* Chiziqli tenglamalar sistemasini yechishning Gauss va teskari matritsa usuli

Biz noma‘lumlar soni tenglamalar soniga teng chiziqli tenglamalar sistemasini yechishning Kramer va matritsa usuli bilan tanishdik. Bu usullarning zaif tomonlari shundaki, noma‘lumlar soni biroz katta bo’lganda juda ko’p hisoblashlarni bajarishga to’g’ri keladi. Masalan to’rt noma‘lumli to’rtta chiziqli tenglamalar sistemasini Kramer usuli bilan yechish uchun beshta to’rtinchi tartibli determinantlarni hisoblashga to’g’ri keladi. To’rtinchi tartibli determinant biror satr yoki ustun elementlari bo’yicha yoyilganda yoyilmada to’rtta uchinchi tartibli determinant qatnashadi. Demak jami $5 \cdot 4 = 20$ ta uchunchi tartibli determinantlarni hisoblashga to’g’ri keladi. Besh va undan ortiq noma‘lumlar qatnashgan sistema haqida gapirmasak ham bo’ladi.

Bunday hollarda chiziqli tenglamalar sistemasini Gauss taklif etgan quyidagi usul bilan yechgan ma‘qul.

Gauss usuli tenglamalardan noma‘lumlarni ketma-ket yo’qotishga asoslangan bo’lib oxirgi tenglamada bitta noma‘lum qoladi xolos. Undan noma‘lumni topib oxirgidan oldingi tenglamaga qo’yib ikkinchi noma‘lum topiladi va hokazo shu jarayon davom ettirilib topilgan noma‘lumlarning qiymatlarini birinchi tenglamaga qo’yib undan birinchi noma‘lum aniqlanadi.

Gauss usuli bilan misolda tanishib chiqamiz.

2-misol. $$\begin{cases} 2x+3y-z+t=-2, \\ 3x-y+2z-3t=-3, \\ 2x+y-z+2t=2, \\ x-2y+z-t=1 \end{cases} \qquad (1)$$

sistema yechilsin.

Yechish. Sistemani Gauss usuli bilan yechamiz.

1-qadam ushbu

$$\left(\begin{array}{cccc|c} 2 & 3 & -1 & 1 & -2 \\ 3 & -1 & 2 & -3 & -3 \\ 2 & 1 & -1 & 2 & 2 \\ 1 & -2 & 1 & -1 & 1 \end{array}\right) \qquad (2)$$

Matritsani birinchi ustunini ikkinchi satridan boshlab barcha elementlarini nolga aylantiramiz. Birinchi satrni ikkiga bo’lib

$$\left(\begin{array}{cccc|c} 1 & 3/2 & -1/2 & 1/2 & -1 \\ 3 & -1 & 2 & -3 & -3 \\ 2 & 1 & -1 & 2 & 2 \\ 1 & -2 & 1 & -1 & 1 \end{array}\right) (3)$$

ko'rinishida yozamiz.

a) (3) matritsaning birinchi satrinini -3 ga ko'paytirib ikkinchi satriga qo'shsak, birinchi satrinini -2 ga ko'paytirib uchinchi satriga qo'shsak, birinchi satrini -1 ga ko'paytirib to'rtinchi satriga qo'shsak:

$$\left(\begin{array}{cccc|c} 1 & 3/2 & -1/2 & 1/2 & -1 \\ 0 & -11/2 & 7/2 & -9/2 & -0 \\ 0 & -2 & 0 & 1 & 4 \\ 0 & -7/2 & 3/2 & -3/2 & 2 \end{array}\right) (4)$$

hosil bo'ladi.

2-qadam. (4) matritsaning uchinchi satrining ikkinchi ustuni elementidan boshlab qolgan barcha elementlarini nolga aylantiramiz. Ikkinchi satrini $-\frac{11}{2}$ ga bo'lib ushbu

$$\begin{pmatrix} 1 & 3/2 & -1/2 & 1/2 & |-1 \\ 0 & 1 & -7/11 & 9/11 & |+0 \\ 0 & -2 & 0 & 1 & |\ 4 \\ 0 & -7/2 & 3/2 & -3/2 & |\ 2 \end{pmatrix} \quad (5)$$

ko'rinishda yozamiz.

(5) matritsaning ikkinchi satrini +2 ga ko'paytirib uchinchi satriga qo'shsak,ikkinchi satrini $\frac{7}{2}$ ga ko'paytirib to'rtinchi satrga qo'shsak:

$$\begin{pmatrix} 1 & 3/2 & -1/2 & 1/2 & |-1 \\ 0 & 1 & -7/11 & 9/11 & |+0 \\ 0 & 0 & -14/11 & 29/11 & |\ 4 \\ 0 & 0 & -8/11 & 15/11 & |\ 2 \end{pmatrix} \quad (6)$$

3-qadam. (6) matritsaning to'rtinchi satrini uchinchi ustun elementini nolga aylantiramiz.Dastlab buning uchun matritsani uchinchi satrini $-\frac{14}{11}$ *ga bo'lib*

$$\begin{pmatrix} 1 & 3/2 & -1/2 & 1/2 & |..-1 \\ 0 & 1 & -7/11 & 9/11 & |.....0 \\ 0 & 0 & 1 & -29/14 & \left|\frac{-22}{7}\right. \\ 0 & 0 & -8/11 & 15/11 & |....\ 2 \end{pmatrix}$$

ko'rinishda yozamiz. Bu matrirsaning uchinchi satrini $\frac{8}{11}$ ga ko'paytirib to'rtinchi satriga qo'shsak :

$$\begin{pmatrix} 1 & 3/2 & -1/2 & 1/2 & |..-1 \\ 0 & 1 & -7/11 & 9/11 & |.....0 \\ 0 & 0 & 1 & -29/14 & \left|\frac{-22}{7}\right. \\ 0 & 0 & 0 & -1/7 & |-2/7 \end{pmatrix} \quad (7)$$

matritsaga ega bo'lamiz. Bu matritsaga mos sistema qo'yidagicha bo'ladi.

$$\begin{cases} x+\frac{3}{2}y-\frac{1}{2}z+\frac{1}{2}t=-1, \\ y-\frac{7}{11}z+\frac{9}{11}t=0, \\ z-\frac{29}{14}t=-\frac{22}{7}, \\ -\frac{1}{7}t=-\frac{2}{7} \end{cases} \qquad (8)$$

oxirgi tenglamasida bitta *t* noma‘lum, undan oldingisida ikkita *z* va *t* noma‘lumlar, ikkinchi tenglamasida uchta *y*, *z*, *t* noma‘lumlar va birinchi tenglamasida barcha noma‘lumlar - *x*, *y*, *z*, *t* lar qatnashadi.

Endi noma‘lumlarni topish unchalik qiyin emas.

4-qadam. (8) sistemaning to’rtinchi tenglamasi $-\frac{1}{7}t=-\frac{2}{7}$ dan *t* ni topamiz. $t=\left(-\frac{2}{7}\right):\left(-\frac{1}{7}\right)=2.$

5-qadam. *t* ning topilgan qiymati 2 ni (8) sistemaning uchinchi tenglamasiga qo’yib *z* noma‘lumni topamiz: $z-\frac{29}{14}\cdot 2=-\frac{22}{7}; \quad z=\frac{29}{7}-\frac{22}{7}=\frac{7}{7}=1.$

6-qadam. $t=2$, $z=1$ qiymatlarni (8) sistemaning ikkinchi tenglamasi $y-\frac{7}{11}z+\frac{9}{11}t=0$ ga qo'yib y noma'lumni topamiz:

$$y-\frac{7}{11}\cdot 1+\frac{9}{11}\cdot 2=0;\quad y+1=0,\ y=-1.$$

7-qadam. Topilgan $y=-1$, $z=1$, $t=2$ qiymatlarni (8) sistemaning birinchi tenglamasi $x+\frac{3}{2}y-\frac{1}{2}z+\frac{1}{2}t=-1$ ga qo'yib x noma'lumni aniqlaymiz:

$$x+\frac{3}{2}(-1)-\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 2=-1;\quad x=0$$

Shunday qilib $x=0$, $y=-1$, $z=1$, $t=2$ ya'ni (0; -1; 1; 2) sonlar to'plami berilgan sistemaning yechimi bo'lar ekan.

Gauss usulining muhim tomoni shundan iboratki sistemani yechishdan oldin uni birgalikda yoki birgalikda emasligini aniqlashning hojati yo'q.

Agar sistema birgalikda va aniq bo'lsa bu usul xuddi yuqoridagi misoldagi singari yagona yechimga olib keladi.

Agar sistema birgalikda bo'lmasa bu usulning qaysidir qadamida yo'qotilishi lozim bo'lgan noma'lum bilan birgalikda barcha noma'lumlar ham yo'qolib ketadi va tenglikning o'ng tomonida esa noldan farqli ozod son qoladi.

3-misol.

$$\begin{cases} 3x - y + 4z = 6, \\ x + 2y - z = 3, \\ 5x + 3y + 2z = 8 \end{cases} \qquad (9)$$

sistema Gauss usuli bilan yechilsin.

Yechish. **1-qadam.** Birinchi va ikkinchi tenglamalarni o'rin almashtirib birinchi tenglamadagi x oldidagi koeffitsientni 1 ga keltiramiz:

$$\begin{cases} x + 2y - z = 3, \\ 3x - y + 4z = 6, \\ 5x + 3y + 2z = 8 \end{cases} \qquad (10)$$

a) bu sistemaning birinchi tenglamasini –3 ga ko'paytirib ikkinchi tenglamasiga qo'shamiz:

$$-3x-6y+3z=-9,$$

$$+$$

$$3x-y+4z=6.$$

$$-----------------$$

$$-7y+7z=-3$$

b) (3.10) sistemaning birinchi tenglamasini –5 ga ko'paytirib uchinchi tenglamasiga qo'shsak

$$-5x-10y+5z=-15,$$

$$+$$

$$5x+3y+2z=8.$$

$$-----------------$$

$$-7y+7z=-7$$

hosil bo’ladi.   Shunday qilib (3.31) sistema

$$\begin{cases} x + 2y - z = 3, \\ -7y + 7z = -3, \\ -7y + 7z = -7. \end{cases} \quad (10)$$

ko’rinishga ega bo’ladi.

2-qadam. (10) sistemaning ikkinchi tenglamasini –1 ga ko’paytirib uchinchisiga qo’shsak uchinchi tenglamasidagi yo’qotilishi lozim bo’lgan y bilan bir qatorda z noma‘lum ham yo’qolib ketadi, ya‘ni.

$$\begin{array}{l} \quad 7y - 7z = 3, \\ + \\ \; -7y + 7z = -7 \\ --------------- \\ \qquad 0 = -4 \end{array}$$

hosil bo’ladi.

Shunday qilib Gauss usuliga binoan sistema birgalikda emas, ya‘ni yechimga ega emas ekan.

Agar sistema birgalikda, ammo aniqmas bo'lsa Gauss usulining qandaydir qadamida ikkita bir xil tenglamalarga ega bo'lamiz.

Ya'ni bu holda tenglamalar soni noma'lumlar sonidan bittaga kam bo'ladi.

4-misol. $\begin{cases} 3x - y + 4z = 6, \\ x + 2y - z = 3, \\ 5x + 3y + 2z = 12 \end{cases}$ sistema Gauss usuli bilan yechilsin.

Yechish. Birinchi tenglamadagi x oldidagi koeffitsientni 1 ga keltirish maqsadida sistemadagi birinchi va ikkinchi tenglamalarni o'rinlarini almashtirib uni

$$\begin{cases} x + 2y - z = 3, \\ 3x - y + 4z = 6, \\ 5x + 3y + 2z = 12 \end{cases} \quad (11)$$

ko'rinishda yozamiz.

a) (11) sistemaning birinchi tenglamasini –3 ga ko’paytirib sistemaning ikkinchi tenglamasiga qo’shamiz:

$$-3x-6y+3z=-9,$$

$$+$$

$$3x-y+4z=6.$$

$$-----------------$$

$$-7y+7z=-3$$

b) (11) sistemaning birinchi tenglamasini –5 ga ko’paytirib uchinchi tenglamaga qo’shamiz:

$$-5x-10y+5z=-15,$$

$$+$$

$$5x+3y+2z=12.$$

$$-----------------$$

$$-7y+7z=-3$$

Shunday qilib

$$\begin{cases} x+2y-z=3, \\ -7y+7z=-3, \\ -7y+7z=-3 \end{cases}$$

sistemaga ega bo’lamiz. Bu sistema uch noma‘lumli ikkita tenglamalar sistemasi

$$\begin{cases} x + 2y - z = 3, \\ -7y + 7z = -3 \end{cases}$$

ga teng kuchli. Oxirgi sistema esa cheksiz ko’p yechimlarga ega.

Chiziqli tenglamalar sistemasini matritsalar yordamida yechish.
Endi matritsalar yordamida chiziqli tenglamalar sistemasini yechishga o’tamiz.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}\right\} \qquad (12)$$

n noma’lumli, n ta tenglamalar sistemasi berilgan bo’lsin.

$$A=\begin{pmatrix} a_{11} & a_{12}\cdots a_{1n} \\ a_{21} & a_{22}\cdots a_{2n} \\ ------ & ---- \\ a_{n1} & a_{n2}\cdots a_{nn} \end{pmatrix},\quad B=\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix},\quad X=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

belgilashlarni kiritamiz. Endi (12) sistemani matritsalarni ko'paytirish qoidasidan foydalanib,

$$AX = B \tag{13}$$

ko'rinishda yozish mumkin. $\det A \neq 0$ bo'lsa, teskari matritsa A^{-1} mavjud va $A^{-1}AX = A^{-1}B$ hosil bo'ladi. SHunday qilib, noma'lum X matritsa $A^{-1}B$ matritsaga teng bo'ladi, yahni

$$X = A^{-1}B.$$

Bu (12) tenglamalar sistemasini yechishning ***<u>matritsaviy yozuvini</u>*** bildiradi.

1-misol. Matritsalar yordamida ushbu tenglamalar sistemasini yeching:

$$\begin{cases} x_1 + x_2 + x_3 = 4, \\ x_1 + 2x_2 + 4x_3 = 4, \\ x_1 + 3x_2 + 9x_3 = 2 \end{cases}.$$

Echish. Quyidagi belgilashlarni kiritamiz:

$$A=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 3 & 9\end{pmatrix};\quad X=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix};\quad B=\begin{pmatrix}4\\ 4\\ 2\end{pmatrix}.$$

Bu matritsalar yordamida berilgan tenglamalar sistemasini

$$AX=B \qquad (14)$$

ko'rinishda yozamiz. Endi A matritsaning determinantini hisoblaymiz.

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 3 & 9\end{vmatrix}=1\cdot2\cdot9+1\cdot4\cdot1+1\cdot3\cdot1-1\cdot2\cdot1-1\cdot1\cdot9-1\cdot4\cdot3=2.$$

A matritsaning determinanti 0 dan farqli bo'lganligi uchun, unga teskari yagona A^{-1} matritsa mavjud va tenglamalar sistemasi yagona yechimga ega bo'ladi. Endi A^{-1} teskari matritsani topish uchun Δ determinant elementlarining hamma algebraik to'ldiruvchilarini hisoblaymiz:

$$A_{11}=\begin{vmatrix}2&4\\3&9\end{vmatrix}=18-12=6,\quad A_{12}=-\begin{vmatrix}1&4\\1&9\end{vmatrix}=-5,\quad A_{13}=\begin{vmatrix}1&2\\1&3\end{vmatrix}=1$$

$$A_{21}=-\begin{vmatrix}1&1\\3&9\end{vmatrix}=-6,\quad A_{22}=\begin{vmatrix}1&4\\1&9\end{vmatrix}=8,\quad A_{23}=-\begin{vmatrix}1&1\\1&3\end{vmatrix}=-2$$

$$A_{31}=\begin{vmatrix}1&1\\2&4\end{vmatrix}=2,\quad A_{32}=-\begin{vmatrix}1&1\\1&4\end{vmatrix}=-3,\quad A_{33}=\begin{vmatrix}1&1\\1&2\end{vmatrix}=1.$$

Teskari A^{-1} matritsani topish formulasiga asosan,

$$A^{-1}=\frac{1}{2}\begin{pmatrix}6&-6&2\\-5&8&-3\\1&-2&1\end{pmatrix}=\begin{pmatrix}3&-3&1\\-2{,}5&4&-1{,}5\\0{,}5&-1&0{,}5\end{pmatrix}$$

(14) tenglikning ikki tomonini chapdan A^{-1} ga ko'paytirsak, $A^{-1}AX=A^{-1}B$ yoki $X=A^{-1}B$ bo'lib, yahni

$$X=\begin{pmatrix}3 & -3 & 1\\ -2{,}5 & 4 & -1{,}5\\ 0{,}5 & -1 & 0{,}5\end{pmatrix}\cdot\begin{pmatrix}4\\4\\2\end{pmatrix}=\begin{pmatrix}3\cdot 4+(-3)\cdot 4+1\cdot 2\\ -2{,}5\cdot 4+4\cdot 4+(-1{,}5)\cdot 2\\ 0{,}5\cdot 4-1\cdot 4+0{,}5\cdot 2\end{pmatrix}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$$

tenglik hosil bo'ladi.

SHunday kilib, $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$ yoki $x_1=2$, $x_2=3$, $x_3=-1$.

(Topilgan yechimlarni tenglamalar sistemasiga bevosita qo'yib, yechimning to'g'riligini tekshirib ko'rish mumkin).