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On the Free Boundary Problem for the Predator-Prey Model

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Abstract. In this article, we investigate the free boundary problem for the classical predator-prey model with double free boundaries. This system mimics the spread of invasive or new predator species, in which free boundaries represent the expanding fronts of predator species and are described by the Stefan condition. For this system, the existence and uniqueness of solutions is checked, and the behavior of positive solutions is also considered. A priori estimates for the required functions are established. For this model, the dichotomy of spread and disappearance has been proven.

INTRODUCTION

Migration of a new or invasive species is one of the most important topics in mathematical ecology. Many mathematicians have tried to develop various invasion models and investigate them from the point of view of mathematical ecology. For example, in [1-8], population models with a free reaction-diffusion boundary are proposed in order to understand the process of creating a new or invasive population. In fact, the spatial distribution of prey and predator is not uniform within a fixed bounded areal, so it is more realistic to introduce reaction-diffusion equations to describe the spatial distribution of each species. H.W. Yin et al. Investigated a modified Leslie-Gower predator-prey model with Crowley-Martin functional response and spatial diffusion under uniform Neumann boundary conditions [11-12]. They obtained the existence of a global positive solution, as well as local and global asymptotic stability of constant equilibria. In addition, they found the presence and absence of intermittent positive stable states.

Our main goal is to investigate the long-term behavior of a predator-prey model with a Leslie-Gower free boundary. In this article, we consider the following model:

To find functions $u(t,x)$, $v(t,x)$, $s(t)$, $h(t)$ in the domain $(D \subset Q)$ $Q = \{(t,x) : 0 < t \le T, -L < x < L\}$, $D = \{(t,x) : 0 < t \le T,\ h(t) < x < s(t)\}$ satisfying the conditions

$$u_t = u_{xx} + c_1 u_x + u(1-u) - v\left(\frac{u}{u+m}\right), \quad (t,x) \in Q, \tag{1}$$

$$v_t = d v_{xx} + c_2 v_x + kv\left(1 - \frac{bv}{u+a}\right), \quad (t,x) \in D, \tag{2}$$

$$u(0,x) = u_0(x), \quad -L \le x \le L, \tag{3}$$

$$v(0,x) = v_0(x), \quad -s_0 \le x \le s_0, \tag{4}$$

$$u_x(t,-L) = u(t,L) = 0, \quad t > 0, \tag{5}$$

$$v(t,h(t)) = v(t,s(t)) = 0 \quad t > 0, \tag{6}$$

$$\dot{s}(t) = -\mu v_x(t,s(t)), \quad t > 0, \tag{7}$$

$$\dot{h}(t) = -\mu v_x(t,h(t)), \quad t > 0. \tag{8}$$

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where - free (unknown) boundaries $h(t)$, $s(t)$, which represent the front of propagation, is determined together with the functions $u(t,x)$, $v(t,x)$; $d-$ diffusion coefficient, s_0, L, μ, m, $c_i (i=1,2)$, a and b- are positive constants.

The initial data (u_0, v_0) satisfy:

i. $u_0(x) \in C^{2+\alpha}[-L,L]$, $u_0(-L)=0$, $u_0(L)=0$;
$0 < u_0(x) < M_1$, $-L \le x \le L$;

ii. $v_0(x) \in C^{2+\alpha}[-s_0,s_0]$, $v_0(-s_0)=0$, $v_0(s_0)=0$;
$0 < v_0(x) < M_2$, $-s_0 \le x \le s_0$.

Here and in what follows, M will denote constants depending on the data of the problem.

A similar problem was studied in [9,10, 16-18].

A PRIORI ESTIMATES

In this section, we establish some a priori estimates of Schauder type that will be used to prove the global solvability of the problem. At the same time, the principles of maximum and comparison theorems are widely accepted.

There are various methods for obtaining a priori estimates. In this paper, we will apply the method of obtaining a priori estimates proposed by S.N. Kruzhkov [13]. Therefore, we will adhere to the notation adopted in [13].

Lemma 1 *Let $u(t,x), v(t,x), s(t)$ be a solution to problem (1) - (8). Then*

$$0 < u(t,x) \le M_1, \quad t>0, \quad \bar{Q}_T,$$

$$0 < v(t,x) \le M_2, \quad t>0, \quad \bar{D}_T,$$

$$0 < \dot{s}(t) \le M_3, \quad 0 < \dot{h}(t) \le M_4, \quad t>0,$$

where M_3, M_4 are constants depending on μ, d, k, M_1, M_2.

We will establish Hölder norm bounds $|\cdot|_{1+\alpha}$ and $|\cdot|_{2+\alpha}$ in $\bar{D}_T$ and $\bar{Q}_T$.

For each equation of the system, we formulate the corresponding problem

$$\begin{cases} u_{xx} + b_1(u,v,u_x) - u_t = 0, \quad Q_T, \\ u(0,x) = u_0(x), \quad -L \le x \le L, \\ u_x(t,-L) = u(t,L) = 0, \quad 0 \le t \le T, \end{cases} \tag{9}$$

$$\begin{cases} dv_{xx} + b_2(u,v,v_x) - v_t = 0, \quad D_T, \\ v(0,x) = v_0(x), \quad -s_0 \le x \le s_0, \\ v(t,h(t)) = v(t,s(t)) = 0, \quad 0 \le t \le T, \\ \dot{s}(t) = -\mu v_x(t,s(t)), \quad 0 \le t \le T, \\ \dot{h}(t) = -\mu v_x(t,h(t)), \quad 0 \le t \le T, \end{cases} \tag{10}$$

where $b_1(u,v,u_x) = c_1 u_x + u(1-u) - v\left(\frac{u}{u+m}\right)$, $b_2(u,v,v_x) = c_2 v_x + kv\left(1 - \frac{bv}{u+a}\right)$.

Under these assumptions on the initial data, we can assume that $u_0(x)=0, v_0(x)=0$.

Theorem 1 *Assume that $u(t,x), u_x(t,x)$ are continuous in Q_T and suppose that $u(x,t)$ is a solution for the problem (9). Then*

$$|u_x(t,x)| \le C_1(M_1), (t,x) \in \bar{Q}_T. \tag{11}$$

Moreover, if the weak second derivatives u_{xx}, u_{tx} are in $L^2(Q_T)$, then there exists $\alpha = \alpha(M_1, s)$, such that

$$|u|_{1+\alpha, Q_T} \le C_2(M_1, C_1). \tag{12}$$

Additionally, assume that, $u(t,x)$ satisfying (9) in $\bar{Q}_T$, is continuous with its derivatives u_t, u_x, u_{xx} and $|u|_{2+\alpha,\bar{Q}_T} < \infty$. Then,

$$|u|_{2+\alpha,\bar{Q}_T} \le C_3(M_1, C_1, C_2). \tag{13}$$

Proof. The estimates (11)-(13) for $(t,x) \in Q$ are immediate consequences of the results of [5].

In the case of problem (10), a priori estimates are constructed as follows. Estimates in the interior of the domain are established as in the case of problem (9). Further, replacing $\tau = t, y = \frac{2s_0 x}{s(\tau)-h(\tau)} - \frac{s(\tau)+h(\tau)}{s(\tau)-h(\tau)} s_0$, we straighten out the boundary. Then domain D_T is mapped to domain $\Omega = \{(\tau,y) : 0 < \tau < T, 0 < y < 1\}$ and for the function $w(\tau,y) = v(\tau,u,s(\tau),h(\tau))$, we obtain an equation with bounded coefficients and the right-hand side. By the results of [13], we establish estimates for $|w_y|, |w|_{1+\gamma}$ up to the right boundary. Estimates for the highest derivatives are obtained from the results for linear equations [14,15].

Now let us prove that the free boundaries do not cross the lateral boundaries in the considered time interval. First, we get a new representation for the free boundary. Integrating (2) over D, we obtain

$$\int_0^t d\eta \int_0^{s(\eta)} \left[\left(d v_\xi + \frac{1}{2} m_2 v^2 \right)_\xi - v_\eta \right] d\xi + k \int_0^t d\eta \int_0^{s(\eta)} v \left(1 - \frac{bv}{u+a} \right) d\xi = 0,$$

$$\frac{d}{\mu}(s(t) - h(t)) = \int_{-s_0}^{s_0} v_0(\xi) d\xi - \int_{h(t)}^{s(t)} v(t,\xi) d\xi + k \int_0^t d\eta \int_{-h(\eta)}^{s(\eta)} v \left(1 - \frac{bv}{u+a} \right) d\xi. \qquad (14)$$

Theorem 2 *Let $u(t,x)$, $v(t,x)$, $s(t)$, $h(t)$ be a solution to (1)-(8). Then $g(t) < L$, where $g(t) = \ |s(t) - h(t)| > 0$.*

Proof. We use relation (9):

$$\frac{d}{\mu} g(t) + \int_{h(t)}^{s(t)} v(t,\xi) d\xi - k \int_0^t d\eta \int_{h(\eta)}^{s(\eta)} v \left(1 - \frac{bv}{u+a} \right) d\xi = \int_{-s_0}^{s_0} v_0(\xi) d\xi.$$

Here,

$$\frac{d}{\mu} g(t) - k \int_0^t d\eta \int_0^{s(\eta)} v(\eta,\xi) d\xi \leq K,$$

where $K = \int_{-s_0}^{s_0} v_0(\xi) d\xi$.

In the same way, taking into account the inequalities $0 \leq \int_{h(\eta)}^{s(\eta)} v(\eta,\xi) d\xi \leq M_2(s(\eta) - h(\eta))$,

$$\frac{d}{\mu} g(t) - k M_2 \int_0^t g(\eta) d\eta \leq K,$$

we have

$$g(t) \leq m + n \int_0^t g(\eta) d\eta,$$

where $m = \frac{\mu K}{d}$, $n = \frac{k \mu M_2}{d}$. Then

$$g(t) = m e^{-nt} < L.$$

Theorem 2 is proved.

Lemma 2 (Comparison principle) *Let $u(t,x)$, $v(t,x)$, $s(t)$, $h(t))$ be a solution to problem (1) - (8) with initial data $(u_0(x), v_0(x))$.*

a) *Suppose that $(w(t,x),\ z_1(t),\ \delta_1(t))$ satisfies*

$$\begin{cases} w_t - dw_{xx} - c_2 w_x \geq kw(1-\frac{bw}{a}), \quad t>0, \quad \delta_1(t)<x<z_1(t), \\ w(t,z_1(t))=0,\ w(t,\delta_1(t))=0, \quad t>0, \\ \quad \dot{z}_1(t) \geq -\mu w_x(t,z_1(t)), \quad t>0. \\ \dot{\delta}_1(t) \leq -\mu w_x(t,\delta_1(t)), \quad t>0. \end{cases}$$

If $w(0,x) \geq v_0(x)$ in $[-L,L]$ and $\begin{cases} z_1(0) \geq s(0), \\ \delta_1(0) \leq h(0), \end{cases}$ then

$\diamond \begin{cases} z_1(t) \geq s(t) \\ \delta_1(t) \leq h(t) \end{cases}$ *for $t \geq 0$,*

$\diamond w(t,x) \geq v(t,x)$ *for $x \in [h(t),s(t)]$.*

b) *Suppose that $(\vartheta(t,x), z_2(t), \delta_2(t))$ satisfies*

$$\begin{cases} \vartheta_t - d\vartheta_{xx} - c_2\vartheta_x \leq k\vartheta(1-\frac{b\vartheta}{M_1+a}) \ \ t>0 \quad \delta_2(t)<x<z_2(t), \\ \vartheta(t,\delta_2(t))=0,\ \vartheta(t,z_2(t))=0, \quad t>0, \\ z_2(t) \leq -\mu\vartheta_x(t,z_2(t)), \quad \delta_2(t) \geq -\mu\vartheta_x(t,\delta_2(t)),\ \ t>0. \end{cases}$$

If $\vartheta(0,x) \leq v_0(x)$ in $[-L,L]$, $\begin{cases} z_2(0) \leq s(0), \\ \delta_2(0) \geq h(0), \end{cases}$ then

$\diamond \begin{cases} z_2(t) \leq s(t) \\ \delta_2(t) \geq h(t) \end{cases}$ *for $t \geq 0$.*

$\diamond \vartheta(t,x) \leq v(t,x)$ *for $x \in [h(t),s(t)]$.*

SOME QUALITATIVE PROPERTIES OF SOLUTIONS

Theorem 3 *Let $u(t,x)$, $v(t,x)$, $s(t)$, $h(t)$ be a solution to problem (1) - (8). If $s_\infty > L$, then*

$$\lim_{t\to+\infty} sup\, u(t,x) \leq \bar{u}(x); \qquad \lim_{t\to+\infty} sup\, u(t,x) \geq \underline{u}(x);$$

$$\lim_{t\to+\infty} sup\, v(t,x) \leq \bar{v}(x); \qquad \lim_{t\to+\infty} sup\, v(t,x) \geq \underline{v}(x),$$

where $\bar{u}(x)$, $\underline{u}(x)$, $\bar{v}(x)$, $\underline{v}(x)$ upper and lower solutions to the problem.

Theorem 4 *Let $u(t,x)$, $v(t,x)$, $s(t)$, $h(t)$ be a solution to problem (1) - (8). If $s_\infty < L$, then $\lim_{t\to+\infty} sup\ u(t,\cdot) \geq \bar{u}(x)$ for $x \in [-L,L]$ and $\lim_{t\to+\infty} sup\ \| v(t,\cdot)\|_{C[h(t),s(t)]} = 0$.*

UNIQUENESS AND EXISTENCE OF A SOLUTION

Let's use the representations for the unknown boundary (14).

Theorem 5 *Let conditions i.- ii hold., Lemma* 1 *and Theorem* 2*. Then the solution to problem (1) - (8) is unique.*

Proof. We first establish the result for smaller values of t, and then extend the proof to the general case of $0 < t < \infty$.

Assume that $s_1(t), h_1(t), u_1(x,t), v_1(x,t)$ and $s_2(t), h_2(t), u_2(x,t),\ v_2(x,t)$ are the solutions of the problem (14) and let $\begin{cases} y_1(t) = \max(h_1(t),h_2(t)), \\ y_2(t) = \min(h_1(t),h_2(t)), \end{cases}$ $\begin{cases} z_1(t) = \max(s_1(t),\ s_2(t)), \\ z_2(t) = \min(s_1(t),\ s_2(t)). \end{cases}$

Then, each pair satisfies the identity (14).

Subtracting, we obtain that

$$\frac{d}{\mu}(g_1(t)-g_2(t))-(h_1(t)-h_2(t)) \leq \int\limits_{y_1(t)}^{z_2(t)} (v_1(t,\xi)-v_2(t,\xi))d\xi + \int\limits_{y_2(t)}^{y_1(t)} v_i(\xi,t)d\xi + \mu \int\limits_{z_2(t)}^{z_1(t)} v_i(\xi,t)d\xi$$

$$+\int\limits_0^t d\eta \int\limits_{y_1(\eta)}^{z_2(\eta)} (f_2(v_1)-f_2(v_2))d\xi + \int\limits_0^t d\eta \int\limits_{z_2(\eta)}^{z_1(\eta)} f_2(v_i)d\xi$$

where $u_i, v_i (i=1,2)$ are the solution between $y(t)$ and $h(t)$.

Further, using the ideas and results of [7], the proof of the theorem is completed.

Theorem 6 *Let the conditions of Lemma* 1 *and Theorems* 5 *be satisfied. Then there exists a solution* $u(x,t) \in C^{2+\gamma}(\bar{D}_T)$, $v(x,t) \in C^{2+\gamma}\left(\bar{Q}_T\right)$, $s(t) \in C^{1+\gamma}([0,T])$, $h(t) \in C^{1+\gamma}([0,T])$ *of problem (1) - (8).*

To prove the existence of a solution to problem (1)-(8), we use the Leray – Schauder theorem [6,8].

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