Mavzu: Trigonometrik funksiyalarni integrallash

$\int R(\sin x, \cos\ x)dx$

(1) ko'rinishdagi integrallarni qaraymiz. Bu integral $tg\frac{x}{2}=t$ almashtirish yordami bilan hamma vaqt ratsional funksiyaning integraliga keltirilishi mumkin ekanini ko'rsatamiz. *Sinx* va *cosx* funksiyalarni $tg\frac{x}{2}=t$ (2) bilan, ya'ni t bilan ifoda etamiz:

$$sinx=\frac{2sin\frac{x}{2}cos\frac{x}{2}}{1}=\frac{2sin\frac{x}{2}cos\frac{x}{2}}{sin^2\frac{x}{2}cos^2\frac{x}{2}}=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{2t}{1+t^2},$$

$$cosx=\frac{cos^2\frac{x}{2}-sin^2\frac{x}{2}}{1}=\frac{cos^2\frac{x}{2}-sin^2\frac{x}{2}}{cos^2\frac{x}{2}+sin^2\frac{x}{2}}=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

Endi (2) tenglikdan:

$$x=2\,arc\,tg\,t,\qquad dx=\frac{2dt}{1+t^2}$$

Shunday qilib, *sinx*, *cosx* va *dx* lar *t* bilan ratsional ifodalandi, ammo ratsional funksiyalarning raysional funksiyasi o'z navbatida yana ratsiona funksiya bo'lgani uchun hosil qilingan ifodalarni berilgan (1) integralga qo'yib ratsional funksiyaning integralini hosil qilamiz:

$$\int R(sinx,cosx)dx=\int R\left[\frac{2t}{1+t^2},\qquad\frac{1-t^2}{1+t^2}\right]\frac{2dt}{1+t^2}$$

1 misol

$\int\frac{dx}{5-3\cos x}=(*)$ trigonometric almashtirish bajarmiz

$$z=tg\frac{x}{2};x=2arctgz\Rightarrow dx=\frac{2dz}{1+z^2};\cos x=\frac{1-z^2}{1+z^2}$$

$$(*)=\int\frac{\frac{2dz}{1+z^2}}{5-\frac{3(1-z^2)}{1+z^2}}=2\int\frac{dz}{5+5z^2-3+3z^2}=2\int\frac{dz}{8z^2+2}=\int\frac{dz}{4z^2+1}=\frac{1}{2}\int\frac{d(2z)}{(2z)^2+1}=$$

$$=\frac{1}{2}arctg\,2z+C=\frac{1}{2}arctg\left(2tg\frac{x}{2}\right)+C,\text{где } C=const$$

Yuqoridagi almashtirish har qanday trigonometric funksiyani integrallash imkonini beradi. Shuning uchun uni ba'zan «universal trigonometric almashtiris» deb ataladi. Lekin amalda bu almashtirish ko'pincha ancha murakkab ratsional funksiyaga olib keladi. Shunung uchun «universal» almashtirish bilan bir qatorda ba'zi hollar uchun maqsadga tez olib keladigan boshqa almashtirishlar ham qo'llaniladi.

1) Agar integral $\int R(\sin x)\cos x dx$ ko'rinishida bo'lsa, u holda sinx = t, cosxdx = dt almashtirish bu integralni $\int R(t)dt$ ko'rinishiga olib keladi.
2) Agar integral $\int R(\cos x)\sin x dx$ ko'rinishida bo'lsa, u holda cosx = t, sinxdx = -dt almashtirish yordamida bu integral ratsional funksiyaning integraliga keltiriladi.
3) Agar integral ostidagi funksiya faqat tgx ga bog'liq bo'lsa, u holda

   tgx = *t*, *x* = *arctgt*, dx = $\frac{dt}{1+t^2}$ almashtirish yordamida bu integral ratsional funksiyaning integraliga keltiriladi.

$$\int R(tgx)dx = \int R(t)\frac{dt}{1+t^2}$$

4) Agar integral ostidagi funksiya *R(sinx, cosx)* ko'rinishida bo'lsa, ammo bunda sinx va cosx larning faqat juft darajalari kirsa, u holda
   tgx = *t* (3) almashtirish tatbiq etiladi. $Sin^2 x$ va $cos^2 x$ lar tgx bilan ratsional ifoda etiladi.

Ba'zi trigonometric funksiyalarni tgx orqali ifodalanishi.

$$cos^2x = \frac{1}{1+tg^2x} = \frac{1}{1+t^2}$$

$$sin^2x = \frac{tg^2x}{1+tg^2} = \frac{t^2}{1+t^2}$$

$$dx = \frac{dt}{1+t^2}$$

2 misol.

$$\int \frac{sin^3x}{2+cosx}dx$$

Integral hisoblansin.

Yechish. Bu integralni

$\int R(\cos x)\sin x dx$

ko'rinishiga keltiriladi.

$$\int \frac{sin^3x}{2+cosx}dx = \int \frac{sin^2x\ sinx\ dx}{2+\cos x} = \int \frac{1-cos^2x}{2+cosx}\sin x\ dx$$

Cosx =z almashtirishni bajaramiz. Bu holda *sinxdx = -dz*

Demak,

$$\int \frac{sin^3x}{2+cosx}dx = \int \frac{1-z^2}{2+z}(-dz) = \int \frac{z^2-1}{z+2}dz$$
$$= \int \left(z-2+\frac{3}{z+2}\right) = \frac{z^2}{2} - 2z + 3\ln(z+2) + C$$
$$= \frac{cos^2x}{2} - 2cosx + 3\ln(cosx+2) + C$$

5) $\int \sin^m x \cos^n xdx$ ko‘rinishdagi integrallarda uchta holni ko‘ramiz.

a) $\int \sin^m x \cos^n xdx$ integralda m va n larning kamida bittasi toq bo’lsin. Aniqliq uchun n toq son deb faraz qilamiz. n = 2p + 1 deb olib, integralni o’zgartiramiz.

$$\int \sin^m x \cos^{2p+1} xdx = \int \sin^m x \cos^{2p} x \cos xdx = \int \sin^m x(1-\sin^2 x)^p \cos xdx$$

o’zgaruvchini almashtiramiz: sinx = t, cosxdx = dt

yangi o’zgaruvchini berilgan integralga qo’yamiz

$\int \sin^m x \cos^n xdx = \int t^m (1-t^2)^p dt$, bu esa t ning ratsional funksiyasining integralidir.

b) $\int \sin^m x \cos^n xdx$ integralda m va n manfiy bo’lmagan juft son. m = 2p, n = 2q deb qaraymiz. Trigonometriyada ma’lum bo’lgan formulalarni yozamiz:

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2} \quad (4)$$

bularni berilgan integralga qo’yamiz:

$$\int \sin^{2p} x \cos^{2q} xdx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^p \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^q dx$$

Darajaga ko’tarib hamda qavslarni ochib, *cos2x* ning juft ba toq darajalarini o’z ichiga olgan hadlarni hosil qilamiz.

Toq darajali hadlar a) holda ko’rsatilgandek, integrallanadi.

Darajaning juft ko’rsatkichlarini (4) formulalariga ko’ra yana pasaytiramiz.

Daraja ko’rsatkichlarini pasaytiriwni oson integrallanadigan $\int \cos\ kxdx$ ko’rinishdagi hadlar hosil bo’lgunicha shunday davom ettiramiz.

3-misol. $\int \sin 2x \cos 7xdx$ integralni hisoblang.

yechish. Yuqoridagi formulalarning birinchisidan

$$\sin 2x\cos 7x = \frac{1}{2}\left[\sin(2x+7x)+\sin(2x-7x)\right] = \frac{1}{2}(\sin 9x \sin 5x),$$

$$\int \sin 2x\cos 7x dx = \frac{1}{2}\int(\sin 9x - \sin 5x)dx = \frac{1}{2}\int \sin 9x dx - \frac{1}{2}\int \sin 5x dx =$$

$$= \frac{1}{2}\cdot\frac{1}{9}(-\cos 9x) - \frac{1}{2}\cdot\frac{1}{5}(-\cos 5x) + C = \frac{1}{10}\cos 5x - \frac{1}{18}\cos 9x + C.$$

natijaga ega bo‘lamiz.

4-misol. $\int \sin^2 x\, dx$ integralni hisoblang.

yechish. Bu integralni izoќlarsiz ќisoblaymiz:

$$\int \sin^2 x dx = \int \frac{1-\cos 2x}{2}dx = \int \frac{1}{2}dx - \int \frac{\cos 2x}{2}dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

5-misol. $\int \sin^2 x \cos^4 x dx$ integralni hisoblang.

yechish. Trigonometrik funksiyalarning darajalarini pasaytirish formulalaridan foydalanib, quyidagi natijaga kelamiz:

$$\int \sin^2 x\cos^4 x dx = \int \frac{1-\cos 2x}{2}\cdot\left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{8}\int(1-\cos 2x)(1+\cos 2x)^2 dx =$$

$$= \frac{1}{8}\int(1-\cos^2 2x)(1+\cos 2x)dx = \frac{1}{8}\int \sin^2 2x dx + \frac{1}{8}\int(\sin^2 2x\cos 2x dx) =$$

$$= \frac{1}{16}\int(1-\cos 4x)dx + \frac{1}{16}\int \sin^2 2x d\sin 2x = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{16}\frac{\sin^3 2x}{3} + C =$$

$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.$$

6-misol. $\int \sin^3 x\cos^4 x dx$ integralni hisoblang.

yechish. $\sin x dx = -d(\cos x)$ va $\sin^2 x = 1-\cos^2 x$ ekanligini hamda

$\cos x = z$ almashtirish kiritib, quyidagini hosil qilamiz:

$$\int \sin^3 x \cos^4 x dx = \int \sin^2 x \cos^4 x \sin x dx = \int (1-\cos^2 x)\cos^4 x(-d\cos x) =$$

$$= -\int (1-z^2)z^4 dz = -\int (z^4 - z^6)dz = -\frac{z^5}{5} + \frac{z^7}{7} + C = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C.$$

Because

$$\sin^5 x = \sin x \sin^4 x = \sin x(1-\cos^2 x)^2 ,$$

choosing $y = \cos x$ has the effect that $dy = -\sin x\, dx$ and

$$\int \sin^5 x\, dx = -\int (1-y^2)^2\, dy = \int (-1+2y^2-y^4)\, dy$$

$$= -y + \frac{2}{3}y^3 - \frac{1}{5}y^5 + c = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c .$$

7-misol. $\int \frac{\sin^3 x}{\cos^2 x} dx$ integralni hisoblang.

yechish. $\sin x dx = -d\cos x$ bo‘lgani uchun, $t = \cos x$ almashtirish olsak,

$$\int \frac{\sin^3 x}{\cos^2 x} dx = \int \frac{\sin^2 x \sin x dx}{\cos^2 x} = \int \frac{1-\cos^2 x}{\cos^2 x} \sin x dx = \int \frac{1-t^2}{t^2}(-dt) =$$

$$= -\int \frac{1}{t^2} dt + \int dt = \frac{1}{t} + t + C = \frac{1}{\cos x} + \cos x + C$$

Har xil argumentli sinus va kosinuslar ko‘paytmalari shaklidagi funksiyalarni integrallash.

$$\int \sin mx \cos nx dx, \quad \int \sin mx \sin nx dx, \quad \int \cos mx \cos nx dx \qquad (1)$$

ko‘rinishdagi integrallarni karaymiz. Trigonometrik funksiyalarning ko‘paytmadan yig‘indiga keltirish formulalaridan foydalanamiz.

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha+\beta)\right], \quad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right],$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

formulalardan foydalanib, (1) ko‘rinishdagi integrallarni

$$\int \sin axdx, \quad \int \cos bxdx$$

integrallardan biriga keltirib itegrallanadi.

8-misol. $\int \sin 2x \cos 7xdx$ integralni hisoblang.

yechish. Yuqoridagi formulalarning birinchisidan foydalanamiz

$$\sin 2x \cos 7x = \frac{1}{2}\left[\sin(2x+7x) + \sin(2x-7x)\right] = \frac{1}{2}(\sin 9x \sin 5x),$$

$$\int \sin 2x \cos 7xdx = \frac{1}{2}\int(\sin 9x - \sin 5x)dx = \frac{1}{2}\int \sin 9xdx - \frac{1}{2}\int \sin 5xdx =$$

$$= \frac{1}{2}\cdot\frac{1}{9}(-\cos 9x) - \frac{1}{2}\cdot\frac{1}{5}(-\cos 5x) + C = \frac{1}{10}\cos 5x - \frac{1}{18}\cos 9x + C.$$

natijaga ega bo‘lamiz.

9-misol. $\int \cos 7x \cos 3xdx, \quad \int \sin 4x \sin 2xdx, \quad \int \sin 5x \cos 3x$ integrallarni mustaqil hisoblang.

Problem 10. Find: $\int \frac{1}{3}\cos 5x \sin 2x\, dx$

$$\int \frac{1}{3}\cos 5x \sin 2x\, dx$$

$$= \frac{1}{3}\int \frac{1}{2}[\sin(5x+2x) - \sin(5x-2x)]\, dx,$$

from 7 of Table 50.1

$$= \frac{1}{6}\int (\sin 7x - \sin 3x)\, dx$$

$$= \frac{1}{6}\left(\frac{-\cos 7x}{7} + \frac{\cos 3x}{3}\right) + c$$

Mustaqil bajarish uchun topshiriqlar

Ushbu integrallarni ќisoblang.

1. $\int \sin 3x \sin 7x dx$.   2. $\int \sin 5x \cos 3x dx$.   3. $\int \sin x \sin 3x dx$.

4. $\int \sin 3x \cos 2x dx$.   5. $\int \cos 4x \cos 2x dx$.   6. $\int \sin 3x \cos x dx$.

7. $\int \sin x \cos^4 x dx$.   8. $\int \sin^3 x \cos^3 x dx$.   9. $\int \sin^2 5x dx$.

10. $\int \cos 7x \cos 3x dx$.   11. $\int \sin 4x \sin 2x dx$.

12. $\int \sin^2 x \cos^2 x dx$.   13. $\int \frac{\cos^5 x}{\sin^2 x} dx$.   14. $\int \frac{\cos^3 x dx}{\sin^2 x}$.

15. $\int tg^3 x dx$.   16. $\int ctg^3 x dx$.   17. $\int \frac{\sin^3 x + 1}{\cos^2 x} dx$.

18. $\int \sin^4 x dx$.   19. $\int \cos^4 x dx$.   20. $\int \sin^5 x dx$.

21. $\int \cos^5 x dx$.   22. $\int \sin^2 x \cos^3 x dx$.   23. $\int \sqrt[3]{cos^2 x} \sin^3 x \, dx$.