Mavzu:

Irratsional funksiyalarni integrallash. Binomial differensiallarni integrallash.

Reja:

1. $\int R(x^{\frac{m_1}{n_1}},\ x^{\frac{m_2}{n_2}},\dots,x^{\frac{m_k}{n_k}})dx$ kurinishdagi integrallar.
2. $\int R(z^{\frac{m_1}{n_1}},\ z^{\frac{m_2}{n_2}},\dots,z^{\frac{m_k}{n_k}})dx,\ z=\frac{ax+b}{cx+d}$ kurinishdagi integrallar.
3. Binomial differensiallarni integrallash.
4. CHebishev teoremasi.

1. $\int R(x^{\frac{m_1}{n_1}},\ x^{\frac{m_2}{n_2}},\dots,x^{\frac{m_k}{n_k}})dx$ kurinishdagi integrallar.

$$\int R(x^{\frac{m_1}{n_1}},x^{\frac{m_2}{n_2}},\dots,x^{\frac{m_k}{n_k}})dx \qquad (1)$$

kurinishdagi integralni karaymiz, bu erda R – uz argumentlarining ratsional funksiyasi. Irratsionallik argumentlarda namoyon buladi. Berilgan integralni ratsional funksiyani integrallashga keltiriladi. Buning uchun kuyidagi uzgaruvchini almashtirishni bajarish kifoya:

$$x=t^n,\quad t=\sqrt[n]{x},\quad dx=nt^{n-1}dt.$$

Bu erda n soni $n_1,n_2,\dots,n_k$ sonlarning eng kichik umumiy bulinuchisi:

$$n=\text{EKUB}(n_1,n_2,\dots,n_k),$$

yoki boshkacha aytganda n soni $\frac{m_1}{n_1},\frac{m_2}{n_2},\dots,\frac{m_k}{n_k}$ kasrlarning eng kichik umumiy maxraji.

Misol 1. $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}$ integralni xisoblang.

Bu erda $\sqrt{x}=x^{\frac{1}{2}},\ \sqrt[3]{x}=x^{\frac{1}{3}}$; $\frac{1}{2}$ va $\frac{1}{3}$ kasrlarning eng kichik umumiymaxraji 6. SHunga kura $x=t^6$ almashtirish bajaramiz:

$$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \left| \begin{matrix} x = t^6 \\ t = \sqrt[6]{x} \\ dx = 6t^5 dt \end{matrix} \right| = \int \frac{6t^5 dt}{t^3 + t^2} = 6 \int \frac{t^3 dt}{t+1} =$$

$$= 6 \int \left(t^2 - t + 1 - \frac{1}{t+1} \right) = 2t^3 - 3t^2 + 6t - 6\ln|t+1| + C =$$

$$= 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left(\sqrt[6]{x} + 1\right) + C.$$

2. $\int R\left(z^{\frac{m_1}{n_1}}, z^{\frac{m_2}{n_2}}, \dots, z^{\frac{m_k}{n_k}}\right)dx$ **kurinishdagi integrallar.**

$$\int R\left(z^{\frac{m_1}{n_1}}, z^{\frac{m_2}{n_2}}, \dots, z^{\frac{m_k}{n_k}}\right)dx, \qquad (2)$$

kurinishdagi integralni karaymiz, bu erda R – uz argumentlarining ratsional funksiyasi, $z = \frac{ax+b}{cx+d}$.

Mazkur integralda irratsionallik xuddi oldingi integral kabi yukotiladi:

$$\frac{ax+b}{cx+d} = t^n, \quad t = \sqrt[n]{\frac{ax+b}{cx+d}}, \quad x = \frac{dt^n - b}{a - ct^n}, \quad dx = \frac{n(ad-bc)t^{n-1}}{(a-ct^n)^2}.$$

Misol 2. $\int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}}$ integralni xisoblang.

$$\int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}} = \int \sqrt[3]{\frac{x+1}{x-1}} \frac{dx}{x+1}.$$

x ni almashtiramiz $t = \sqrt[3]{\frac{x+1}{x-1}}$, $x = \frac{t^3+1}{t^3-1}$, $dx = \frac{6t^2 dt}{(t^3-1)^2}$; bundan

$$\int \sqrt[3]{\frac{x+1}{x-1}} \frac{dx}{x+1} = \int \frac{-3dt}{t^3-1} = \int \left(-\frac{1}{t-1} + \frac{t+2}{t^2+t+1} \right) dt =$$

$$= \frac{1}{2}\ln\frac{t^2+t+1}{(t-1)^2} + \sqrt{3}\,\mathrm{arctg}\,\frac{2t-1}{\sqrt{3}} + C,$$

Bu erda $t = \sqrt[3]{\frac{x+1}{x-1}}$.

3. Binomial differensiallarni integrallash.

Binomal deb

$$x^m(a+bx^n)^p dx \qquad (3)$$

kurinishdagi differensiallarga aytiladi; bu erda *a*, *b* – ixtiyoriy sonlar, *m*, *n*, *p* – ratsional sonlar. Bu ifodalarni kaysi xollarda integrallashini kurib chikamiz.

1) *r* - butun son (musbat, nol yoki manfiy). Bu xol (1) integralda kurilgan.Agar *m* va *n* kasrlahyb eng kichik umumiy maxrajini λ deb belgilasak ifoda (3) $R\left(\sqrt[\lambda]{x}\right)dx$ kurinishni oladi. Bu erda

$$t=\sqrt[\lambda]{x}$$

almashtirish masalani xal etadi.

2) Karalayotgan ifodada $z=x^n$ almashtirish bajaramiz. U xolda

$$x^m(a+bx^n)^p dx=\frac{1}{n}(a+bz)^p z^{\frac{m+1}{n}-1}dz$$

va, bu erda $\frac{m+1}{n}-1=q$ belgilash kiritib, kuyidagiga ega bulamiz

$$\int x^m(a+bx^n)^p dx=\frac{1}{n}\int(a+bz)^p z^q dz. \qquad (4)$$

Agar *q* butun son bulsa, avval kurilgan xol integral (2) vujudga keladi. Bu erda υ deb *r* ning maxrajini belgilasak

$$t=\sqrt[v]{a+bz}=\sqrt[v]{a+bx^n}$$

almashtirish masalani xal etadi.

3) (4) dagi ikkinchi ifodani kuyidagicha yozamiz:

$$\frac{1}{n}\int\left(\frac{a+bz}{z}\right)^p z^{p+q}\ .$$

Agar *p* + *q* butun son bulsa, avval kurilgan xol integral (2) vujudga keladi. Bu erda υ deb *r* ning maxrajini belgilasak

$$t=\sqrt[v]{\frac{a+bz}{z}}=\sqrt[v]{ax^{-n}+b}$$

almashtirish masalani xal etadi.

4. CHebishev teoremasi.

(4) dagi ikkala integral fakat karalgan uchta xolda integrallanadi, kolgan xolatlar bundan mustusno.

Misol 3. $\int \frac{dx}{\sqrt[4]{1+x^4}}$ integralni xisoblang.

$$\int \frac{dx}{\sqrt[4]{1+x^4}} = \int x^0 (1+x^4)^{-\frac{1}{4}} dx.$$

Bu erda $m = 0$, $n = 4$, $p = -\frac{1}{4}$; uchinchi xolat $\frac{m+1}{n} + p = 0$, $\upsilon = 4$. Kursatilgan almashtirishni bajaramiz:

$t = \sqrt[4]{x^{-4}+1} = \frac{\sqrt[4]{1+x^4}}{x}$, $x = (t^4-1)^{-\frac{1}{4}}$, $dx = -t^3(t^4-1)^{-\frac{5}{4}} dt$, bundan

$$\sqrt[4]{1+x^4} = tx = t(t^4-1)^{-\frac{1}{4}},$$

$$\int \frac{dx}{\sqrt[4]{1+x^4}} = -\int \frac{t^2 dt}{t^4-1} = \frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| - \frac{1}{2}\operatorname{arctg} t + C.$$