.20 kg/m^3 and kinematic viscosity of 1.49×10^{-5} m^2/s. Calculate the drag force on the plate.

[**Ans.** 0.00716 N]

12. Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by,

$$\frac{u}{U}=2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2$$ where δ = boundary layer thickness. $\left[\textbf{Ans.} \frac{\delta}{3}; \frac{2}{15}\delta; \frac{22}{105}\delta\right]$