Possibilities of solving two-dimensional hydrodynamic problems on the basis of the non-divergent form of recording the transport and conservation equations

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Abstract. Existing and applied in practice to solve the aerodynamics and hydrodynamics problems finite-difference schemes ensure that the laws of conservation of matter and energy only in limited configurations of velocity fields. In other cases, there are balance errors, which are currently accepted as the norm, and special algorithms of recalculation have been developed to reduce them. Additional calculations are labor-intensive, and when solving small-scale problems, balance errors can completely distort the calculation results. This article attempts to solve two-dimensional hydrodynamic problems using a new finite-difference computational scheme previously developed by the authors, based on the non-divergent form of recording the transfer and conservation equations. Initially, the scheme was developed and tested in one-dimensional space and showed complete conservativity, stability, transportability and adequacy. To solve two-dimensional problems, a transformation of the proposed scheme was performed. The solution of the test problems and comparison with the calculation results of other known schemes showed that in two-dimensional space the proposed scheme surpasses the results obtained by the HEC-RAS and Courant-Isakson-Reese schemes. The proposed scheme makes it possible to use the maximum possible time steps in the calculations, and the resulting scheme viscosity has minimal values. This property of the scheme makes it possible to apply it to solve small-scale aerodynamic and hydrodynamic problems.

1. Introduction

When solving problems about the motion of a coherent medium, the main requirement is to ensure the laws of conservation of matter [2, 4, 10, 13, 14, 15, 24]. Also, the calculation schemes used must have computational stability, adequacy, and transportability [3, 7, 8, 23, 24]. For calculations of this type, equations based on the laws of conservation of momentum of motion and mass are used [5, 6, 8]. Usually, finite-difference schemes [12, 22, 27, 28, 29], usually the Courant-Isakson-Reese scheme [11] and its modifications, are used as the mathematical expression of these equations. However, all existing and applied in practice finite-difference schemes are limited in their application, since they do not provide an exact execution of the laws of conservation of momentum and matter at any configurations of velocity fields.

This limitation has led to the introduction of a number of simplifications in approximate calculations, and the calculation error is taken for granted, although the presence of such an error causes a loss of conservativity and puts the results of calculations in doubt.

The concept of "balance error" has even been introduced into various calculation tools and an algorithm of steps to reduce the mass-energy balance error has been developed. One of the methods of non-conservativity reduction used in fluid mechanics [1, 5, 24] is the reduction of the computational grid step in the whole computational domain or only in the area of conservativity loss.

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Hydrodynamicists in many countries of the world use the HEC-RAS system to solve two-dimensional hydrodynamic problems [5, 9], which is officially used in the USA (https://www.hec.usace.army.mil) for calculation of hydromechanical processes and subsequent justification of insurance payments.
Let's consider how accurate results this system can give. Figure 1 shows a scan of the automatically generated listing for the calculation of possible flooding of territories in case of a dam break on the Kravfish River. The step of the calculation grid is 15 by 15 meters.

```
System
Project File:  C:**************************\Fall-River\HEC-
RAS\NorthBranchCrawfish.prj
Project Name:  Fall Creek DFA. North Br. Crawfish River
Plan Name:     Dam failure 2D
Short ID:      dam failure 2D
Starting Time:   02Jan2020 1500
Ending Time:     10Jan2020 1900
                ################################################
                #       1D and 2D Unsteady Flow Module         #
                #          HEC-RAS 6.0 Beta                    #
                #             07MAY21 at 08:45:27              #
                ################################################
          US Inflow      DS Outflow       Lat Hydro        SA Hydro
Groundwater      Diversions   Precip Excess
****        ********     ***********      **********   *************
    Start 1D Reach  Final 1D Reach   Starting SA's      Final SA's
    **************  **************   *************      **********
 ***   Volume Accounting for 2D Flow Area in Acre Feet   ***
           2D Area    Starting Vol      Ending Vol      Cum Inflow
Cum Outflow           Error   Percent Error
***********           *****   *************
      NBCR-2D-Mesh           1112.           480.0          21206.
21838.          0.2640        0.001209

                ***   Total Volume Accounting (for the entire model) in
Acre Feet   ***
Total Boundary Flux of Water In           21206.
Total Boundary Flux of Water Out          21838.
Starting Volume                            1112.
Ending Volume                               480.0
                           Error   Percent Error
                           *****   *************
                          0.2640        0.001183
```

Fig. 1. A scan of the HEC-RAS program listing for the calculation of the dam break on the Kravfish River

When examining the output of the text information in detail, a serious error can be detected. In Fig. 1 we can notice that in the last 7 lines a water balance error appeared: the amount of water at the outlet of the calculation area is greater than the amount of water at the inlet. Since the listing comes out in automatic mode, we conclude that this error is inherent and legitimized in this program, despite the obvious violation of the law of conservation of mass. If the error is very large, recalculations are performed on the grid with a smaller step. Repeated successive calculations take an enormous amount of labor and time. In particular, it took 6.5 hours to perform the calculation of the Wisconsin River levee breach using a 9th generation computer. The number of calculations attempts before a satisfactory result is obtained depends on the intuition and skill of the computational fluid mechanic.
When using such computational schemes to solve the aerodynamics problem, especially at small scales (for example, estimates of air circulation in urban areas), the main parameters are the minimum changes in air density and humidity. Therefore, an error in the calculation for such problems can lead to incorrect determination of the type of air mass circulation, i.e. the whole calculation can become erroneous.
Thus, we can conclude that calculations performed with a predetermined error can give only approximate results, and with violations of the fundamental laws of physics on the conservation of matter and energy.

A finite-difference scheme providing an exact execution of the laws of conservation of mass [25, 26] was proposed at TIIAME NRU and tested on calculations for one-dimensional space. A comparison of the results with the best known and most common calculation schemes [1, 24] showed that the proposed scheme is fully conservative, stable, transportable, and adequate [25, 26]. The purpose of this paper was to check the possibility of applying the proposed scheme for calculations in two-dimensional space.

2. Materials and Methods

The basic calculation method was the previously published scheme [25, 26] for one-dimensional space. The equation of matter transfer can be written in the form:

$$\frac{\partial S}{\partial t} = -\frac{\partial (SV)}{\partial x} \tag{1}$$

Or in the form:

$$\frac{\partial S}{\partial t} = -\left(V\frac{\partial S}{\partial x} + S\frac{\partial V}{\partial x}\right) \tag{2}$$

Where: S - concentration of substance in a unit of space, V - velocity of substance movement, t - time, x - spatial coordinate.

A recording form of equation (1) is a basis for most used finite difference schemes [12, 16-18, 19] which are called "divergent" [16, 17, 18], because they are based on a divergent recording form of equations of conservation of substance and energy. The basis of the new scheme for calculating the transfer and conservation of matter [25, 26] was not the divergent, but the expanded form of recording (2) of the transfer and conservation equations. Usually, the finite-difference analogues of this form of notation led to an exponential growth of accumulated errors, which caused an emergency stoppage of calculations. For this reason, this form of notation is practically not used in calculations and is very poorly studied. However, it is on the basis of this form of recording the conservation and transfer equations that a new finite-difference scheme of the TIIAME NRU [24] was developed.

In [25, 26] it is shown how equation (2) can be solved for one-dimensional space. To solve it, scheme (3) was applied:

$$\begin{aligned}
&U_{t,i} = V_{t,i} \quad if\ V_{t,i} \geq 0,\\
&U_{t,i} = 0 \quad if\ V_{t,i} > 0,\\
&W_{t,i} = V_{t,i} \quad if\ V_{t,i} \leq 0,\\
&W_{t,i} = 0 \quad if\ V_{t,i} > 0,\\
&S_{t+1,i} = S_{t,i} - \frac{\Delta t}{\Delta x} \cdot \begin{bmatrix} U_{t,i} * (S_{t,i} - S_{t,i-1}) + S_{t,i} * (U_{t,i+1} - U_{t,i}) \\ + \\ W_{t,i} * (S_{t,i+1} - S_{t,i}) + S_{t,i} * (W_{t,i} - W_{t,i-1}) \end{bmatrix}
\end{aligned} \tag{3}$$

Where: Ut,i, Wt,i - some newly introduced variables computed through the value of the velocity of matter movement, depending on the sign of that velocity.

Testing of scheme (3) has shown that its computational characteristics exceed those of the most widely used finite-difference Courant-Isakson-Reese schemes [6, 20, 21, 24] and their modifications.

To analyze the behavior of the scheme (3) in two-dimensional space, the following conditions were assumed.

The computational domain has a rectangular shape (Fig. 2). In it, we set two contiguous circular trajectories of motion, their velocity field is shown by arrows 3 in the figure. The velocity in the field of motion is taken equal to 1m/s. In one of the points on each circular trajectory of motion, set the concentration of substance equal to 1 and located symmetrically on each circular trajectory of motion. The cells in which, at the initial moment of time, the concentration of substance was set equal to 1 are highlighted in red. Complete symmetry of circular trajectories in the test problem is introduced to check the most important property of any finite-difference schemes - invariance. Approximate calculation schemes should give results that do not depend on the direction of coordinate axes. The loss of invariance makes any calculation scheme unacceptable for practical use.

Let us perform the calculation with a time step slightly smaller than the Courant-Levy condition allows, so that we can observe the manifestation of the scheme viscosity. We take 0.9 part of the maximum possible time step.

3. Results and Discussion

Figures 2, 3 and 4 show the distribution of matter in circular motions. Cells in which, at the initial moment of time, the presence of a unit of substance was set are marked in red. Obviously, the sum of the substance in the calculation area must always be equal to two.

Figure 2 shows the distribution of substance at 30-time steps (27 sec. of process) when applying the finite-difference directed difference scheme for the two-dimensional transfer and conservation of substance equation written in the form

$$\frac{\partial S}{\partial t} = -\left(V\frac{\partial S}{\partial x} + S\frac{\partial V}{\partial x}\right) - \left(V\frac{\partial S}{\partial y} + S\frac{\partial V}{\partial y}\right) \quad (4)$$

In the results of the calculation (Fig. 2) we can detect a loss of conservativity. The initial amount of matter in two units increased to 6.22. However, this result is adequate to the physics of the process and the matter moves along circular trajectories. The loss of conservativity goes in the direction of appearance of matter in the system and this means an emergency stop of the calculation process in the future due to the appearance of very large numbers. This is one of the main reasons, which immediately forced the computational fluid mechanics to avoid the form of the conservation equation in the form (4) for the development of approximate finite-difference calculation schemes.

Next, let us perform the solution using the Courant-Isakson-Reese finite-difference scheme written in the divergent form (5):

$$\frac{\partial S}{\partial t} = -\frac{\partial (SV)}{\partial x} - \frac{\partial (SV)}{\partial y} \quad (5)$$

Figure 3 shows the result of the calculation by formula (5) of the distribution of the substance for 30 steps in time. In the results of the calculation (Fig. 3) one can see both the loss of conservatism and the loss of adequacy of the calculation to the real physical process. Indeed, in the beginning the substance was 2 units, but after 30 iterations the substance became 0.66 units. Loss of adequacy is very noticeable on cells adjacent to cells of flow reversal. There the concentration becomes a negative value.

Fig. 2. Application of directed differences for the equation of transport and conservation of matter written in the form (4)

Figure 4 shows the distribution of matter by 30 steps in time when solving the problem with the help of a new finite-difference scheme of directional differences for the equation of transfer and conservation of matter [25, 26]. To perform the solution, the TIIAME NRU scheme written in the form (4) was generalized to the two-dimensional space in the form (6). The same generalization can be performed similarly for three-dimensional space.

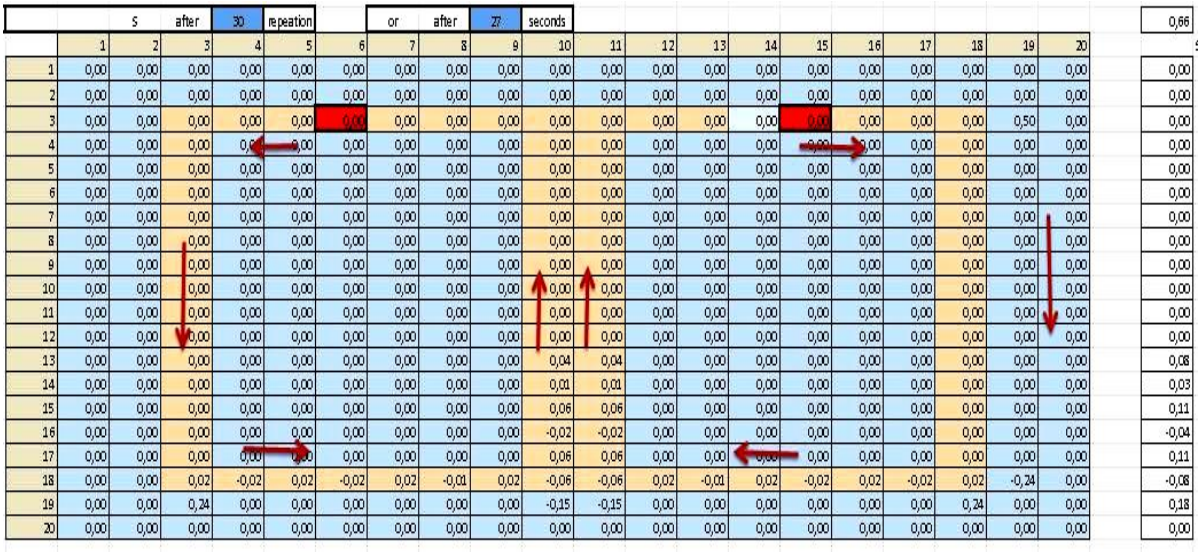

Fig. 3. Application of directed differences to the transfer equation written in divergent form (5)

$$
\begin{aligned}
Ux_{t,i,j} &= Vx_{t,i,j} \quad if\ Vx_{t,i,j} > 0, \\
Ux_{t,i,j} &= 0 \quad if\ Vx_{t,i,j} \le 0, \\
Wx_{t,i,j} &= Vx_{t,i,j} \quad if\ Vx_{t,i,j} \le 0, \\
Wx_{t,i,j} &= 0 \quad if\ Vx_{t,i,j} > 0,
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
Uy_{t,i,j} &= Vy_{t,i,j} \quad if\ Vy_{t,i,j} > 0, \\
Uy_{t,i,j} &= 0 \quad if\ Vy_{t,i,j} \le 0, \\
Wy_{t,i,j} &= Vy_{t,i} \quad if\ Vy_{t,i,j} \le 0, \\
Wy_{t,i,j} &= 0 \quad if\ Vy_{t,i,j} > 0,
\end{aligned}
\tag{6}
$$

$$
S_{t+1,i,j} = S_{t,i,j} - \frac{\Delta t}{\Delta x} \cdot \begin{bmatrix} Ux_{t,i,j} * \left(S_{t,i,j} - S_{t,i-1,j}\right) + S_{t,i,j} * \left(Ux_{t,i+1,j} - Ux_{t,i,j}\right) \\ + \\ Wx_{t,i,j} * \left(S_{t,i+1,j} - S_{t,i,j}\right) + S_{t,i,j} * \left(Wx_{t,i,j} - Wx_{t,i-1,j}\right) \end{bmatrix}
$$

$$
- \frac{\Delta t}{\Delta y} \cdot \begin{bmatrix} Uy_{t,i,j} * \left(S_{t,i,j} - S_{t,i,j-1}\right) + S_{t,i,j} * \left(Uy_{t,i,j+1} - Uy_{t,i,j}\right) \\ + \\ Wy_{t,i,j} * \left(S_{t,i,j+1} - S_{t,i,j}\right) + S_{t,i,j} * \left(Wy_{t,i,j} - Wy_{t,i,j-1}\right) \end{bmatrix}
$$

where UXt,i,j, WXt,i,j, UYt,i,j, WYt,i,j are some newly introduced variables, calculated through the value of the velocity of the substance, depending on the sign of this velocity.

According to the calculation results shown in Fig. 4 shows that the application of the investigated calculation scheme (6) provides accurate implementation of the law of conservation of matter, has transportability, stability in calculations, adequacy and invariance.

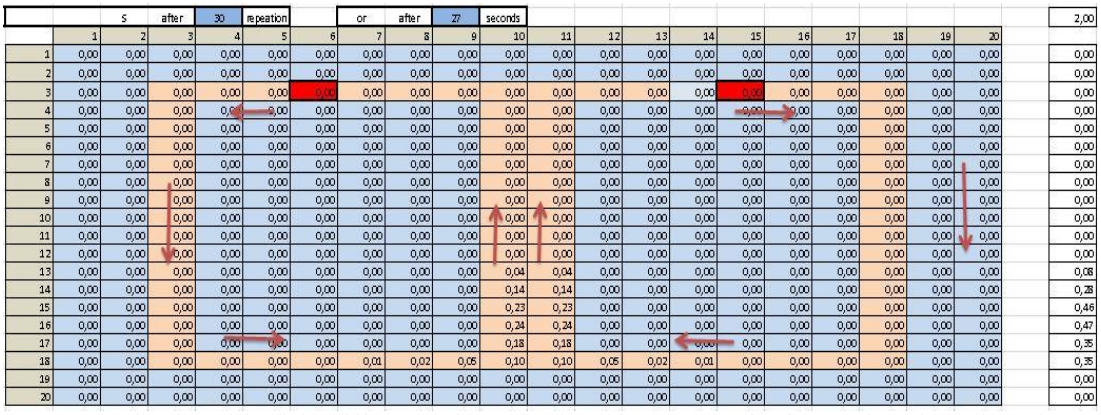

Fig. 4. Application of the finite-difference scheme developed in SRIIMSH for the equation of transfer and conservation of matter (2) in two-dimensional space.

Comparison of results obtained for three variants of solution is presented in fig. 5. Combined diagram of substance distribution for 30 intervals of time.

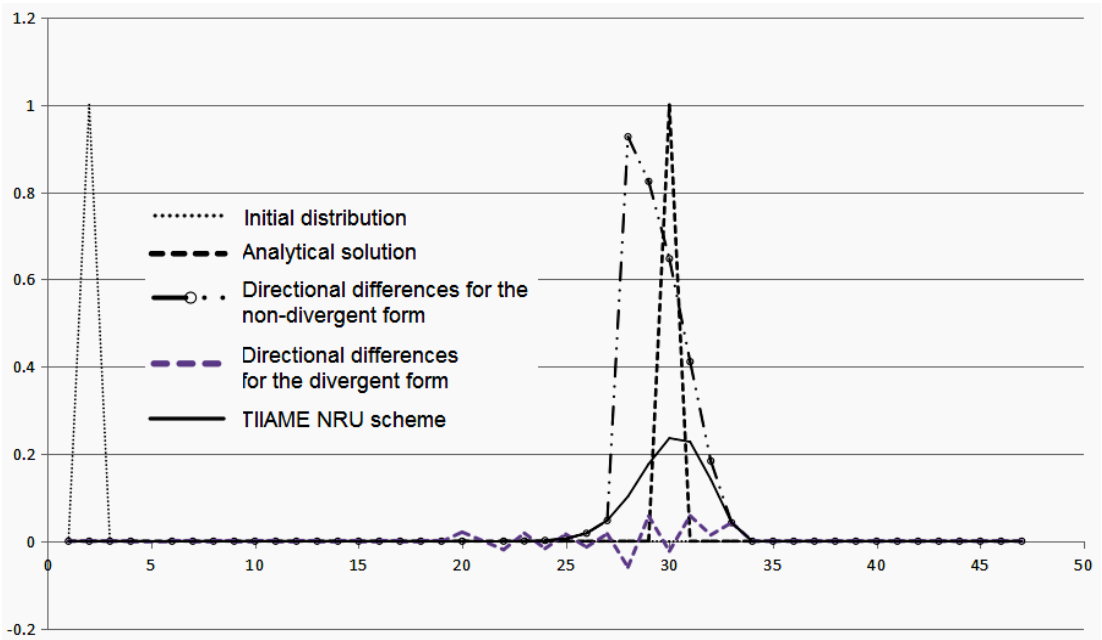

Fig. 5. Combined graph of the substance distribution for the 30th time interval calculated by three different finite-difference schemes (4), (5) and (6) along their circular motion path.

A time step of 0.9 of the maximum possible steps determined by the Courant-Levy criterion provided a manifestation of the scheme viscosity. But this time step turned out to be too large for the Courant-Isakson-Reese scheme. Figure 5 clearly shows the appearance of "cogged" instability, which disappears when the time step is reduced to half of the maximum possible. However, the circuit viscosity at such a small time step significantly distorts the result. The matter is as if smeared along the trajectory of its motion. That is, obtaining an exact solution when applying the Courant-Izakson-Reese scheme in two-dimensional space and in this configuration of the velocity field is in principle impossible already at a time step equal to 0.9 of the maximum possible.
The scheme (6) proposed by the authors easily copes with the task of obtaining the exact solution by approximate calculation.

4. Conclusions

1. Testing of the finite-difference scheme adapted for the two-dimensional space previously developed at TIIAME NRU has shown that this scheme fully ensures observance of the laws of conservation of matter and energy. This

scheme meets all the requirements to the schemes for approximate calculation of the equations of transfer and conservation of mass and energy, has the properties of conservativeness, transportability, stability in calculations, adequacy and invariance. The results of calculations according to the proposed scheme exceed the capabilities of usually used similar schemes for the two-dimensional space.

2. The stability of calculations according to the calculation scheme proposed by the authors at the maximum possible time steps with a minimum manifestation of the scheme viscosity allows using this scheme for calculations of air microcirculation in urban areas. It is expected that the proposed finite-difference scheme will be especially effective in calculating small-scale atmospheric circulation determined by small deviations of humidity, temperature, and pressure from mean values.
3. In addition, since the proposed scheme does not produce errors in the mass balances, its use will significantly reduce the time and labor intensity of the calculations, as there will be no need for additional calculations with changes in the size of the calculation grid.

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