Mavzu: Ratsional kasrlar. Eng sodda ratsional kasrlar va ularni integrallash.

Har qanday ratsional funksiyani ratsional kasr ko'rinishida, ya'ni ikki ko'phadning nisbati ko'rinishida tasvirlash mumkin:

$$\frac{Q(x)}{f(x)}=\frac{B_0x^m+B_1x^{m-1}+...+B_m}{A_0x^n+A_1x^{n-1}+...+B_n}$$

Muhokamaning umumiyligini cheklamasdan, bu ko'phadlar umumiy ildizga ega emas deb faraz qilamiz.

Agar suratinng darajasi mahrajning darajasidan past bo'lsa, kasr *to'gri kasr deb ataladi,* aks holda *noto'g'ri kasr* deb ataladi.

Agar kasr no'to'g'ri bo'lsa, suratni maxrajga (ko'phadni ko'phadga bo'lish qoidasi boyicha) bo'lib, berilgan kasrni ko'phad bilan biror to'g'ri kasrning yig'indisi ko'rinishida tasvirlash mumkin:

$$\frac{Q(x)}{f(x)}=M(x)+\frac{F(x)}{f(x)}$$

bu yerda *M(x)* – ko'phad, $\frac{F(x)}{f(x)}$ – to'g'ri kasr.

1 – misol. Noto'g'ri ratsional kasr berilgan bo'lsin:

$$\frac{x^4-3}{x^2+2x+1}.$$

Suratni maxrajiga (ko'phadlarni bo'lish qoidasi boyicha) bo'lib, shuni hosil qilamiz:

$$\frac{x^4-3}{x^2+2x+1}=x^2-2x+3-\frac{4x-6}{x^2+2x+1}.$$

Ko'phadlarni integrallash hech qanday qiyinchilik tug'dirmagani uchun, ratsional kasrlarni integrallashdagi asosiy qiyinchilik to'g'ri ratsional kasrlarni integrallashdan iboratdir.

T a ‘ r i f. Ushbu ko’rinishdagi to’g’ri ratsional kasrlar

I. $\frac{A}{x-a}$

II. $\frac{A}{(x-a)^k}$ (*k* –butun musbat son *k*≥2)

III. $\frac{Ax+B}{x^2+px+q}$ (mahrajning ildizlari kompleks sonlar, ya’ni $\frac{p^2}{4}-q<0$)

IV. $\frac{Ax+B}{(x^2+px+q)^k}$ (*k*-butun musbat son, *k*≥2;

mahrajning ildizlar kompleks sonlar); I, II, II va IV tipdagi *eng sodda ratsional kasrlar* deb ataladi.

I, II va III tipdagi eng sodda kasrlarni integrallashda katta qiyinchilikka uchramaymiz, shuning uchun ularning integrallarini hech qanday izohsiz keltiramiz:

I. $\int \frac{A}{x-a}dx = A\ln|x-a|+C$

II. $\int \frac{A}{(x-a)^k}dx = A\ln\int (x-a)^{-k}dx = A\frac{(x-a)^{-k+1}}{-k+1}+C =$

$= \frac{A}{(1-k)(x-a)^{k-1}}+C$

III.

$$\int \frac{Ax+B}{x^2+px+q}dx = \int \frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{x^2+px+q}dx = \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{x^2+px+q} =$$
$$\frac{A}{2}ln|x^2+px+q| + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)} = \frac{A}{2}ln|x^2+px+q| +$$
$$\frac{2B-Ap}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}} + C$$

.

IV tipdagi eng sodda kasrlarni integrallash murakkabroq hisoblashni talab qiladi. Shu tipdagi integral berilgan bo'lsin:

IV. $\int \frac{Ax+B}{(x^2+px+q)^k}dx.$

Quyidagicha almashtiramiz:

$$\int \frac{Ax+B}{(x^2+px+q)^k}dx = \int \frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{(x^2+px+q)^k}dx =$$
$$\frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{(x^2+px+q)^k}$$

.

Birinchi integral $x^2+px+q=t,\ (2x+p)dx=dt$ almashtirish yordami bilan olinadi:

$$\int \frac{2x+p}{(x^2+px+q)^k}dx = \int \frac{dt}{t^k} = \int t^{-k}dt = \frac{t^{-k+1}}{1-k}+C = \frac{1}{(1-k)(x^2+px+q)^{k-1}}+C.$$

Ikkinchi integralni (uni I_k bilan berlgilaymiz) ushbu ko'rinishda yozamiz:

$$I_k = \int \frac{dx}{(x^2+px+q)^k} = \int \frac{dx}{\left[\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right]^k} = \int \frac{dt}{(t^2+m^2)^k},$$

bu yerda

$$x+\frac{p}{2}=t,\ \ dx=dt,\ \ q-\frac{p^2}{4}=m^2$$

deb faraz qilamiz (maxrajning ildizlari farazga ko'ra kompleks sonlar, demak $q-\frac{p^2}{4}>0$). Endi quyidagicha kirishamiz:

$$I_k = \int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2}\int \frac{(t^2+m^2)-t^2}{(t^2+m^2)^k}dt = \frac{1}{m^2}\int \frac{dt}{(t^2+m^2)^{k-1}} - \frac{1}{m^2}\int \frac{t^2}{(t^2+m^2)^k}d\,t. \quad (1)$$

So'nggi integralni bunday almashtiramiz:

$$\int \frac{t^2\,dt}{(t^2+m^2)^k} = \int \frac{t*tdt}{(t^2+m^2)^k} = \frac{1}{2}\int t\frac{d(t^2+m^2)}{(t^2+m^2)^k} = -\frac{1}{2(k-1)}\int td\left(\frac{1}{(t^2+m^2)^{k-1}}\right).$$

Bo'laklab integrallab, quyidagini hosil qilamiz:

$$\int \frac{t^2 dt}{(t^2+m^2)^k} = -\frac{1}{2(k-1)}\left[t\frac{1}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}}\right].$$

Bu ifodani (1) tenglikka qoysak, quyidagi hosil bo'ladi:

$$I_k \int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2}\int \frac{dt}{(t^2+m^2)^{k-1}} + \frac{1}{m^2}\frac{1}{2(k-1)}\left[\frac{t}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}}\right] = \frac{t}{2m^2(k-1)(t^2+m^2)^{k-1}} + \frac{2k-3}{2m^2(k-1)}\int \frac{dt}{(t^2+m^2)^{k-1}}$$

.

O'ng tomonda I_k tipidagi integral bor, lekin integral ostidagi funksiya maxrajining daraja ko'rsatkichi uning daraja ko'rsatkichidan bitta birlik past *(k–1)*, shunday qilib, I_k ni I_{k-1} orqali ifodaladik.

Shu yo'l bilan davom etib, ma'lum integral

$$I_1 = \int \frac{dt}{t^2+m^2} = \frac{1}{m} arctg\frac{t}{m} + C$$

ga yetib boramiz. So'gra *t* va *m* o'rniga ularning qiymatlarini qo'yib, IV integralning *x* va berilgan A,B,p,q sonlar orqali ifodasini topamiz.

2 – misol.

$$\int \frac{x-1}{(x^2+2x+3)^2}dx = \int \frac{\frac{1}{2}(2x+2)+(-1-1)}{(x^2+2x+3)^2}dx = \frac{1}{2}\int \frac{2x+2}{(x^2+2x+3)^2}dx - 2\int \frac{dx}{(x^2+2x+3)^2} = -\frac{1}{2}\frac{1}{(x^2+2x+3)} - 2\int \frac{dx}{(x^2+2x+3)^2}$$

.

Oxirgi integralga $x+1=t$ almashtirishni qo'llaymiz:

$$\int \frac{dx}{(x^2+2x+3)^2} = \int \frac{dx}{[(x+1)^2+2]^2} = \int \frac{dt}{(t^2+2)^2} = \frac{1}{2}\int \frac{(t^2+2)-t^2}{(t^2+2)^2}dt = \frac{1}{2}\int \frac{dt}{t^2+2} - \frac{1}{2}\int \frac{t^2}{(t^2+2)^2}dt = \frac{1}{2}\frac{1}{\sqrt{2}}arctg\frac{t}{\sqrt{2}} - \frac{1}{2}\int \frac{t^2}{(t^2+2)^2}dt$$

.

Oxirgi integralni qaraymiz:

$$\int \frac{t^2}{(t^2+2)^2} dt = \frac{1}{2}\int \frac{td(t^2+2)}{(t^2+2)^2} = -\frac{1}{2}\int td\left(\frac{1}{t^2+2}\right) = -\frac{1}{2}\frac{t}{t^2+2} + \frac{1}{2}\int \frac{dt}{t^2+2} = -\frac{t}{2(t^2+2)} + \frac{1}{\sqrt{2}} arctg\frac{t}{\sqrt{2}}$$

Demak,

$$\int \frac{dx}{(x^2+2x+3)^2} = \frac{1}{2\sqrt{2}} arctg\frac{x+1}{\sqrt{2}} - \frac{1}{2}\left[-\frac{x+1}{2(x^2+2x+3)} + \frac{1}{2\sqrt{2}} arctg\frac{x+1}{\sqrt{2}}\right].$$

oxirgi natija quyidagi ko'rinishda bo'ladi:

$$\int \frac{x-1}{(x^2+2x+3)^2} dx = -\frac{x+2}{2(x^2+2x+3)^2} - \frac{\sqrt{2}}{4} arctg\frac{x+1}{\sqrt{2}} + C.$$

Endi har qanday to'gri ratsional kasrni eng sodda kasrlar yig'indisiga ajratish mumkinligini ko'rsatamiz.

Ushbu to'g'ri ratsional kasr

$$\frac{F(x)}{f(x)}$$

berilgan bo'lsin. Bu kasrga kirgan ko'phadlarning koeffisentlari haqiqiy sonlar va berilgan kasr qisqarmaydigan kasr deb faraz qilamiz (bu esa surat va mahraj umumiy ildizga ega emas degan so'zdir).

1 – t e o r e m a. *x=a mahrajning k karrali ildizi, ya'ni $f(x)=(x-a)^k f_1(x)$ bo'lsin, bu yerda $f_1(a)\neq 0$; u holda .... To''ri kasrni boshqa ikki mo'g'ri kasr yig'indisi shaklida quyidagicha tasvirlash mumkin*

$$\frac{F(x)}{f(x)} = \frac{A}{(x-a)^k} + \frac{F_1(x)}{(x-a)^{k-1} f_1(x)}$$

bu yerda A nolga teng bo'lmagan o'zgarmas son, $F_1(x)$ ko'phad, buning darajasi $(x-a)^{k-1} f_1(x)$ mahrajning darajasidan past.

2 – t e o r e m a. *Agar $f(x)=(x^2+px+q)\varphi_1(x)$ bo'lsa (bu yerda $\varphi_1(x)$ ko'phad x^2+px+q ga bo'linmaydi), ... to'g'ri ratsional kasrni boshqa ikki too'g'ri kasrning yig'indisi ko'rinishida tasvirlash mumkin:*

$$\frac{F(x)}{f(x)}=\frac{Mx+N}{(x^2+px+q)^k}+\frac{\Phi_1(x)}{(x^2+px+q)^{\mu-1}\varphi_1(x)}$$

Isbot. Ushbu ayniyatni yozamiz:

$$\frac{F(x)}{f(x)}=\frac{f(x)}{(x^2+px+q)^{\mu}\varphi_1(x)}=\frac{Mx+N}{(x^2+px+q)^{\mu}}+\frac{F(x)-(Mx+N)\varphi_{1(x)}}{(x^2+px+q)^{\mu}\varphi_1(x)} \qquad (4)$$

bu ayniyat har qanday M va N uchun to'g'ri; M va N ning shunday qiymatini topamizki, unda $F(x)-(Mx+N)\varphi_{1(x)}$ ko'phad x^2+px+q ga bo'linsin. Buning uchun

$$F(x)-(Mx+N)\varphi_{1(x)}=0$$

x^2+px+q ning $\alpha\pm i\beta$ ildizlariga teng ildizlarga ega bo'lishi zarur va yetarlidir. Demak,

$$F(\alpha+i\beta)-[M(\alpha+i\beta)+N]\varphi_1(\alpha+i\beta)=0$$

yoki

$$M(\alpha+i\beta)+N=\frac{F(\alpha+i\beta)}{\varphi_1(\alpha+i\beta)}$$

lekin $\frac{F(\alpha+i\beta)}{\varphi_1(\alpha+i\beta)}$ aniq bir kompleks son, buni K+iL ko'rinishda yozish mumkin, bu yerda K va L haqiqiy sonlar. Shunday qilib,

$$M(\alpha+i\beta)+N=K+iL,$$

bundan

$$M\alpha+N=K,\ \ M\beta=L$$

yoki

$$M = \frac{L}{\beta}, \quad N = \frac{K\beta - L\alpha}{\beta}.$$

M va *N* ning shu qiymatlarida $F(x) - (Mx + N)\varphi_1(x)$ ko'phad $\alpha + i\beta$ ildizga, demak, $\alpha - i\beta$ qo'shma ildizga ham ega bo'ladi. Bu holda ko'phad $x - (\alpha + i\beta)$ va $x - (\alpha - i\beta)$ ayirmalarga, demak, ularning ko'paytmasiga, ya'ni x^2+px+q ga qoldiqsiz bo'linadi. Bo'linmani $\Phi_1(x)$ bilan belgilab, quyidagicha yoza olamiz:

$$F(x) - (Mx + N)\varphi_1(x) = (x^2 + px + q)\Phi_1(x).$$

(4) tenglikdagi ohirgi kasrni x^2+px+q ga qisqartirib, (3) tenglikni hosil qilamiz, bunda $\Phi_1(x)$ ning darajasi maxrajning darajasidan past ekani ravshan.

Shuni isbot qilish talab qilingan edi.

Foydalanilgan adabiyotlar

1. Claudio Canuto, Mathematical Analysis I. 2008.
2. PETER V. O’NEIL, ADVANCED ENGINEERING MATHEMATICS. 2010.
3. Crowell and Slesnick, Calculus with Analytic Geometry. 2008