CHIZIQLI ALGEBRAIK TENGLAMALAR SISTEMASI

Iqtisodiy masalalarni (rejalastirish, boshqarish va boshqa masalalarni) yechishda ko‘p noma‘lumli chiziqli algebraik tenglamalar sistemasi qo‘llaniladi. m ta tenglamalardan tuzilgan n ta noma’lumli chiziqli tenglamalar sistemasini quyidagicha yozish mumkin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases} \qquad (1)$$

bu yerda $a_{11}, a_{12}, \dots, a_{mn}$- sistema koeffitsiyentlari, $x_1, x_2, \dots, x_n$ - noma’lum koeffitsyentlar, $b_1, b_2, \dots, b_m$ –ozod hadlar.

Chiziqli algebraik tenglamalar sistemasini quyidagi ixchamroq ko‘rinishda ham yozish mumkin:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1,2, \dots, m.$$

Agar $x_1^0, x_2^0, \dots, x_n^0$ sonlarni mos ravishda (1) tenglamalar sistemasidagi $x_1, x_2, \dots, x_n$ - noma’lum koeffitsiyentlar o‘rniga qo‘yilganda sistemaning har bir tenglamasi ayniyatga aylansa, $x_1^0, x_2^0, \dots, x_n^0$ sonlar ***sistemaning yechimi*** deyiladi. (1) chiziqli algebraik tenglamalar sistemasi yechimga ega bo‘lsa, ***sistema birgalikda***, aks holda, ya’ni yechimga ega bo‘lmasa, ***sistema birgalikda emas*** deyiladi. Agar birgalikdagi sistema bitta yechimga ega bo‘lsa, ***sistema aniqlangan***, aks holda, ya’ni ko‘p yechimga ega bo‘lsa, ***sistema aniqlanmagan*** deyiladi.

Tenglamalar sistemasida ozod hadlar nolga teng bo‘lsa, sistema ***bir jinsli*** deyiladi.

1-§. Chiziqli algebraik tenglamalar sistemasini yechishning Kramer usuli

Uch noma’lumli tenglamalar sistemasi berilgan bo‘lsin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \qquad (2)$$

Noma’lumlar oldidagi koeffitsiyentlardan 3-tartibli determinantni tuzamiz va uni determinant xossalariga ko‘ra 1-ustun bo‘yicha yoyamiz:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot A_{11} + a_{21} \cdot A_{21} + a_{31} \cdot A_{31}. \qquad (3)$$

Determinantning birinchi, ikkinchi va uchinchi ustunlarini mos ravishda ozod hadlar almashtirib uchda Δ_1, Δ_2, Δ_3 determinantlarni hosil qilamiz hamda ularni ham 1-ustun bo‘yicha yoyamiz:

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = b_1 \cdot A_{11} + b_2 \cdot A_{21} + b_3 \cdot A_{31}, \qquad (4)$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} = b_1 \cdot A_{12} + b_2 \cdot A_{22} + b_3 \cdot A_{32}, \qquad (5)$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} = b_1 \cdot A_{13} + b_2 \cdot A_{23} + b_3 \cdot A_{33}. \qquad (6)$$

(2) sistemaning 1-tenglamasini A_{11} algebraik to‘ldiruvchiga, 2-tenglamasini A_{21}ga, 3- tenglamasini A_{31}ga ko‘paytirib hadlab qo‘shamiz:

$$(A_{11}a_{11} + A_{21}a_{21} + A_{31}a_{31})x_1 +$$
$$+(A_{11}a_{12} + A_{21}a_{22} + A_{31}a_{32})x_2 +$$
$$+(A_{11}a_{13} + A_{21}a_{23} + A_{31}a_{33})x_3 =$$
$$= A_{11}b_1 + A_{21}b_2 + A_{31}b_3 \qquad (7)$$

Yuqoridagi (3) va (4) munosabatlar hamda determinantlarning xossalariga ko‘ra:

$$a_{11} \cdot A_{11} + a_{21} \cdot A_{21} + a_{31} \cdot A_{31} = \Delta,$$
$$A_{11}a_{12} + A_{21}a_{22} + A_{31}a_{32} = 0,$$
$$A_{11}a_{13} + A_{21}a_{23} + A_{31}a_{33} = 0,$$
$$A_{11}b_1 + A_{21}b_2 + A_{31}b_3 = \Delta_1.$$

Natijada (7) tenglama quyidagi ko‘rinishga keladi:

$$\Delta \cdot x_1 = \Delta_1.$$

Xuddi yuqoridagidek, (2) sistemaning 1-tenglamasini A_{12} ga, 2-tenglamasini A_{22} ga va 3-tenglamasini A_{32} ga ko‘paytirib hadlab qo‘shamiz:

$$\Delta \cdot x_2 = \Delta_2.$$

(2) sistemaning 1-tenglamasini A_{13} ga, 2-tenglamasini A_{23} ga va 3-tenglamasini A_{33} ga ko‘paytirib hadlab qo‘shamiz:

$$\Delta \cdot x_3 = \Delta_3.$$

Natijada (2) sistemaga teng kuchli bo‘lgan

$$\begin{cases} \Delta \cdot x_1 = \Delta_1 \\ \Delta \cdot x_2 = \Delta_2 \\ \Delta \cdot x_3 = \Delta_3 \end{cases} \quad (8)$$

sistemani hosil qilamiz.

(8) sistemaning yechimi unda qatnashgan determinantlarga boq‘liqdir.

1^0. $\Delta \neq 0$ bo‘lsin. U holda, (8) sistemadan noma‘lumlarning

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta} \quad (9)$$

qiymatini topamiz. (x_1, x_2, x_3) qiymatlar (1) sistemaning yagona yechimi bo’ladi. (9) formulaga ***Kramer formulasi***[1] deyiladi.

2^0. $\Delta = 0$ bo‘lib, Δ_1, Δ_2 va Δ_3 lardan hech bo‘lmaganda bittasi noldan farqli bo‘lsin. Bu holda (2) sistema yechimga ega bo‘lmaydi.

3^0. $\Delta = 0$ bo‘lib, $\Delta_1 = \Delta_2 = \Delta_3 = 0$ bo’lsin. Bu holda (2) sistema yoki cheksiz ko‘p yechimga ega bo‘ladi yoki bitta ham yechimga ega bo‘lmaydi.

1-misol. Tenglamalar sistemasini yeching.

$$\begin{cases} x_1 - x_3 = 2, \\ 2x_1 - x_2 + 3x_3 = -1, \\ 3x_1 + 2x_2 - 2x_3 = 5. \end{cases}$$

Yechish. Noma’lumlar oldidagi koeffitsentlardan 3-tartibli determinantni tuzamiz va uning qiymatini topamiz:

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[1] Gabriel Kramer (1704-1752) –fransuz, chiziqli algebraning asoschilaridan biri.

$$\Delta = \begin{vmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 3 & 2 & -2 \end{vmatrix} = 2 + 0 - 4 - 3 - 6 - 0 = -11 \neq 0.$$

$\Delta \neq 0$ bo‘lgani uchun berilgan tenglamalar sistemasi yagona yechimga ega bo‘ladi. Kramer formulasiga ko‘ra sistemaning yechimini topamiz.

Δ_1, Δ_2 va Δ_3 larning qiymatini aniqlaymiz:

$$\Delta_1 = \begin{vmatrix} 2 & 0 & -1 \\ -1 & -1 & 3 \\ 5 & 2 & -2 \end{vmatrix} = 4 + 0 + 2 - 5 - 12 - 0 = -11,$$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 5 & -2 \end{vmatrix} = 2 + 18 - 10 - 3 - 15 + 8 = 0,$$

$$\Delta_3 = \begin{vmatrix} 1 & 0 & 2 \\ 2 & -1 & -1 \\ 3 & 2 & 5 \end{vmatrix} = -5 + 0 + 8 + 6 + 2 - 0 = 11.$$

Kramer formulasiga ko‘ra,

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{-11}{-11} = 1, \quad x_2 = \frac{\Delta_2}{\Delta} = \frac{0}{-11} = 0,$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{11}{-11} = -1.$$

2-misol. Tenglamalar sistemasini yeching.

$$\begin{cases} 2x_1 + x_2 + x_3 = 1, \\ x_1 - 2x_2 + 3x_3 = 2, \\ 4x_1 + 2x_2 + 2x_3 = -2. \end{cases}$$

Yechish. $\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 3 \\ 4 & 2 & 2 \end{vmatrix} = -8 + 12 + 2 + 8 -$

$$-12 - 2 = 0.$$

Δ_1, Δ_2 va Δ_3 larning qiymatini aniqlaymiz:

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -2 & 3 \\ -2 & 2 & 2 \end{vmatrix} = -4 + 0 + 4 - 4 - 6 - 0 = -10,$$

$$\Delta_2 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & -2 & 2 \end{vmatrix} = 8 + 12 + 2 + 8 + 12 - 2 = 40,$$

$$\Delta_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 2 \\ 4 & 2 & -2 \end{vmatrix} = 8 + 2 + 8 + 8 - 8 + 2 = 20.$$

$\Delta = 0$ bo‘lgani uchun 2-xossaga ko‘ra sistema yechimga ega emas.

2-§. Chiziqli algebraik tenglamalar sistemasini yechishning Gauss[2] usuli.

Uch noma’lumli tenglamalar sistemasi berilgan bo‘lsin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Noma‘lumlar oldidagi koeffisiyentlardan A matritsani tuzamiz:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

A matritsaning uchinchi ustunidan so‘ng ozod hadlardan iborat to‘rtinchi ustunni vertikal chiziq bilan ajratgan holda yozamiz va hosil bo‘lgan kengaytirilgan matritsani $\overline{A}$ bilan belgilaymiz:

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[2] Gauss Iogann Karl Fridrix (Johann Carl Friedrich, 1777-1855) – nemis matematigi.

$$\overline{A} = \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)$$

Kengaytirilgan $\overline{A}$ matritsada diagonal elementlaridan pastda joylashgan a_{21}, a_{31} va a_{32} elementlar o‘rnida nol hosil qilishimiz kerak. Birinchi yo‘l elementlarini a_{21} ga va ikkinch yo‘l elementlarini a_{11} ga ko‘paytirib, mos ravishda ayiramiz hamda hosil bo‘lgan natijani ikkinchi yo‘lga yozamiz. Xuddi shuningdek, birinchi yo‘l elementlarini a_{31} ga va uchinchi yo‘l elementlarini a_{11} ga ko‘paytirib, mos ravishda ayiramiz hamda hosil bo‘lgan natijani uchinchi yo‘lga yozamiz:

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right)$$

bu yerda $a'_{22} = a_{12} \cdot a_{21} - a_{22} \cdot a_{11}$, $a'_{23} = a_{13} \cdot a_{21} - a_{23} \cdot a_{11}$, $b'_2 = b_1 \cdot a_{21} - b_2 \cdot a_{11}$ va h.k.

Hosil bo‘lgan matritsaning ikkinchi yo‘l elementlarini $\acute{a}_{32}$ ga va uchinchi yo‘l elementlarini $\acute{a}_{32}$ ga ko‘paytirib, mos ravishda ayiramiz hamda hosil bo‘lgan natijani uchinchi yo‘lga yozamiz:

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \acute{a}_{22} & \acute{a}_{23} & \acute{b}_2 \\ 0 & 0 & \ddot{a}_{33} & \ddot{b}_3 \end{array}\right)$$

bu yerda $\ddot{a}_{33}$=$\acute{a}_{23} \cdot \acute{a}_{32} - \acute{a}_{33} \cdot \acute{a}_{22}$, $\ddot{b}_3$=$\acute{b}_2 \cdot \acute{a}_{32} - \acute{b}_3 \cdot \acute{a}_{22}$.

Hosil bo‘lgan matritsani noma’lumlar orqali ifodalaymiz:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & | & b_1 \\ 0 & \acute{a}_{22} & \acute{a}_{23} & | & \acute{b}_2 \\ 0 & 0 & \ddot{a}_{33} & | & \ddot{b}_3 \end{pmatrix} \sim \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ \acute{a}_{22}x_2 + \acute{a}_{33}x_3 = \acute{b}_2 \\ \ddot{a}_{33}x_3 = \ddot{b}_3 \end{cases}$$

Sistemaning uchinchi tenglamasidan noma'lum koeffitsiyent x_3 ning qiymatini topamiz:

$$x_3 = \frac{\ddot{b}_3}{\ddot{a}_{33}}.$$

x_3 ning qiymatini sistemaning ikkinchi tenglamasiga qoyamiz va noma'lum koeffitsiyent x_2 ning qiymatini aniqlaymiz. Shuningdek, 1-tenglamadan x_1 ning qiymati aniqlanadi.

3-misol. Tenglamalar sistemasini yeching.

$$\begin{cases} x_1 - x_3 = 2, \\ 2x_1 - x_2 + 3x_3 = -1, \\ 3x_1 + 2x_2 - 2x_3 = 5. \end{cases}$$

Yechish. Tenglamalar sistemasini Gauss usulida yechamiz. Ushbu kengaytirilgan matritsani tuzamiz:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & -1 & 3 & -1 \\ 3 & 2 & -2 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -5 & 5 \\ 0 & -2 & -1 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -5 & 5 \\ 0 & 0 & 11 & -11 \end{array}\right) \sim \begin{cases} x_1 - x_3 = 2, \\ x_2 - 5x_3 = 5, \\ 11x_3 = -11. \end{cases}$$

Sistemaning uchinchi tenglamasidan noma'lum koeffitsiyent x_3 ning qiymatini topamiz:

$$11x_3 = -11 \quad \Rightarrow \quad x_3 = \frac{-11}{11} \quad \Rightarrow \quad x_3 = -1.$$

x_3 ning qiymatini sistemaning ikkinchi tenglamasiga qoyamiz va noma’lum koeffitsiyent x_2 ning qiymatini aniqlaymiz:

$$x_2 - 5 \cdot (-1) = 5 \implies x_2 + 5 = 5 \implies x_2 = 0.$$

Birinchi tenglamadan x_1 ning qiymatini aniqlaymiz:

$$x_1 - x_3 = 2 \implies x_1 - (-1) = 2 \implies x_1 = 1.$$

Demak, sistemaning yechimi $\{1;\ 0;\ -1\}$**.**

3-§. Chiziqli algebraik tenglamalar sistemasini teskari matritsa yordamida yechish

Uch noma’lumli tenglamalar sistemasi berilgan bo‘lsin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \qquad (10)$$

Noma‘lumlar oldidagi koeffisientlardan A matritsani, noma’lumlardan tashkil topgan X – ustun matritsani va ozod hadlardan B-ustun matritsani tuzamiz:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

u holda (10) tenglamalar sistemasini matritsali tenglama, ya’ni

$$AX=B \qquad (11)$$

ko‘rinishda ifodalash mumkin.

Agar A matritsa xosmas matritsa bo‘lsa, u holda (11) tenglama quyidagicha yechiladi. (11) tenglamaning o‘ng va chap qismini A matritsaga teskarisi matritsa A^{-1} ni ko`paytiramiz:

$$A^{-1}(AX) = A^{-1}B \quad \text{yoki } (A^{-1}A)X = A^{-1}B,$$

$A^{-1}A = E$ va $EX = X$ bo‘lgani uchun tenglamaning

$$X = A^{-1}B \qquad (12)$$

ko‘rinishidagi yechimiga ega bo‘lamiz.

8-misol. Ushbu tenglamalar sistemasini matritsalar yordamida yeching.

$$\begin{cases} x_1 - x_3 = 2, \\ 2x_1 - x_2 + 3x_3 = -1, \\ 3x_1 + 2x_2 - 2x_3 = 5. \end{cases}$$

Yechish.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 3 & 2 & -2 \end{pmatrix}; \; X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

A matritsa determinantini hisoblaymiz:

$$detA = \begin{vmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 3 & 2 & -2 \end{vmatrix} = 2 + 0 - 4 - 3 - 6 - 0 = -11 \neq 0.$$

Demak, $detA \neq 0 \;\Rightarrow\; A^{-1} -$ mavjud. A^{-1} ni topamiz:

$$A^{-1} = \frac{1}{detA}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix},$$

bu yerda

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ 2 & -2 \end{vmatrix} = -4,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = 13,$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 7,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & -1 \\ 2 & -2 \end{vmatrix} = -2,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = 1,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = -2,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & -1 \\ -1 & 3 \end{vmatrix} = -1,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = -5,$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1.$$

Demak, berilgan matritsaga teskari matritsa quyidagi ko‘rinishga bo‘ladi:

$$A^{-1} = \frac{1}{-11}\begin{pmatrix} -4 & -2 & -1 \\ 13 & 1 & -5 \\ 7 & -2 & -1 \end{pmatrix}.$$

$$A^{-1} = \begin{pmatrix} \frac{4}{11} & \frac{2}{11} & \frac{1}{11} \\ -\frac{13}{11} & -\frac{1}{11} & \frac{5}{11} \\ -\frac{7}{11} & \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

(12) tenglikdan sistemaning yechimini topamiz:

$$X = \begin{pmatrix} \frac{4}{11} & \frac{2}{11} & \frac{1}{11} \\ -\frac{13}{11} & -\frac{1}{11} & \frac{5}{11} \\ -\frac{7}{11} & \frac{2}{11} & \frac{1}{11} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{8}{11} - \frac{2}{11} + \frac{5}{11} \\ -\frac{26}{11} + \frac{1}{11} + \frac{25}{11} \\ -\frac{14}{11} - \frac{2}{11} + \frac{5}{11} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Demak, tenglamalar sistemasining yechimi:

$$x_1 = 1;\ x_2 = 0;\ x_3 = -1.$$

4-§. Bir jinsli chiziqli algebraik tenglamalar sistemasi

Ozod hadlari nolga teng bo‘lgan chiziqli algebraik tenglamalar sistemasini qaraylik.

Ta’rif. Agar har bir tenglamada ozod hadlar nolga teng bo‘lsa, birinchi darajali ***tenglamalar sistemasi bir jinsli*** deyiladi.

Ushbu bir jinsli chiziqli algebraik tenglamalar sistemasini berilgan bo‘lsin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0. \end{cases} \qquad (13)$$

Ma’lumki, $x_1 = 0;\ x_2 = 0;\ x_3 = 0$ sonlar (13) sistemaning har bir tenglamasini qanoatlantiradi. Bu yechim (13) sistemaning ***trivial yechimi*** deyiladi.

Agar (13) sistemaning asosiy determinanti $detA \neq 0$ bo‘lsa, sistema *trivial* yechimga ega bo‘ladi. (13) sistemada ozod hadlar nolga teng bo‘lgani uchun $\Delta_1 = \Delta_2 = \Delta_3 = 0$ bo‘ladi. Kramer formulasiga ko‘ra, $x_1 = 0;\ x_2 = 0;\ x_3 = 0.$

Demak, (13) sistema noldan farqli yechimga ega bo‘lishi uchun $detA = 0$ bo‘lishi zarur ekan.

Ikkita tenglamadan iborat sistema berilgan bo‘lsin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0. \end{cases} \quad (14)$$

1 −holat. Noma’lumlar oldidagi koeffitsiyentlar proporsional emas, ya’ni quyidagi uchta determinantlarning kamida bittasi nolga teng bo‘lmaydi:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix}. \quad (15)$$

U holda yechimni simmetrik ko‘rinishda yoshish mumkin:

$$x_1 = k\begin{vmatrix} a_{12} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad x_2 = -k\begin{vmatrix} a_{11} & a_{31} \\ a_{21} & a_{31} \end{vmatrix},$$

$$x_3 = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad (16)$$

bu yerda *k* – ixtiyoriy o‘zgarmas son.

2−holat. Noma’lumlar oldidagi koeffitsiyentlar o‘zaro proporsional bo‘lsin, ya’ni (15) determinantlarning hammasi nolga bo‘lsin. Sistema bitta tenglamaga keltiriladi.

9-misol. Tenglamalar sistemasini yeching.

$$\begin{cases} 2x_1 - x_2 + 3x_3 = 0, \\ x_1 + 2x_2 - 5x_3 = 0. \end{cases}$$

Yechish.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5,$$

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 2 & -5 \end{vmatrix} = -1,$$

$$\begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ -5 & 1 \end{vmatrix} = 13.$$

(16) formulaga asosan: $x_1 = -k;\ x_2 = -13k;\ x_3 = 5k.$

10-misol. Tenglamalar sistemasini yeching.

$$\begin{cases} 3x_1 + x_2 - 2x_3 = 0, \\ 6x_1 + 2x_2 - 4x_3 = 0. \end{cases}$$

Yechish. Noma'lumlar oldidagi koeffitsiyentlar o'zaro proporsionaldir. U holda, (15) determinantlarning hammasi nolga teng bo'ladi. Sistemani bitta tenflama bilan ifodalash mumkin. Tenglamadagi ixtiyoriy ikkita noma'lumga ixtiyoriy qiymatlar beriladi va uchinchi noma'lum aniqlanadi.

11-misol.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 3x_1 + x_2 + 2x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 0 \end{cases}$$

Yechish. Sistemaning asosiy determinantini hisoblaymiz:

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 1 + 8 + 27 - 6 - 6 - 6 = 18 \neq 0.$$

Demak, sistema $x_1 = 0;\ x_2 = 0;\ x_3 = 0$ trivial yechimga egadir.

12-misol.

$$\begin{cases} -2x_1 - 2x_2 + x_3 = 0, \\ x_1 + x_2 + 4x_3 = 0, \\ 3x_1 + 3x_2 - x_3 = 0. \end{cases}$$

Yechish. Sistemaning asosiy determinantini hisoblaymiz:

$$\Delta = \begin{vmatrix} -2 & -2 & 1 \\ 1 & 1 & 4 \\ 3 & 3 & -1 \end{vmatrix} = 2 - 24 + 3 - 3 + 24 - 2 = 0.$$

Demak, sistema noldan farqli yechimga egadir. Sistemaning ixtiyoriy ikkita tenglamasini olib, (16) formulaga asosan, sistemaning yechimni topamiz. Bu yerda k – ixtiyoriy o‘zgarmas son.

$$x_1 = k\begin{vmatrix} -2 & 1 \\ 1 & 4 \end{vmatrix} = -9k, \quad x_2 = -k\begin{vmatrix} -2 & 1 \\ 1 & 4 \end{vmatrix} = 9k,$$

$$x_3 = k\begin{vmatrix} -2 & -2 \\ 1 & 1 \end{vmatrix} = 0.$$

Sistemaning echimi: $\{-9k;\ 9k;\ 0\}$.

5-§. Chiziqli algebraik tenglamalar sistemasining tatbiqlari

Chiziqli algebraik tenglamalar sistemasi yordamida iqtisodiyot tarmoqlariga tegishli masalalarning yechimlarini toppish mumkin. Chiziqli tenglamalar sistemasining amaliy masalalarga tatbiqini quyidagi masalalarda ko‘rib chiqamiz.

1-masala. Zavodda 3 xil turdagi temir-buyum mahsulotlari ishlab chiqariladi. Mahsulotlar uchun 3 turdagi S_1, S_2 va S_3 xom-ashyo ishlatiladi. Bitta mahsulot uchun har bir xom-ashyodan ishlatish me’yori va bir oylik xom-ashyo ishlatish hajmi 1-jadvalda berilgan. Zavodning har bir mahsulot bo‘yicha bir oylik ishlab chiqarish hajmini toping.

1-jadval

Xom-ashyo turlari	Bitta mahsulot ishlab chiqarish uchun xom-ashyo islatilishi me'yori (shartli birlikda)			Bir oylik xom-ashyo islatilishi (shartli birlikda)
	darvoza	deraza panjarasi	zinapoya to'siqlari	
S_1	2	0	3	69
S_2	1	2	1	60
S_3	5	0	4	120

Yechish. Masalani chiziqli algebraik tenglamalar sistemasi yordamida yechamiz.

Faraz qilaylik, zavod bir oyda x dona darvoza, y dona deraza panjarasi, z dona zinapoya to'siqlari ishlab chiqarsin. U holda, har bir turdagi mahsulot uchun xom-ashyo sarflanishiga mos holda, quyidagi sistemani hosil qilamiz:

$$\begin{cases} 2x + 3z = 69, \\ x + 2y + z = 60, \\ 5x + 4z = 120. \end{cases}$$

Bu sistemani turli usullar bilan yechish mumkin. Biz Kramer usulidan foydalanamiz. Buning uchun asosiy determinantni tuzamiz va hisoblaymiz:

$$\Delta = \begin{vmatrix} 2 & 0 & 3 \\ 1 & 2 & 1 \\ 5 & 0 & 4 \end{vmatrix} = -14.$$

Asosiy determinant noldan farqli, demak, sistema birgalikda va yagona yechimga ega. Yordamchi determinantlarni tuzamiz va hisoblaymiz:

$$\Delta x = \begin{vmatrix} 69 & 0 & 3 \\ 60 & 2 & 1 \\ 120 & 0 & 4 \end{vmatrix} = -168,$$

$$\Delta y = -231, \quad \Delta z = -210.$$

Kramer formulasiga asosan, masalaning matematik nuqtai-nazardan yechimini topamiz:

$$x = \frac{-168}{-14} = 12, \quad y = \frac{-231}{-14} = 16.5, \quad z = \frac{-210}{-14} = 15.$$

Masala yechimi butun bo‘lishini hisobga olsak, sistemaning noma’lumlari qiymatidan quyidagi xulosaga kelamiz, ya’ni zavod bir oyda 12 ta darvoza, 16 ta deraza va 15 ta zinapoya to‘siqlarini ishlab chiqaradi.

2-masala. Ma’lum bir sondagi o‘ramli materialdan fabrikada A- ko‘rinishda 360 ta, B-ko‘rinishdagi – 300 ta va C-ko‘rinishdagi 675 ta mahsulot tikiladi. 3- xil usuldagi bichishdan foydalanish mumkin. Har bir material o‘ramidan bichish usullari bo‘yicha mahsulotlar tayyorlash miqdori 2-jadvalda berilgan. Reja bajarilishi shartini matematik shaklda yozing.

Yechish. x, y va z bilan mos ravichda birinchi, ikkinchi va uchinchi bichish usullari bo‘yicha ishlatilgan material o‘ramlari bo‘lsin. U holda, 1-bichish usulida x ta o‘ramda $3x$ ta, 2-bichish usulida – $2y$ ta, 3-bichish usulida – z ta A

turdagi mahsulotlar rejasini bajarish uchun quyidagi tenglama o‘rinli bo‘lishi kerak: $3x + 2y + z = 360$. Xuddi shu yo‘l bilan $x + 6y + 2z = 300$, $4x + y + 5z = 675$ tenglamalarni hosil qilamiz. Ularni ushbu sistema ko‘rinishida ifodalaymiz:

$$\begin{cases} 3x + 2y + z = 360, \\ x + 6y + 2z = 300, \\ 4x + y + 5z = 675. \end{cases}$$

2-jadval

Mahsulot turlari	Bichish shakllari		
	1	2	3
A	3	2	1
B	1	6	2
C	4	1	5

Sistemani Gauss usulida yechamiz:

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 360 \\ 1 & 6 & 2 & 300 \\ 4 & 1 & 5 & 675 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 6 & 2 & 300 \\ 3 & 2 & 1 & 360 \\ 4 & 1 & 5 & 675 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 6 & 2 & 300 \\ 0 & -16 & -5 & -550 \\ 0 & -7 & 2 & 15 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 6 & 2 & 300 \\ 0 & 16 & 5 & 550 \\ 0 & -14 & 4 & 30 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 6 & 2 & 300 \\ 0 & 16 & 5 & 550 \\ 0 & 2 & 9 & 570 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 6 & 2 & 300 \\ 0 & 2 & 9 & 570 \\ 0 & 16 & 5 & 540 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 6 & 2 & 300 \\ 0 & 2 & 9 & 570 \\ 0 & 0 & -67 & -4020 \end{array}\right) \sim \begin{cases} x + 6y + 2z = 300, \\ 2y + 9z = 570, \\ -67z = -4020. \end{cases}$$

Bu tenglamalar sistemasi yuqorida masala shartiga binoan tuzilgan tenglamalar sistemasiga teng kuchlidir. Hosil bo‘lgan sistemadan $x = 90,\ y = 15, z = 60$ qiymatlarni aniqlaymiz.

Yuqorida ko‘rilgan masalalardan ma’lumki, chiziqli algebraik tenglamalar sistemasining iqtisodiy masalalarni yechishda o‘rni kattadir.

Mustaqil yechish uchun misollar

1. Tenglamalar sistemasini yeching:

1) $\begin{cases} 2x - 3y + z - 2 = 0, \\ x + 5y - 4z + 5 = 0, \\ 4x + y - 3z + 4 = 0. \end{cases}$ 2) $\begin{cases} 7x + 2y + 3z = 15, \\ 5x - 3y + 2z = 15, \\ 10x - 11y + 5z = 36. \end{cases}$

3) $\begin{cases} x + 2y + 3z = 4, \\ 2x + y - z = 3, \\ 3x + 3y + 2z = 10. \end{cases}$ 4) $\begin{cases} 2x - y + z = 2, \\ 3x + 2y + 2z = -2, \\ x - 2y + z = 1. \end{cases}$

2. Sistemalarni Gauss usuli bilan yeching:

1) $\begin{cases} x - y + 3z = -4, \\ 2x + 3y - 2z = 5, \\ 3x + 5y + z = 4. \end{cases}$ 2) $\begin{cases} x - 2y - 3z = 8, \\ 3x + y + z = 3, \\ 4x + 3y - 2z = -1. \end{cases}$

3) $\begin{cases} x - 2y + 3z = 6, \\ 2x + 3y - 4z = 16, \\ 3x - 2y - 5z = 12. \end{cases}$ 4) $\begin{cases} 2x + y - 3z = 3, \\ 3x + 4y - 5z = 9, \\ 2y + 7z = 11. \end{cases}$

3. Tenglamalar sistemasini matritsalar yordamida yeching:

1) $\begin{cases} x + 2y + z = 4, \\ 3x - 5y + 3z = 1, \\ 2x + 7y - z = 8. \end{cases}$ 2) $\begin{cases} 2x - 4y + 9z = 28, \\ 7x + 3y - 6z = -1, \\ 7x + 9y - 9z = 5. \end{cases}$

3) $\begin{cases} 2x + y + 2z = 6, \\ x - 3y - z = -5, \\ 5x - 2y + z = -1. \end{cases}$ 4) $\begin{cases} x - 2y + 3z = 5, \\ 2x + 3y - z = -4, \\ 3x + y - 2z = -1. \end{cases}$

5) $\begin{cases} 2x + y + 2z = 6, \\ x - 3y - z = -5, \\ 5x - 2y + z = -1. \end{cases}$ 6) $\begin{cases} x - 2y + 3z = 5, \\ 2x + 3y - z = -4, \\ 3x + y - 2z = -1. \end{cases}$

4. Bir jinsli chiziqli algeraik tenglamalar sistemasini yeching:

1) $\begin{cases} 3x + 2y - z = 0, \\ 2x - y + 3z = 0, \\ x + 3y - 4z = 0. \end{cases}$ 2) $\begin{cases} 3x + 2y - z = 0, \\ 2x - y + 3z = 0, \\ x + y - z = 0. \end{cases}$

3) $\begin{cases} x + 2y - 4z = 0, \\ 2x - y - 3z = 0, \\ x + 3y + z = 0. \end{cases}$ 4) $\begin{cases} 5x - 3y + 2z = 0, \\ 2x + 4y - 3z = 0, \\ 3x - 7y + 5z = 0. \end{cases}$

5) $\begin{cases} 3x - y + 2z = 0, \\ 2x + 3y - 5z = 0, \\ x + y + z = 0. \end{cases}$ 6) $\begin{cases} 2x - y + 3z = 0, \\ x + 2y - 5z = 0, \\ 3x + y - 2z = 0. \end{cases}$