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**Contents**

# Euler type integral representations for certain confluent hypergeometric functions in three variables

**Hasanov Anvar**
*V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences*
**Tashkent, Uzbekistan**
*and*
*Ghent University*
**Ghent, Belgium**
anvarhasanov@yahoo.com

**Ergashev Tuhtasin**
*National Research University "TIIAME"*
*and*
*V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences*
**Tashkent, Uzbekistan**
*and*
*Ghent University*
**Ghent, Belgium**
ergashev.tukhtasin@gmail.com

**Tulakova Ziyodakhon**
*Fergana State Technical University*
**Fergana, Uzbekistan**
ziyodacoders@gmail.com

## Abstract

Hypergeometric functions are devided into the complete and confluent functions. Srivastava and Karlsson have defined, in 1985, 205 complete triple series. At present, integral representations for these functions are known. Recently, all possible 395 hypergeometric series of three variables have been published in the scientific literature, all of which are analogous to the double confluent series of Horn and Humbert (20 functions in total) and are confluent forms of the known complete hypergeometric functions of three variables. This paper firstly presents some integral representations of the Euler type for new 28 confluent hypergeometric functions of three variables. The main results were obtained using the properties of the gamma and beta functions. Thus, all derived integrals can be considered as generalized representations of the Euler type for classical hypergeometric functions of one and two variables.

**Keywords:** Complete and confluent hypergeometric functions; Euler type integral representations; Horn and Humbert hypergeometric functions of two variables; confluent hypergeometric functions of three variables, beta function.

**MSC 2020:** 33C15, 33C65, 44A20

## 1. Introduction

The great interest in the theory of hypergeometric functions (including functions of one, two or more variables) is primarily due to the fact that hypergeometric functions allow us to find solutions to various applied problems related to thermal conductivity and dynamic processes, electromagnetic oscillations, aerodynamics, quantum mechanics and potential theory. These functions, which relate to higher and special (or transcendental) functions [1], [2], [3], are often called special functions of mathematical physics.

With a view to introducing formally the Gaussian hypergeometric series and its generalizations, we recall here some definitions and identities involving the beta function $B(x, y)$, gamma function $\Gamma(z)$ and Pochhammer's symbol $(\lambda)_n$.

The gamma function $\Gamma(z)$ is defined by

$$\Gamma(z)=\begin{cases}\int\limits_0^\infty t^{z-1}e^{-t}dt, \;\; Re(z)>0,\\ \\ \dfrac{\Gamma(z+1)}{z}, \;\; Re(z)<0; \;\; z\neq -1,-2,-3,....\end{cases} \tag{1}$$

The definition (1) was used by Euler and there are other definitions of the gamma function (see [4]).

Throughout this work we shall find it convenient to employ the Pochhammer symbol $(\lambda)_n$ defined by

$$(\lambda)_n=\begin{cases}1, \text{ if } \; n=0,\\ \\ \lambda(\lambda+1)...(\lambda+n-1), \text{ if } \; n=1,2,3,....\end{cases} \tag{2}$$

Since $(1)_n = n!$, $(\lambda)_n$ may be looked upon as a generalization of the elementary factorial; hence the symbol $(\lambda)_n$ is also referred to as thew factorial function.

The following property is very important in our research:

$$(\lambda)_{m+n}=(\lambda)_m(\lambda+m)_n. \tag{3}$$

In terms of gamma functions, we have

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \;\; \lambda\neq 0,-1,-2,\dots, \tag{4}$$

which can easily be verified.

The beta function is defined by the integral

$$B(\alpha,\beta):=\int\limits_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \;\; Re\alpha>0, \, Re\beta>0. \tag{5}$$

A function

$$F(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \, c\neq 0,-1,-2,... \tag{6}$$

is known as the Gaussian hypergeometric function.

The Gaussian hypergeometric series $F(a,b;c;z)$ includes two numerator parameters $a$ and $b$, and one denominator parameter $c$. Its natural generalization is the introduction of an arbitrary number of parameters in both the numerator and the denominator. The resulting series

$${}_pF_q\left[(a_p);(b_q);z\right]:=\sum_{k=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_k}{\prod\limits_{j=1}^{q}(b_j)_k}\frac{z^k}{k!}$$

is known as the generalized Gauss series [4, p. 182], or simply, the generalized hypergeometric series. Here $p$ and $q$ are positive integers or zero (interpreting an empty product as 1), and we assume that the variable $z$, the numerator parameters $a_1$,..., $a_p$, and the denominator parameters $b_1$, ..., $b_q$ take on complex values, provided that $b_j \neq 0,\, -1,\, -2,\, \dots;\, j=1,\, \dots,\, q$.

Gauss' series (6) in the present notation is ${}_2F_1(a,b;c;z)=F(a,b;c;z)$.

The double Appell hypergeometric functions are defined as following [5]:

$$F_1\left(a,b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_{m+n}m!n!}x^m y^n, \tag{7}$$

$$F_2\left(a,b,b';c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_m\left(c'\right)_n m!n!}x^m y^n, \tag{8}$$

$$F_3\left(a,a',b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m\left(a'\right)_n(b)_m\left(b'\right)_n}{(c)_{m+n}m!n!}x^m y^n, \tag{9}$$

$$F_4\left(a,b;c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m\left(c'\right)_n m!n!}x^m y^n, \tag{10}$$

here, in all definitions (7) – (10), as usual, the denominator parameters $c$ and $c'$ are neither zero nor a negative integer.

Seven confluent forms of the four Appell series were introduced by Humbert [6], who denoted these confluent hypergeometric series of two variables as follows:

$$\Phi_1\left(\alpha,\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}m!n!}x^m y^n,\ \ |x|<1, \tag{11}$$

$$\Phi_2\left(\beta,\beta';\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m\left(\beta'\right)_n}{(\gamma)_{m+n}m!n!}x^m y^n, \tag{12}$$

$$\Phi_3\left(\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m}{(\gamma)_{m+n}m!n!}x^m y^n, \tag{13}$$

$$\Psi_1\left(\alpha,\beta;\gamma,\gamma';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n m!n!}x^m y^n,\ \ |x|<1, \tag{14}$$

$$\Psi_2\left(\alpha;\gamma,\gamma';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m\left(\gamma'\right)_n m!n!}x^m y^n, \tag{15}$$

$$\Xi_1\left(\alpha,\alpha',\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m\left(\alpha'\right)_n(\beta)_m}{(\gamma)_{m+n}m!n!}x^m y^n,\ \ |x|<1, \tag{16}$$

$$\Xi_2\left(\alpha,\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_{m+n}m!n!}x^m y^n,\ \ |x|<1, \tag{17}$$

where the denominator parameters $\gamma$ and $\gamma'$ are neither zero nor a negative integer. A hypergeometric functions defined in (11) to (17) are called *Humbert functions.*

The great success of the theory of hypergeometric series in one and two variables stimulated the development of a corresponding theory in three and more variables. Srivastava and Karlsson [7] have defined, in 1985, 205 series which are all analogous to four Appell and ten Horn functions. In particular, the authors compiled a table of 205 different complete triple hypergeometric Gauss series, accompanied by references to their sources, if known. Realizing the importance of integral representations of multiple hypergeometric functions for solving applied problems, Hasanov and Ruzhansky [8] developed Euler-type integral representations for all 205 complete triple hypergeometric functions. Later, these same authors [9] constructed systems of partial differential equations that the indicated functions satisfy, and found all their linearly independent solutions near the origin in those cases where such solutions exist.

However, comparatively less attention has been paid to the study of the confluent hypergeometric functions of three variables. In the works of Exton [10] and Jain [11], individual functions representing confluent forms of complete hypergeometric functions of three variables were investigated. In his paper [12] Ergashev identified 395 confluent hypergeometric functions of three variables, denoting them as $\mathrm{E}_1$, ..., $\mathrm{E}_{395}$. He thus probably completed the classification of all possible degenerate second-order hypergeometric functions for three variables. The study also includes an analysis of systems of partial differential equations associated with these 395 functions. In addition, particular solutions of some systems of differential equations near the origin were found, if such solutions exist (for details, see [13]).

A unification of hypergeometric series in three variables was introduced by Srivastava [14] who defined a general triple hypergeometric series $F^{(3)}[x,y,z]$ by

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array}\ x,y,z\right]=\sum_{m,n,p=0}^{\infty}\Lambda(x,y,z)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\tag{18}$$

where, for convenience,

$$\Lambda(x,y,z)=\frac{\prod\limits_{j=1}^{A}(a_j)_{m+n+p}\prod\limits_{j=1}^{B}(b_j)_{m+n}\prod\limits_{j=1}^{B'}(b'_j)_{n+p}\prod\limits_{j=1}^{B''}(b''_j)_{p+m}\prod\limits_{j=1}^{C}(c_j)_{m}\prod\limits_{j=1}^{C'}(c'_j)_{n}\prod\limits_{j=1}^{H''}(c''_j)_{p}}{\prod\limits_{j=1}^{E}(e_j)_{m+n+p}\prod\limits_{j=1}^{G}(g_j)_{m+n}\prod\limits_{j=1}^{G'}(g'_j)_{n+p}\prod\limits_{j=1}^{G''}(g''_j)_{p+m}\prod\limits_{j=1}^{H}(h_j)_{m}\prod\limits_{j=1}^{H'}(h'_j)_{n}\prod\limits_{j=1}^{H''}(h''_j)_{p}},$$

and, as also in the definition, $(a)$ abbreviates the array of $A$ parameters $a_1$, ..., $a_A$, with similar interpretations for $(b)$, $(b')$, $(b'')$, et cetera.

In this paper we will establish integral representations for the general hypergeometric function in three variables $F^{(3)}[x,y,z]$ and the following confluent hypergeometric functions:

$$\mathrm{E}_{22}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_m(a_3)_n(a_4)_p}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{23}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_m(a_3)_m(a_4)_p}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{24}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_m(a_3)_m(a_4)_n}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{25}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_m(a_3)_m}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{26}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_m(a_3)_n}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{27}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_m(a_3)_p}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{28}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_n(a_3)_p}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{29}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_m}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{30}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_n}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{31}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_p}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{32}\left(a;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{n+p}}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{33}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_{n+p}(a_3)_m}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{34}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_{n+p}}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{35}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_m}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{36}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{37}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{n+p}}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{38}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+p}(a_2)_{n+p}(a_3)_m}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{39}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+p}(a_2)_{n+p}}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{40}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_m(a_3)_n}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{41}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_m(a_3)_p}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{42}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_m}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{43}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_{n+p}}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{62}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_m}{(c_1)_m(c_2)_n(c_3)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{63}\left(a_1,a_2;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{64}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_m(a_3)_n}{(c_1)_m(c_2)_n(c_3)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{65}\left(a_1,a_2;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_m}{(c_1)_m(c_2)_n(c_3)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{66}\left(a;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}}{(c_1)_m(c_2)_n(c_3)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{67}\left(a_1,a_2;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_{m+n}}{(c_1)_m(c_2)_n(c_3)_p m!n!p!}x^m y^n z^p.$$

Note, that the confluent hypergeometric functions $\mathrm{E}_{22}-\mathrm{E}_{43}$ and $\mathrm{E}_{62}-\mathrm{E}_{67}$, except the functions $\mathrm{E}_{24}$, $\mathrm{E}_{28}$, $\mathrm{E}_{30}$, $\mathrm{E}_{35}$ and $\mathrm{E}_{40}$, were first introduced by Jain [11]. The confluent hypergeometric functions $\mathrm{E}_{24}$, $\mathrm{E}_{28}$, $\mathrm{E}_{30}$, $\mathrm{E}_{35}$ and $\mathrm{E}_{40}$ are found in [15] (for details, see [12], [13]).

## 2. Euler type integral representations

**Theorem 2.1.** *If* $\mathrm{Re}\,(\beta) > \mathrm{Re}\,(\alpha) > 0$, *then the following integral representation formulas*

$$F^{(3)}\left[\begin{array}{l}\alpha,(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} x,y,z\right]$$
$$=\kappa\int\limits_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-\alpha-1}F^{(3)}\left[\begin{array}{l}\beta,(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} x\xi,y\xi,z\xi\right]d\xi, \tag{19}$$

$$F^{(3)}\left[\begin{array}{l}(a)::\alpha,(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} x,y,z\right]$$
$$=\kappa\int\limits_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-\alpha-1}F^{(3)}\left[\begin{array}{l}(a)::\beta,(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} x\xi,y\xi,z\right]d\xi, \tag{20}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);\alpha,(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} x,y,z\right]$$
$$=\kappa\int\limits_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-\alpha-1}F^{(3)}\left[\begin{array}{l}(a)::(b);\beta,(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array} x,y\xi,z\xi\right]d\xi, \tag{21}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::\beta,(g);(g');(g''):(h);(h');(h'');\end{array} x,y,z\right]$$
$$=\kappa\int\limits_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-\alpha-1}F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):\alpha,(h);\beta-\alpha,(h');(h'');\end{array} x\xi,y(1-\xi),z\right]d\xi \tag{22}$$

*are valid. Hereinafter,* $\kappa=\dfrac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}$.

Proof. The equalities (19) to (22) follow easily from the definition of the triple series (18), if we use the formula (5) for calculating the beta function. □

**Corollary 2.1.** *If* $\mathrm{Re}\,(\beta) > \mathrm{Re}\,(\alpha) > 0$, *then the following integral representation formulas are valid:*

$$\mathrm{E}_{22}\left(\alpha,a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{22}\left(\beta,a_1,a_2,a_3;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{23}$$

$$\mathrm{E}_{22}\left(a_1,a_2,a_3,a_4;\beta,c_2;x,y,z\right)$$
$$=\kappa\int\limits_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}{}_1F_1\left(a_2;\alpha;x\xi\right)F_2\left(a_1,a_3,a_4;\beta-\alpha,c_2;y\left(1-\xi\right),z\right)d\xi, \tag{24}$$

$$\mathrm{E}_{22}\left(a_1,\alpha,a_3,a_4;\beta,c;x,y,z\right)=\kappa\int\limits_0^1 e^{x\xi}\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}F_2\left(a_1,a_3,a_4;\beta-\alpha,c;y\left(1-\xi\right),z\right)d\xi, \tag{25}$$

$$\mathrm{E}_{22}\left(a_1,a_2,\alpha,a_4;\beta,c;x,y,z\right)=\kappa\int\limits_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-y\xi\right)^{a_1}}{}_1F_1\left(a_2;\beta-\alpha;x-x\xi\right){}_2F_1\left(a_1,a_4;c;\frac{z}{1-y\xi}\right)d\xi, \tag{26}$$

$$\mathrm{E}_{22}\left(a_1,a_2,a_3,\alpha;c,\beta;x,y,z\right)=\kappa\int\limits_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-z\xi\right)^{a_1}}\Xi_1\left(a_1,a_2,a_3;c;\frac{y}{1-z\xi},x\right)d\xi, \tag{27}$$

$$\mathrm{E}_{23}\left(\alpha, a_1, a_2, a_3; c_1, c_2; x, y, z\right) = \kappa \int\limits_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1} \mathrm{E}_{23}\left(\beta, a_1, a_2, a_3; c_1, c_2; x, y\xi, z\xi\right) d\xi, \tag{28}$$

$$\mathrm{E}_{23}\left(a_1, \alpha, a_3, a_4; \beta, c_2; x, y, z\right) = \kappa \int\limits_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_3}} \Psi_1\left(a_1, a_4; c_2, \beta-\alpha; z, y\left(1-\xi\right)\right) d\xi, \tag{29}$$

$$\mathrm{E}_{23}\left(a_1, a_2, a_3, \alpha; c, \beta; x, y, z\right) = \kappa \int\limits_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-z\xi\right)^{a_1}} \Xi_1\left(a_2, a_1, a_3; c; x, \frac{y}{1-z\xi}\right) d\xi, \tag{30}$$

$$\mathrm{E}_{24}\left(\alpha, a_1, a_2, a_3; c_1, c_2; x, y, z\right) = \kappa \int\limits_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1} \mathrm{E}_{24}\left(\beta, a_1, a_2, a_3; c_1, c_2; x, y\xi, z\xi\right) d\xi, \tag{31}$$

$$\mathrm{E}_{24}\left(a_1, \alpha, a_3, a_4; \beta, c_2; x, y, z\right) = \kappa \int\limits_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_3}} \Psi_1\left(a_1, a_4; \beta-\alpha, c_2; y\left(1-\xi\right), z\right) d\xi, \tag{32}$$

$$\mathrm{E}_{25}\left(\alpha, a_1, a_2; c_1, c_2; x, y, z\right) = \kappa \int\limits_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1} \mathrm{E}_{25}\left(\beta, a_1, a_2; c_1, c_2; x, y\xi, z\xi\right) d\xi, \tag{33}$$

$$\mathrm{E}_{25}\left(a_1, \alpha, a_3; \beta, c_2; x, y, z\right) = \kappa \int\limits_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_3}} \Psi_2\left(a_1; \beta-\alpha, c_2; y\left(1-\xi\right), z\right) d\xi, \tag{34}$$

$$\mathrm{E}_{26}\left(\alpha, a_1, a_2; c_1, c_2; x, y, z\right) = \kappa \int\limits_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1} \mathrm{E}_{26}\left(\beta, a_1, a_2; c_1, c_2; x, y\xi, z\xi\right) d\xi, \tag{35}$$

$$\mathrm{E}_{26}\left(a_1, \alpha, a_3; \beta, c; x, y, z\right) = \kappa \int\limits_0^1 e^{x\xi}\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1} \Psi_1\left(a_1, a_3; \beta-\alpha, c; y\left(1-\xi\right), z\right) d\xi, \tag{36}$$

$$\mathrm{E}_{27}\left(\alpha, a_1, a_2; c_1, c_2; x, y, z\right) = \kappa \int\limits_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1} \mathrm{E}_{27}\left(\beta, a_1, a_2; c_1, c_2; x, y\xi, z\xi\right) d\xi, \tag{37}$$

$$\mathrm{E}_{27}\left(a_1, \alpha, a_3; \beta, c; x, y, z\right) = \kappa \int\limits_0^1 e^{x\xi}\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1} \Psi_1\left(a_1, a_3; c, \beta-\alpha; z, y\left(1-\xi\right)\right) d\xi, \tag{38}$$

$$\mathrm{E}_{27}\left(a_1, a_2, \alpha; c, \beta; x, y, z\right) = \kappa \int\limits_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-z\xi\right)^{a_1}} \Phi_2\left(a_2, a_1; c; x, \frac{y}{1-z\xi}\right) d\xi, \tag{39}$$

$$\mathrm{E}_{28}\left(\alpha, a_1, a_2; c_1, c_2; x, y, z\right) = \kappa \int\limits_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1} \mathrm{E}_{28}\left(\beta, a_1, a_2; c_1, c_2; x, y\xi, z\xi\right) d\xi, \tag{40}$$

$$\mathrm{E}_{28}\left(a_1, a_2, \alpha; c, \beta; x, y, z\right) = \kappa \int\limits_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-z\xi\right)^{a_1}} \Xi_2\left(a_1, a_2; c; \frac{y}{1-z\xi}, x\right) d\xi, \tag{41}$$

$$\mathrm{E}_{29}\left(\alpha, b; c_1, c_2; x, y, z\right) = \kappa \int\limits_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1} \mathrm{E}_{29}\left(\beta, b; c_1, c_2; x, y\xi, z\xi\right) d\xi, \tag{42}$$

$$\mathrm{E}_{29}\left(a, \alpha; \beta, c; x, y, z\right) = \kappa \int\limits_0^1 e^{x\xi}\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1} \Psi_2\left(a; \beta-\alpha, c; y\left(1-\xi\right), z\right) d\xi, \tag{43}$$

$$\mathrm{E}_{30}\left(\alpha,b;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{30}\left(\beta,b;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{44}$$

$$\mathrm{E}_{30}\left(a,\alpha;\beta,c;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-y\xi\right)^{a}}{}_0F_1\left(\beta-\alpha;x\left(1-\xi\right)\right){}_1F_1\left(a;c;\frac{z}{1-y\xi}\right)d\xi, \tag{45}$$

$$\mathrm{E}_{31}\left(\alpha,b;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{31}\left(\beta,b;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{46}$$

$$\mathrm{E}_{31}\left(a,\alpha;c,\beta;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-z\xi\right)^{a}}\Phi_3\left(a,c;\frac{y}{1-z\xi},x\right)d\xi, \tag{47}$$

$$\mathrm{E}_{32}\left(\alpha;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{32}\left(\beta;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{48}$$

$$\mathrm{E}_{33}\left(\alpha,a_2,a_3;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{33}\left(\beta,a_2,a_3;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{49}$$

$$\mathrm{E}_{33}\left(a_1,a_2,\alpha;\beta,c;x,y,z\right)=\kappa\int\limits_0^1 e^{x\xi}\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}F_4\left(a_1,a_2;\beta-\alpha,c;y\left(1-\xi\right),z\right)d\xi, \tag{50}$$

$$\mathrm{E}_{34}\left(\alpha,a,b;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{34}\left(\beta,b;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{51}$$

$$\mathrm{E}_{35}\left(a_1,\alpha,a_3;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{35}\left(a_1,\beta,a_3;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{52}$$

$$\mathrm{E}_{35}\left(a_1,a_2,\alpha;\beta,c;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_1}}\Psi_1\left(a_2,a_1;\beta-\alpha,c;\frac{y\left(1-\xi\right)}{1-x\xi},z\right)d\xi, \tag{53}$$

$$\mathrm{E}_{36}\left(a_1,\alpha,a_3;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{36}\left(a_1,\beta,a_3;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{54}$$

$$\mathrm{E}_{36}\left(\alpha,a_2,a_3;\beta,c;x,y,z\right)=\kappa\int\limits_0^1 e^{x\xi}\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-y\xi\right)^{a_2}}F\left(a_2,a_3;c;\frac{z}{1-y\xi}\right)d\xi, \tag{55}$$

$$\mathrm{E}_{36}\left(a_1,a_2,\alpha;c,\beta;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-z\xi\right)^{a_2}}\Phi_1\left(a_1,a_2;c;\frac{y}{1-z\xi},x\right)d\xi, \tag{56}$$

$$\mathrm{E}_{37}\left(a_1,\alpha;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{37}\left(a_1,\beta;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{57}$$

$$\mathrm{E}_{37}\left(\alpha,b;\beta,c;x,y,z\right)=\kappa\int\limits_0^1 e^{x\xi}\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-y\xi\right)^{b}}{}_1F_1\left(b;c;\frac{z}{1-y\xi}\right)d\xi, \tag{58}$$

$$\mathrm{E}_{38}\left(a_1,\alpha,a_3;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{38}\left(a_1,\beta,a_3;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{59}$$

$$\mathrm{E}_{38}\left(a_1,a_2,\alpha;\beta,c;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_1}}\Psi_1\left(a_2,a_1;c,\beta-\alpha;\frac{z}{1-x\xi},y\left(1-\xi\right)\right)d\xi, \tag{60}$$

$$\mathrm{E}_{39}\left(a_1,\alpha;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{39}\left(a_1,\beta;c_1,c_2;x,y\xi,z\xi\right)d\xi, \tag{61}$$

$$\mathrm{E}_{40}\left(\alpha,a_2,a_3;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{40}\left(\beta,a_2,a_3;c_1,c_2;x\xi,y\xi,z\xi\right)d\xi, \tag{62}$$

$$\mathrm{E}_{41}\left(\alpha,a_2,a_3;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{41}\left(\beta,a_2,a_3;c_1,c_2;x\xi,y\xi,z\xi\right)d\xi, \tag{63}$$

$$\mathrm{E}_{41}\left(a_1,a_2,\alpha;c_1,\beta;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-z\xi\right)^{a_1}}\Phi_1\left(a_1,a_2;c;\frac{x}{1-z\xi},\frac{y}{1-z\xi}\right)d\xi, \tag{64}$$

$$\mathrm{E}_{41}\left(a_1,\alpha,a_3;\beta,c;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_1}}\Psi_1\left(a_1,a_3;c,\beta-\alpha;\frac{z}{1-x\xi},\frac{y\left(1-\xi\right)}{1-x\xi}\right)d\xi, \tag{65}$$

$$\mathrm{E}_{42}\left(\alpha,b;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{42}\left(\beta,b;c_1,c_2;x\xi,y\xi,z\xi\right)d\xi, \tag{66}$$

$$\mathrm{E}_{42}\left(a,\alpha;\beta,c;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a}}\Psi_2\left(a;\beta-\alpha,c;\frac{y\left(1-\xi\right)}{1-x\xi},\frac{z}{1-x\xi}\right)d\xi, \tag{67}$$

$$\mathrm{E}_{43}\left(\alpha,b;c_1,c_2;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{43}\left(\beta,b;c_1,c_2;x\xi,y\xi,z\xi\right)d\xi, \tag{68}$$

$$\mathrm{E}_{62}\left(a_1,\alpha,a_3;c_1,c_2,c_3;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{62}\left(a_1,\beta,a_3;c_1,c_2,c_3;x,y\xi,z\xi\right)d\xi, \tag{69}$$

$$\mathrm{E}_{62}\left(a_1,a_2,\alpha;\beta,c_2,c_3;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}(1-\xi)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_1}}\Psi_1\left(a_2,a_1;c_2,c_3;\frac{y}{1-x\xi},z\right)d\xi, \tag{70}$$

$$\mathrm{E}_{63}\left(a_1,\alpha;c_1,c_2,c_3;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{63}\left(a_1,\beta;c_1,c_2,c_3;x,y\xi,z\xi\right)d\xi, \tag{71}$$

$$\mathrm{E}_{63}\left(\alpha,b;c_1,c_2,c_3;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{63}\left(\beta,b;c_1,c_2,c_3;x\xi,y\xi,z\right)d\xi, \tag{72}$$

$$\mathrm{E}_{64}\left(\alpha,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{64}\left(\beta,a_2,a_3;c_1,c_2,c_3;x\xi,y\xi,z\xi\right)d\xi, \tag{73}$$

$$\mathrm{E}_{64}\left(a_1,\alpha,a_3;\beta,c_2,c_3;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}(1-\xi)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_1}}\Psi_1\left(a_1,a_3;c_2,c_3;\frac{y}{1-x\xi},\frac{z}{1-x\xi}\right)d\xi, \tag{74}$$

$$\mathrm{E}_{65}\left(\alpha,b;c_1,c_2,c_3;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{65}\left(\beta,b;c_1,c_2,c_3;x\xi,y\xi,z\xi\right)d\xi, \tag{75}$$

$$\mathrm{E}_{65}\left(a,\alpha;\beta,c_2,c_3;x,y,z\right)=\kappa\int\limits_0^1\frac{\xi^{\alpha-1}(1-\xi)^{\beta-\alpha-1}}{(1-x\xi)^{a}}\Psi_2\left(a;c_2,c_3;\frac{y}{1-x\xi},\frac{z}{1-x\xi}\right)d\xi, \tag{76}$$

$$\mathrm{E}_{66}\left(\alpha;c_1,c_2,c_3;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{66}\left(\beta;c_1,c_2,c_3;x\xi,y\xi,z\xi\right)d\xi, \tag{77}$$

$$\mathrm{E}_{67}\left(\alpha,b;c_1,c_2,c_3;x,y,z\right)=\kappa\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{67}\left(\beta,b;c_1,c_2,c_3;x\xi,y\xi,z\xi\right)d\xi. \tag{78}$$

Proof. The equalities (23) to (78) follow easily from the definition of the series which are in the single integrals, if we use the formula (5) for calculating the beta function. To give an example, by definition (14) of the Humbert function $\Psi_1$, in the right-hand side of the integral (65) we have

$$\begin{aligned} I_1 &= \kappa\int\limits_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{(1-x\xi)^{a_1}}\Psi_1\left(a_1,a_3;c,\beta-\alpha;\frac{z}{1-x\xi},\frac{y\left(1-\xi\right)}{1-x\xi}\right)d\xi \\ &= \kappa\sum_{p,n=0}^{\infty}\frac{\left(a_1\right)_{p+n}\left(a_3\right)_p}{\left(c\right)_p\left(\beta-\alpha\right)_n p!n!}y^n z^p\int\limits_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha+n-1}}{(1-x\xi)^{a_1+p+n}}d\xi. \end{aligned} \tag{79}$$

Using in (79) the Taylor expansion

$$\frac{1}{(1-t)^{\nu}}=\sum_{m=0}^{\infty}\frac{(\nu)_m}{m!}t^m,\quad |t|<1, \tag{80}$$

we obtain

$$I_1=\kappa\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{p+n}\left(a_3\right)_p\left(a_1+p+n\right)_m}{\left(c\right)_p\left(\beta-\alpha\right)_n m!n!p!}x^m y^n z^p\int\limits_0^1\xi^{\alpha+m-1}\left(1-\xi\right)^{\beta-\alpha+n-1}d\xi. \tag{81}$$

Next, applying to the integral in (81) the definition of beta function (5), relations (3) and (4), we get

$$I_1=\kappa\frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(\alpha)_m(a_3)_p}{(\beta)_{m+n}(c)_p m!n!p!}x^m y^n z^p=\mathrm{E}_{41}\left(a_1,\alpha,a_3;\beta,c;x,y,z\right).$$

The integral representation (65) is proved. □

**Theorem 2.2**. *If* $\mathrm{Re}\left(\alpha\right)>0$ *and* $\mathrm{Re}\left(\beta\right)>0$, *then the following integral representation formulas*

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b''):(c);\alpha,(c');\beta,(c'');\\(e)::\beta,(g);(g');(g''):(h);(h');(h'');\end{array}x,y,z\right]$$

$$=\lambda\int\limits_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}F^{(3)}\left[\begin{array}{l}(a)::(b);\alpha+\beta,(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array}x,y\xi,z(1-\xi)\right]d\xi, \tag{82}$$

$$F^{(3)}\left[\begin{array}{l}(a)::(b);\alpha,(b');(b''):\beta,(c);(c');(c'');\\(e)::\beta,(g);(g');(g''):(h);(h');(h'');\end{array}x,y,z\right]$$

$$=\lambda\int\limits_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}F^{(3)}\left[\begin{array}{l}\alpha+\beta,(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array}x(1-\xi),y\xi,z\xi\right]d\xi \tag{83}$$

*are valid. Hereinafter,* $\lambda=\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$.

Proof. The equalities (82) and (83) follow easily from the definition of the triple series (18), if we use the formula (5) for calculating the beta function. □

**Corollary 2.2**. *If* $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\beta) > 0$, *then the following integral representation formulas are valid:*

$$\mathrm{E}_{22}(a_1, a_2, \alpha, \beta; c_1, c_2; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,\mathrm{E}_{33}(a_1, \alpha+\beta, a_2; c_1, c_2; x, y\xi, z(1-\xi))\,d\xi, \tag{84}$$

$$\mathrm{E}_{22}(a_1, \alpha, \beta, a_4; c_1, c_2; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,\mathrm{E}_{36}(\alpha+\beta, a_1, a_4; c_1, c_2; x\xi, y(1-\xi), z)\,d\xi, \tag{85}$$

$$\mathrm{E}_{22}(\alpha, \beta, a_3, a_4; c_1, c_2; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,\mathrm{E}_{41}(\alpha+\beta, a_3, a_4; c_1, c_2; y\xi, x(1-\xi), z\xi)\,d\xi, \tag{86}$$

$$\mathrm{E}_{23}(a_1, a_2, a_3, a_4; \alpha+\beta, c; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,F(a_2, a_3; \alpha; x\xi)\,\Psi_1(a_1, a_4; c, \beta; z, y(1-\xi))\,d\xi, \tag{87}$$

$$\mathrm{E}_{23}(\alpha, \beta, a_3, a_4; c_1, c_2; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,\mathrm{E}_{41}(\alpha+\beta, a_3, a_4; c_1, c_2; x(1-\xi), y\xi, z\xi)\,d\xi, \tag{88}$$

$$\mathrm{E}_{24}(a_1, a_2, \alpha, \beta; c_1, c_2; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,\mathrm{E}_{35}(\alpha+\beta, a_1, a_2; c_1, c_2; x\xi, y(1-\xi), z)\,d\xi, \tag{89}$$

$$\mathrm{E}_{24}(\alpha, \beta, a_3, a_4; c_1, c_2; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,\mathrm{E}_{40}(\alpha+\beta, a_3, a_4; c_1, c_2; x(1-\xi), y\xi, z\xi)\,d\xi, \tag{90}$$

$$\mathrm{E}_{24}(a_1, a_2, a_3, a_4; \alpha+\beta, c; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,F(a_2, a_3; \alpha; x\xi)\,\Psi_1(a_1, a_4; \beta, c; y(1-\xi), z)\,d\xi, \tag{91}$$

$$\mathrm{E}_{25}(a_1, a_2, a_3; \alpha+\beta, c; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,F(a_2, a_3; \alpha; x\xi)\,\Psi_2(a_1; \beta, c; y(1-\xi), z)\,d\xi, \tag{92}$$

$$\mathrm{E}_{25}(\alpha, \beta, a; c_1, c_2; x, y, z) = \lambda \int_0^1 \xi^{\beta-1}(1-\xi)^{\alpha-1}\,\mathrm{E}_{42}(\alpha+\beta, a; c_1, c_2; x\xi, y(1-\xi), z(1-\xi))\,d\xi, \tag{93}$$

$$\mathrm{E}_{26}(a_1, a_2, a_3; \alpha+\beta, c; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,{}_1F_1(a_2; \alpha; x\xi)\,\Psi_1(a_1, a_3; \beta, c; y(1-\xi), z)\,d\xi, \tag{94}$$

$$\mathrm{E}_{26}(a, \alpha, \beta; c_1, c_2; x, y, z) = \lambda \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1}\,\mathrm{E}_{37}(\alpha+\beta, a; c_1, c_2; x\xi, y(1-\xi), z)\,d\xi, \tag{95}$$

$$\mathrm{E}_{26}(\alpha, \beta, a; c_1, c_2; x, y, z) = \lambda \int_0^1 \xi^{\beta-1}(1-\xi)^{\alpha-1}\,\mathrm{E}_{42}(\alpha+\beta, a; c_1, c_2; x\xi, y(1-\xi), z(1-\xi))\,d\xi, \tag{96}$$

$$\mathrm{E}_{27}(\alpha, \beta, a; c_1, c_2; x, y, z) = \lambda \int_0^1 \xi^{\beta-1}(1-\xi)^{\alpha-1}\,\Psi_1(\alpha+\beta, a; c_2, c_1; z(1-\xi), x\xi + y(1-\xi))\,d\xi, \tag{97}$$

$$\mathrm{E}_{27}\left(a_1,a_2,a_3;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}{}_1F_1\left(a_2;\alpha;x\xi\right)\Psi_1\left(a_1,a_3;c,\beta;z,y\left(1-\xi\right)\right)d\xi, \tag{98}$$

$$\mathrm{E}_{28}\left(a_1,a_2,a_3;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}{}_0F_1\left(\alpha;x\xi\right)F_2\left(a_1;a_2,a_3;\beta,c;y\left(1-\xi\right),z\right)d\xi, \tag{99}$$

$$\mathrm{E}_{29}\left(\alpha,\beta;c_1,c_2;x,y,z\right)=\lambda\int\limits_0^1\xi^{\beta-1}\left(1-\xi\right)^{\alpha-1}\Psi_2\left(\alpha+\beta;c_1,c_2;x\xi+y\left(1-\xi\right),z\left(1-\xi\right)\right)d\xi, \tag{100}$$

$$\mathrm{E}_{29}\left(a,b;\alpha+\beta,c_3;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}{}_1F_1\left(b;\alpha;x\xi\right)\Psi_2\left(a;\beta,c_3;y\left(1-\xi\right),z\right)d\xi, \tag{101}$$

$$\mathrm{E}_{30}\left(a,b;\alpha+\beta,c_3;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}{}_0F_1\left(-;\alpha;x\xi\right)\Psi_1\left(a,b;\beta,c_3;y\left(1-\xi\right),z\right)d\xi, \tag{102}$$

$$\mathrm{E}_{31}\left(a,b;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}{}_0F_1\left(-;\alpha;x\xi\right)\Psi_1\left(a,b;c,\beta;z,y\left(1-\xi\right)\right)d\xi, \tag{103}$$

$$\mathrm{E}_{32}\left(a;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}{}_0F_1\left(-;\alpha;x\xi\right)\Psi_2\left(a;\beta,c;y\left(1-\xi\right),z\right)d\xi, \tag{104}$$

$$\mathrm{E}_{33}\left(a_1,a_2,a_3;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}{}_1F_1\left(a_3;\alpha;x\xi\right)F_4\left(a_1,a_2;\beta,c;y\left(1-\xi\right),z\right)d\xi, \tag{105}$$

$$\mathrm{E}_{33}\left(\alpha,a,\beta;c_1,c_2;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{43}\left(\alpha+\beta,a;c_1,c_2;x\left(1-\xi\right),y\xi,z\xi\right)d\xi, \tag{106}$$

$$\mathrm{E}_{34}\left(a,b;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}{}_0F_1\left(\alpha;x\xi\right)F_4\left(a,b;\beta,c;y\left(1-\xi\right),z\right)d\xi, \tag{107}$$

$$\mathrm{E}_{35}\left(a,\alpha,\beta;c_1,c_2;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\Psi_1\left(\alpha+\beta,a;c_1,c_2;x\left(1-\xi\right)+y\xi,z\xi\right)d\xi, \tag{108}$$

$$\mathrm{E}_{35}\left(a_1,a_2,a_3;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{62}\left(a_1,a_2,a_3;\alpha,\beta,c;x\xi,y\left(1-\xi\right),z\right)d\xi, \tag{109}$$

$$\mathrm{E}_{36}\left(\alpha,a,\beta;c_1,c_2;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{43}\left(\alpha+\beta,a;c_1,c_2;x\xi,y\xi,z\left(1-\xi\right)\right)d\xi, \tag{110}$$

$$\mathrm{E}_{36}\left(a_1,a_2,a_3;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}E_{62}\left(a_2,a_1,a_3;c,\beta,\alpha;z,y\left(1-\xi\right),x\xi\right)d\xi, \tag{111}$$

$$\mathrm{E}_{37}\left(a,b;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{63}\left(a,b;\alpha,\beta,c;x\xi,y\left(1-\xi\right),z\right)d\xi, \tag{112}$$

$$\mathrm{E}_{38}\left(a,\alpha,\beta;c_1,c_2;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{43}\left(\alpha+\beta,a;c_1,c_2;y\xi,x\left(1-\xi\right),z\xi\right)d\xi, \tag{113}$$

$$\mathrm{E}_{38}\left(\alpha,\beta,a;c_1,c_2;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{60}\left(\alpha+\beta,a;c_1,c_2;x\xi,y\left(1-\xi\right),z\xi\left(1-\xi\right)\right)d\xi, \tag{114}$$

$$\mathrm{E}_{39}\left(\alpha,\beta;c_1,c_2;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{H}_7\left(\alpha+\beta;c_2,c_1;z\xi\left(1-\xi\right),x\xi+y\left(1-\xi\right)\right)d\xi, \tag{115}$$

$$\mathrm{E}_{39}\left(a,b;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}E_{63}\left(a,b;\alpha,c,\beta;x\xi,z,y\left(1-\xi\right)\right)d\xi, \tag{116}$$

$$\mathrm{E}_{40}\left(a,\alpha,\beta;c_1,c_2;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\Psi_1\left(a,\alpha+\beta;c_1,c_2;x\xi+y\left(1-\xi\right),z\right)d\xi, \tag{117}$$

$$\mathrm{E}_{40}\left(a_1,a_2,a_3;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{64}\left(a_1,a_2,a_3;\alpha,\beta,c;x\xi,y\left(1-\xi\right),z\right)d\xi, \tag{118}$$

$$\mathrm{E}_{41}\left(\alpha,a,\beta;c_1,c_2;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{60}\left(\alpha+\beta,a;c_1,c_2;x\xi,y\xi,z\xi\left(1-\xi\right)\right)d\xi, \tag{119}$$

$$\mathrm{E}_{41}\left(a_1,a_2,a_3;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{64}\left(a_1,a_2,a_3;\alpha,c,\beta;x\xi,z,y\left(1-\xi\right)\right)d\xi, \tag{120}$$

$$\mathrm{E}_{42}\left(a,b;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{65}\left(a,b;\alpha,\beta,c;x\xi,y\left(1-\xi\right),z\right)d\xi, \tag{121}$$

$$\mathrm{E}_{43}\left(\alpha,\beta;c_1,c_2;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{61}\left(\alpha+\beta;c_1,c_2;x\xi,y\xi\left(1-\xi\right),z\xi\left(1-\xi\right)\right)d\xi, \tag{122}$$

$$\mathrm{E}_{43}\left(a,b;\alpha+\beta,c;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{67}\left(a,b;c,\beta,\alpha;z,y\left(1-\xi\right),x\xi\right)d\xi, \tag{123}$$

$$\mathrm{E}_{62}\left(a_1,\alpha,\beta;c_1,c_2,c_3;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{67}\left(\alpha+\beta,a_1;c_1,c_2,c_3;x\left(1-\xi\right),y\xi,z\xi\right)d\xi, \tag{124}$$

$$\mathrm{E}_{62}\left(\alpha,\beta,a_3;c_1,c_2,c_3;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{69}\left(\alpha+\beta,a_3;c_2,c_1,c_3;y\xi\left(1-\xi\right),x\xi,z\left(1-\xi\right)\right)d\xi, \tag{125}$$

$$\mathrm{E}_{63}\left(\alpha,\beta;c_1,c_2,c_3;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{70}\left(\alpha+\beta;c_2,c_1,c_3;y\xi\left(1-\xi\right),x\xi,z\left(1-\xi\right)\right)d\xi, \tag{126}$$

$$\mathrm{E}_{64}\left(\alpha,\beta,a_3;c_1,c_2,c_3;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{69}\left(\alpha+\beta,a_3;c_1,c_2,c_3;x\xi\left(1-\xi\right),y\xi,z\xi\right)d\xi, \tag{127}$$

$$\mathrm{E}_{65}\left(\alpha,\beta;c_1,c_2,c_3;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{70}\left(\alpha+\beta;c_1,c_2,c_3;x\xi\left(1-\xi\right),y\xi,z\xi\right)d\xi, \tag{128}$$

$$\mathrm{E}_{67}\left(\alpha,\beta;c_1,c_2,c_3;x,y,z\right)=\lambda\int\limits_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{71}\left(\alpha+\beta;c_1,c_2,c_3;x\xi\left(1-\xi\right),y\xi\left(1-\xi\right),z\xi\right)d\xi, \quad (129)$$

where $\mathrm{E}_{60}$, $\mathrm{E}_{61}$, $\mathrm{E}_{69}$, $\mathrm{E}_{70}$ and $\mathrm{E}_{71}$ are confluent hypergeometric functions defined in [12, 15, 13]:

$$\mathrm{E}_{60}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+2p}(a_2)_m}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{61}\left(a;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+2p}}{(c_1)_{m+n}(c_2)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{69}\left(a_1,a_2;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_n}{(c_1)_m(c_2)_n(c_3)_p m!n!p!}x^m y^n z^p,$$

$$\mathrm{E}_{70}\left(\alpha;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$\mathrm{E}_{71}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+2n+p}}{(c_1)_m(c_2)_n(c_3)_p m!n!p!}x^m y^n z^p.$$

Proof. The equalities (84) to (129) follow easily from the definition of the series which are in the single integrals, if we use the formula (5) for calculating the beta function. To give an example, by definition (15) of the Humbert function $\Psi_2$, in the right-hand side of the integral (100) we have

$$\begin{aligned} I_2 &= \lambda\int\limits_0^1\xi^{\beta-1}\left(1-\xi\right)^{\alpha-1}\Psi_2\left(\alpha+\beta;c_1,c_2;x\xi+y\left(1-\xi\right),z\left(1-\xi\right)\right)d\xi \\ &= \lambda\sum_{k,p=0}^{\infty}\frac{(\alpha+\beta)_{k+p}}{(c_1)_k\,(c_2)_p\,k!p!}z^p\int\limits_0^1\xi^{\beta-1}\left(1-\xi\right)^{\alpha+p-1}\left[x\xi+y(1-\xi)\right]^k d\xi. \end{aligned} \quad (130)$$

Using in (130) a famous relation [16, p. 52]

$$\sum_{k=0}^{\infty}\frac{f(k)}{k!}(x+y)^k=\sum_{m,n=0}^{\infty}\frac{f(m+n)}{m!n!}x^m y^n,$$

we obtain

$$I_2=\lambda\sum_{m,n,p=0}^{\infty}\frac{(\alpha+\beta)_{m+n+p}}{(c_1)_{m+n}\,(c_2)_p\,m!n!p!}x^m y^n z^p\int\limits_0^1\xi^{\beta+m-1}\left(1-\xi\right)^{\alpha+n+p-1}d\xi. \quad (131)$$

Next, applying to (131) the definition of beta function (5), relations (3) and (4), we get

$$I_2=\lambda\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{n+p}(\beta)_m}{(c_1)_{m+n}\,(c)_p\,m!n!p!}x^m y^n z^p=\mathrm{E}_{29}\left(\alpha,\beta;c_1,c_2;x,y,z\right).$$

The integral representation (100) is proved. □

## 3. Double and triple integral representations

**Theorem 3.1**. *The following double integral representation formulas are valid:*

$$\mathrm{E}_{22}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)\Gamma\left(a_3+a_4\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)}\times$$

$$\times\int\limits_0^1\int\limits_0^1\xi^{a_1-1}\left(1-\xi\right)^{a_2-1}\eta^{a_3-1}\left(1-\eta\right)^{a_4-1}\mathrm{E}_{43}\left(a_1+a_2,a_3+a_4;c_1,c_2;x,y,z\right)d\xi d\eta, \quad (132)$$

$$\mathrm{Re}\,(a_1)>0,\mathrm{Re}\,(a_2)>0,\mathrm{Re}\,(a_3)>0,\mathrm{Re}\,(a_4)>0;$$

$$\mathrm{E}_{23}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\frac{\Gamma\left(c_2\right)}{\Gamma\left(a_4\right)\Gamma\left(c_2-a_4\right)}\int\limits_0^1\int\limits_0^1\xi^{a_1-1}\left(1-\xi\right)^{a_2-1}\times$$
$$\times\frac{\eta^{a_4-1}\left(1-\eta\right)^{c_2-a_4-1}}{\left(1-z\xi\eta\right)^{a_1+a_2}}\Phi_1\left(a_1+a_2,a_3;c_1;\frac{x\left(1-\xi\right)}{1-z\xi\eta},\frac{y\xi}{1-z\xi\eta}\right)d\xi d\eta, \tag{133}$$
$$\mathrm{Re}\,(a_1)>0,\mathrm{Re}\,(a_2)>0,\mathrm{Re}\,(c_2)>\mathrm{Re}\,(a_4)>0;$$

$$\mathrm{E}_{23}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\frac{\Gamma\left(c_2\right)}{\Gamma\left(a_4\right)\Gamma\left(c_2-a_4\right)}\int\limits_0^1\int\limits_0^1 e^{\frac{y\xi}{1-z\xi\eta}}\xi^{a_1-1}\left(1-\xi\right)^{a_2-1}\times$$
$$\times\frac{\eta^{a_4-1}\left(1-\eta\right)^{c_2-a_4-1}}{\left(1-z\xi\eta\right)^{a_1+a_2}}\Xi_1\left(a_1+a_2,c_1-a_1-a_2,a_3;c_1;\frac{x\left(1-\xi\right)}{1-z\xi\eta},-\frac{y\xi}{1-z\xi\eta}\right)d\xi d\eta, \tag{134}$$
$$\mathrm{Re}\,(a_1)>0,\mathrm{Re}\,(a_2)>0,\mathrm{Re}\,(c_2)>\mathrm{Re}\,(a_4)>0;$$

$$\mathrm{E}_{23}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_2+a_3+a_4\right)}{\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)}\int\limits_0^1\int\limits_0^1\xi^{a_3-1}\left(1-\xi\right)^{a_4-1}\times$$
$$\times\eta^{a_3+a_4-1}\left(1-\eta\right)^{a_2-1}\mathrm{E}_{50}\left(a_2+a_3+a_4,a_1;c_1,c_2;x\xi\eta\left(1-\eta\right),y,z\left(1-\xi\right)\eta\right)d\xi d\eta, \tag{135}$$
$$\mathrm{Re}\,(a_2)>0,\mathrm{Re}\,(a_3)>0,\mathrm{Re}\,(a_4)>0;$$

$$\mathrm{E}_{24}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(c_1\right)}{\Gamma\left(a_2\right)\Gamma\left(a_4\right)\Gamma\left(c_1-a_2-a_4\right)}\int\limits_0^1\int\limits_0^1\xi^{a_2-1}\left(1-\xi\right)^{c_1-a_2-1}\times$$
$$\times\frac{\eta^{a_4-1}\left(1-\eta\right)^{c_1-a_2-a_4-1}}{\left(1-x\xi\right)^{a_3}\left(1-y\left(1-\xi\right)\eta\right)^{a_1}}{}_1F_1\left(a_1;c_2;\frac{z}{1-y\left(1-\xi\right)\eta}\right)d\xi d\eta, \tag{136}$$
$$\mathrm{Re}\,(c_1)>\mathrm{Re}\,(a_2)>0,\mathrm{Re}\,(c_1)>\mathrm{Re}\,(a_2+a_4)>0,\mathrm{Re}\,(a_4)>0;$$

$$\mathrm{E}_{24}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)\Gamma\left(a_3+a_4\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)}\int\limits_0^1\int\limits_0^1\xi^{a_3-1}\left(1-\xi\right)^{a_4-1}\times$$
$$\times\eta^{a_1-1}\left(1-\eta\right)^{a_2-1}\Psi_1\left(a_1+a_2,a_3+a_4;c_1,c_2;x\xi\left(1-\eta\right)+y\left(1-\xi\right)\eta,z\eta\right)d\xi d\eta, \tag{137}$$
$$\mathrm{Re}\,(a_1)>0,\mathrm{Re}\,(a_2)>0,\mathrm{Re}\,(a_3)>0,\mathrm{Re}\,(a_4)>0;$$

$$\mathrm{E}_{24}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_2+a_3+a_4\right)}{\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)}\int\limits_0^1\int\limits_0^1\xi^{a_3-1}\left(1-\xi\right)^{a_4-1}\times$$
$$\times\eta^{a_3+a_4-1}\left(1-\eta\right)^{a_2-1}\mathrm{E}_{49}\left(a_2+a_3+a_4,a_1;c_1,c_2;x\xi\eta\left(1-\eta\right),y\eta\left(1-\xi\right),z\right)d\xi d\eta, \tag{138}$$
$$\mathrm{Re}\,(a_2)>0,\mathrm{Re}\,(a_3)>0,\mathrm{Re}\,(a_4)>0;$$

$$\mathrm{E}_{25}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)\Gamma\left(c_1\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(c_1-a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{a_2-1}\left(1-\xi\right)^{a_1-1}\times$$
$$\times\frac{\eta^{a_3-1}\left(1-\eta\right)^{c_1-a_3-1}}{\left(1-\xi\eta\right)^{a_1+a_2}}\Psi_2\left(a_1+a_2;c_1-a_3,c_2;\frac{y\left(1-\xi\right)\left(1-\eta\right)}{1-x\xi\eta},\frac{z\left(1-\xi\right)}{1-x\xi\eta}\right)d\xi d\eta, \tag{139}$$
$$\mathrm{Re}\,(a_1)>0,\mathrm{Re}\,(a_2)>0,\mathrm{Re}\,(c_1)>\mathrm{Re}\,(a_3)>0;$$

$$\mathrm{E}_{25}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2+a_3\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{a_3-1}\left(1-\xi\right)^{a_1-1}\eta^{a_2-1}\left(1-\eta\right)^{a_1+a_3-1}\times$$
$$\times\mathrm{E}_{53}\left(a_1+a_2+a_3;c_1,c_2;x\xi\eta\left(1-\eta\right),y\left(1-\xi\right)\left(1-\eta\right),z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{140}$$
$$\mathrm{Re}\,(a_1)>0,\mathrm{Re}\,(a_2)>0,\mathrm{Re}\,(a_3)>0;$$

$$\mathrm{E}_{26}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)\Gamma\left(c_1\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(c_1-a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{a_2-1}\left(1-\xi\right)^{a_1-1}\times$$

$$\times\frac{\eta^{a_3-1}\left(1-\eta\right)^{c_1-a_3-1}}{\left[1-y\eta+y\xi\eta\right]^{a_1+a_2}}\Psi_2\left(a_1+a_2;c_1-a_3,c_2;\frac{x\xi\left(1-\eta\right)}{1-y\eta+y\xi\eta},\frac{z\left(1-\xi\right)}{1-y\eta+y\xi\eta}\right)d\xi d\eta,\tag{141}$$

$$\mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0,\mathrm{Re}\left(c_1\right)>\mathrm{Re}\left(a_3\right)>0;$$

$$\mathrm{E}_{27}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2+a_3\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{a_1-1}\left(1-\xi\right)^{a_2-1}\times$$

$$\times\eta^{a_1+a_2-1}\left(1-\eta\right)^{a_3-1}\mathrm{H}_7\left(a_1+a_2+a_3;c_2,c_1;z\xi\eta\left(1-\eta\right),x\eta\left(1-\xi\right)+y\xi\eta\right)d\xi d\eta,\tag{142}$$

$$\mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0,\mathrm{Re}\left(a_3\right)>0;$$

$$\mathrm{E}_{27}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)\Gamma\left(c_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(c_2-a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{a_2-1}\left(1-\xi\right)^{a_1-1}\times$$

$$\times\frac{\eta^{a_3-1}\left(1-\eta\right)^{c_2-a_3-1}}{\left[1-z\eta+z\xi\eta\right]^{a_1+a_2}}\ {}_1F_1\left(a_1+a_2;c_1;\frac{x\xi+y\left(1-\xi\right)}{1-z\eta+z\xi\eta}\right)d\xi d\eta,\tag{143}$$

$$\mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0,\mathrm{Re}\left(a_3\right)>0,\mathrm{Re}\left(c_2\right)>\mathrm{Re}\left(a_3\right);$$

$$\mathrm{E}_{28}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)\Gamma\left(c_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(c_2-a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{a_1-1}\left(1-\xi\right)^{a_2-1}\times$$

$$\times\frac{\eta^{a_3-1}\left(1-\eta\right)^{c_2-a_3-1}}{\left(1-z\xi\eta\right)^{a_1+a_2}}\Xi_2\left(\frac{a_1+a_2}{2},\frac{a_1+a_2+1}{2};c_1;\frac{4y\xi\left(1-\xi\right)}{\left(1-z\xi\eta\right)^2},x\right)d\xi d\eta,\tag{144}$$

$$\mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0,\mathrm{Re}\left(c_2\right)>\mathrm{Re}\left(a_3\right)>0;$$

$$\mathrm{E}_{28}\left(a_1,a_2,a_3;c_1,c_3;x,y,z\right)=\frac{\Gamma\left(c_1+c_2\right)\Gamma\left(c_3\right)}{\Gamma\left(a_3\right)\Gamma\left(c_1\right)\Gamma\left(c_2\right)\Gamma\left(c_3-a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{c_1-1}\left(1-\xi\right)^{c_2-1}\times$$

$$\times\frac{\eta^{a_3-1}\left(1-\eta\right)^{c_3-a_3-1}}{\left(1-z\eta\right)^{a_1}}{}_0F_1\left(c_1;x\xi\right)F\left(a_1,a_2;c_2;\frac{y\left(1-\xi\right)}{1-z\eta}\right)d\xi d\eta,\tag{145}$$

$$\mathrm{Re}\left(c_1\right)>0,\mathrm{Re}\left(c_2\right)>0,\mathrm{Re}\left(c_3\right)>\mathrm{Re}\left(a_3\right)>0;$$

$$\mathrm{E}_{28}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(a_1+a_2+a_3\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{a_1-1}\left(1-\xi\right)^{a_2-1}\eta^{a_1+a_2-1}\left(1-\eta\right)^{a_3-1}\times$$

$$\times\mathrm{E}_{34}\left(\frac{a_1+a_2+a_3}{2},\frac{a_1+a_2+a_3+1}{2};c_1,c_2;x,4y\xi\left(1-\xi\right)\eta^2,4z\xi\eta\left(1-\eta\right)\right)d\xi d\eta,\tag{146}$$

$$\mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0,\mathrm{Re}\left(a_3\right)>0;$$

$$\mathrm{E}_{31}\left(a,b;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(c_1\right)\Gamma\left(c_2\right)}{\Gamma\left(a\right)\Gamma\left(b\right)\Gamma\left(c_1-a\right)\Gamma\left(c_2-b\right)}\int\limits_0^1\int\limits_0^1 e^{\frac{y\eta}{1-z\xi}}\xi^{b-1}\eta^{a-1}\times$$

$$\times\left(1-\xi\right)^{c_2-b-1}\left(1-\eta\right)^{c_1-a-1}\left(1-z\xi\right)^{-a}{}_0F_1\left(c_1-a;x\left(1-\eta\right)\right)d\xi d\eta,\tag{147}$$

$$\mathrm{Re}\left(c_1\right)>\mathrm{Re}\left(a\right)>0,\mathrm{Re}\left(c_2\right)>\mathrm{Re}\left(b\right)>0;$$

$$\mathrm{E}_{31}\left(a,b;c_1+c_2,c_3;x,y,z\right)=\frac{\Gamma\left(c_1+c_2\right)\Gamma\left(c_3\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)\Gamma\left(b\right)\Gamma\left(c_3-b\right)}\int\limits_0^1\int\limits_0^1\xi^{c_1-1}\left(1-\xi\right)^{c_2-1}\times$$

$$\times\frac{\eta^{b-1}\left(1-\eta\right)^{c_3-b-1}}{\left(1-z\eta\right)^{a}}\ {}_0F_1\left(c_1;x\xi\right)\ {}_1F_1\left(a;c_2;\frac{y\left(1-\xi\right)}{1-z\eta}\right)d\xi d\eta,\tag{148}$$

$$\mathrm{Re}\,(c_1) > 0, \mathrm{Re}\,(c_2) > 0, \mathrm{Re}\,(c_3) > \mathrm{Re}\,(b) > 0;$$

$$\mathrm{E}_{35}\left(a_1, a_2, a_3; c_1 + c_2, c_3; x, y, z\right) = \frac{\Gamma\left(c_1 + c_2\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)} \frac{\Gamma\left(c_1\right)}{\Gamma\left(a_3\right)\Gamma\left(c_1 - a_3\right)} \int\limits_0^1 \int\limits_0^1 \xi^{c_1 - 1}\left(1 - \xi\right)^{c_2 - 1} \times$$

$$\times \frac{\eta^{a_3 - 1}\left(1 - \eta\right)^{c_1 - a_3 - 1}}{\left(1 - x\xi\eta\right)^{a_1}} \Psi_1\left(a_2, a_1; c_2, c_3; \frac{y\left(1 - \xi\right)}{1 - x\xi\eta}, z\right) d\xi d\eta, \qquad (149)$$

$$\mathrm{Re}\,(c_1) > 0, \mathrm{Re}\,(c_2) > 0, \mathrm{Re}\,(c_1) > \mathrm{Re}\,(a_3) > 0;$$

$$\mathrm{E}_{36}\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right) = \frac{\Gamma\left(c_1\right)}{\Gamma\left(a_1\right)\Gamma\left(c_1 - a_1\right)} \frac{\Gamma\left(c_2\right)}{\Gamma\left(a_3\right)\Gamma\left(c_2 - a_3\right)} \times$$

$$\times \int\limits_0^1 \int\limits_0^1 e^{x\xi} \xi^{a_1 - 1} \eta^{a_3 - 1}\left(1 - \xi\right)^{c_1 - a_1 - 1}\left(1 - \eta\right)^{c_2 - a_3 - 1}\left(1 - y\xi - z\eta\right)^{-a_2} d\xi d\eta, \qquad (150)$$

$$\mathrm{Re}\,(c_1) > \mathrm{Re}\,(a_1) > 0, \mathrm{Re}\,(c_2) > \mathrm{Re}\,(a_3) > 0;$$

$$\mathrm{E}_{36}\left(a_1, a_2, a_3; c_1 + c_2, c_3; x, y, z\right) = \frac{\Gamma\left(c_1 + c_2\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)} \frac{\Gamma\left(c_3\right)}{\Gamma\left(a_3\right)\Gamma\left(c_3 - a_3\right)} \int\limits_0^1 \int\limits_0^1 \xi^{c_1 - 1}\left(1 - \xi\right)^{c_2 - 1} \times$$

$$\times \frac{\eta^{a_3 - 1}\left(1 - \eta\right)^{c_3 - a_3 - 1}}{\left(1 - z\eta\right)^{a_2}} \Psi_1\left(a_1, a_2; c_2, c_1; \frac{y\left(1 - \xi\right)}{1 - z\eta}, x\xi\right) d\xi d\eta, \qquad (151)$$

$$\mathrm{Re}\,(c_1) > 0, \mathrm{Re}\,(c_2) > 0, \mathrm{Re}\,(c_3) > \mathrm{Re}\,(a_3) > 0;$$

$$\mathrm{E}_{38}\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right) = \frac{\Gamma\left(c_1\right)\Gamma\left(c_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_3\right)\Gamma\left(c_1 - a_3\right)\Gamma\left(c_2 - a_1\right)} \int\limits_0^1 \int\limits_0^1 \xi^{a_3 - 1}\left(1 - \xi\right)^{c_1 - a_3 - 1} \times$$

$$\times \frac{\eta^{a_1 - 1}\left(1 - \eta\right)^{c_2 - a_1 - 1}}{\left(1 - x\xi\right)^{a_1 - a_2}\left(1 - x\xi - z\eta\right)^{a_2}} \; {}_1F_1\left(a_2; c_1 - a_3; \frac{y\left(1 - \xi\right)\left(1 - x\xi\right)}{1 - x\xi - z\eta}\right) d\xi d\eta, \qquad (152)$$

$$\mathrm{Re}\,(c_1) > \mathrm{Re}\,(a_3) > 0, \mathrm{Re}\,(c_2) > \mathrm{Re}\,(a_1) > 0;$$

$$\mathrm{E}_{38}\left(a_1, a_2, a_3; c_1 + c_2, c_3; x, y, z\right) = \frac{\Gamma\left(c_1 + c_2\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)} \frac{\Gamma\left(c_1\right)}{\Gamma\left(a_3\right)\Gamma\left(c_1 - a_3\right)} \int\limits_0^1 \int\limits_0^1 \xi^{c_1 - 1}\left(1 - \xi\right)^{c_2 - 1} \times$$

$$\times \frac{\eta^{a_3 - 1}\left(1 - \eta\right)^{c_1 - a_3 - 1}}{\left(1 - x\xi\eta\right)^{a_1}} \Psi_1\left(a_2, a_1; c_3, c_2; \frac{z}{1 - x\xi\eta}, y\left(1 - \xi\right)\right) d\xi d\eta, \qquad (153)$$

$$\mathrm{Re}\,(c_1) > 0, \mathrm{Re}\,(c_2) > 0, \mathrm{Re}\,(c_1) > \mathrm{Re}\,(a_3) > 0;$$

$$\mathrm{E}_{38}\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right) = \frac{\Gamma\left(a_1 + a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)} \frac{\Gamma\left(c_1\right)}{\Gamma\left(a_3\right)\Gamma\left(c_1 - a_3\right)} \int\limits_0^1 \int\limits_0^1 \xi^{a_1 - 1}\left(1 - \xi\right)^{a_2 - 1} \times$$

$$\times \frac{\eta^{a_3 - 1}\left(1 - \eta\right)^{c_1 - a_3 - 1}}{\left(1 - x\xi\eta\right)^{a_1 + a_2}} \mathrm{H}_7\left(a_1 + a_2; c_2, c_1 - a_3; \frac{z\xi\left(1 - \xi\right)}{\left(1 - x\xi\eta\right)^2}, \frac{y\left(1 - \xi\right)\left(1 - \eta\right)}{1 - x\xi\eta}\right) d\xi d\eta, \qquad (154)$$

$$\mathrm{Re}\,(a_1) > 0, \mathrm{Re}\,(a_2) > 0, \mathrm{Re}\,(c_1) > \mathrm{Re}\,(a_3) > 0;$$

$$\mathrm{E}_{39}\left(a, b; c_1 + c_2, c_3; x, y, z\right) = \frac{\Gamma\left(a + b\right)}{\Gamma\left(a\right)\Gamma\left(b\right)} \frac{\Gamma\left(c_1 + c_2\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)} \int\limits_0^1 \int\limits_0^1 \xi^{a - 1}\left(1 - \xi\right)^{b - 1} \times$$

$$\times \eta^{c_1 - 1}\left(1 - \eta\right)^{c_2 - 1} \mathrm{E}_{70}\left(a + b; c_3, c_2, c_1; z\xi\left(1 - \xi\right), y\left(1 - \xi\right)\left(1 - \eta\right), x\xi\eta\right) d\xi d\eta, \qquad (155)$$

$$\mathrm{Re}\,(a) > 0, \mathrm{Re}\,(b) > 0, \mathrm{Re}\,(c_1) > 0, \mathrm{Re}\,(c_2) > 0;$$

$$\mathrm{E}_{40}\left(a_1, a_2, a_3; c_1 + c_2, c_3; x, y, z\right) = \frac{\Gamma\left(c_1 + c_2\right)}{\Gamma\left(a_2\right)\Gamma\left(c_1 - a_2\right)\Gamma\left(c_2\right)} \int\limits_0^1 \int\limits_0^1 \xi^{c_1 - 1}\left(1 - \xi\right)^{c_2 - 1} \times$$

$$\times \frac{\eta^{a_2-1}(1-\eta)^{c_1-a_2-1}}{(1-x\xi\eta)^{a_1}} \Psi_1\left(a_1, a_3; c_2, c_3; \frac{y(1-\xi)}{1-x\xi\eta}, \frac{z}{1-x\xi\eta}\right) d\xi d\eta, \tag{156}$$

$$\operatorname{Re}(c_1) > \operatorname{Re}(a_2) > 0, \operatorname{Re}(c_2) > 0;$$

$$\mathrm{E}_{40}(a_1, a_2, a_3; c_1, c_2; x, y, z) = \frac{\Gamma(a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^1\int_0^1 \xi^{a_1+a_2-1}(1-\xi)^{a_3-1} \times$$

$$\times \eta^{a_1-1}(1-\eta)^{a_2-1}\mathrm{H}_7\left(a_1+a_2+a_3; c_1, c_2; x\xi^2\eta(1-\eta)+y\xi(1-\xi)\eta, z\xi\eta(1-\eta)\right) d\xi d\eta, \tag{157}$$

$$\operatorname{Re}(a_1) > 0, \operatorname{Re}(a_2) > 0, \operatorname{Re}(a_3) > 0;$$

$$\mathrm{E}_{41}(a_1, a_2, a_3; c_1, c_2; x, y, z) = \frac{\Gamma(c_1)}{\Gamma(a_2)\Gamma(c_1-a_2)} \frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)} \int_0^1\int_0^1 \xi^{a_2-1}(1-\xi)^{c_1-a_2-1} \times$$

$$\times \frac{\eta^{a_3-1}(1-\eta)^{c_2-a_3-1}}{(1-x\xi-z\eta)^{a_1}} {}_1F_1\left(a_1; c_1-a_2; \frac{y(1-\xi)}{1-x\xi-z\eta}\right) d\xi d\eta, \tag{158}$$

$$\operatorname{Re}(c_1) > \operatorname{Re}(a_2) > 0, \operatorname{Re}(c_2) > \operatorname{Re}(a_3) > 0;$$

$$\mathrm{E}_{41}(a_1, a_2, a_3; c_1, c_2; x, y, z) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} \frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)} \int_0^1\int_0^1 \xi^{a_1-1}(1-\xi)^{a_2-1} \times$$

$$\times \frac{\eta^{a_3-1}(1-\eta)^{c_2-a_3-1}}{(1-z\xi\eta)^{a_1+a_2}} \mathrm{H}_6\left(a_1+a_2; c_1; \frac{x\xi(1-\xi)}{(1-z\xi\eta)^2}, \frac{y\xi}{1-z\xi\eta}\right) d\xi d\eta, \tag{159}$$

$$\operatorname{Re}(a_1) > 0, \operatorname{Re}(a_2) > 0, \operatorname{Re}(c_2) > \operatorname{Re}(a_3) > 0;$$

$$\mathrm{E}_{41}(a_1, a_2, a_3; c_1+c_2, c_3; x, y, z) = \frac{\Gamma(c_1+c_2)}{\Gamma(c_2)\Gamma(a_2)\Gamma(c_1-a_2)} \int_0^1\int_0^1 \xi^{c_1-1}(1-\xi)^{c_2-1} \times$$

$$\times \frac{\eta^{a_2-1}(1-\eta)^{c_1-a_2-1}}{(1-x\xi\eta)^{a_1}} \Psi_1\left(a_1, a_3; c_3, c_2; \frac{z}{1-x\xi\eta}, \frac{y(1-\xi)}{1-x\xi\eta}\right) d\xi d\eta, \tag{160}$$

$$\operatorname{Re}(c_1) > 0, \operatorname{Re}(c_2) > 0, \operatorname{Re}(c_1) > \operatorname{Re}(a_2) > 0;$$

$$\mathrm{E}_{42}(a, b; c_1+c_2, c_3; x, y, z) = \frac{\Gamma(c_1+c_2)}{\Gamma(c_2)\Gamma(b)\Gamma(c_1-b)} \int_0^1\int_0^1 \xi^{c_1-1}(1-\xi)^{c_2-1} \times$$

$$\times \frac{\eta^{b-1}(1-\eta)^{c_1-b-1}}{(1-x\xi\eta)^{a}} \Psi_2\left(a; c_2, c_3; \frac{y(1-\xi)}{1-x\xi\eta}, \frac{z}{1-x\xi\eta}\right) d\xi d\eta, \tag{161}$$

$$\operatorname{Re}(c_1) > 0, \operatorname{Re}(c_2) > 0, \operatorname{Re}(c_1) > \operatorname{Re}(b) > 0;$$

$$\mathrm{E}_{62}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(c_1-a_3)\Gamma(c_2-a_1)} \int_0^1\int_0^1 \xi^{a_3-1}(1-\xi)^{c_1-a_3-1} \times$$

$$\times \frac{\eta^{a_1-1}(1-\eta)^{c_2-a_1-1}}{(1-x\xi)^{a_1-a_2}(1-x\xi-y\eta)^{a_2}} {}_1F_1\left(a_2; c_3; \frac{(1-x\xi)z}{1-x\xi-y\eta}\right) d\xi d\eta, \tag{162}$$

$$\operatorname{Re}(c_1) > \operatorname{Re}(a_3) > 0, \operatorname{Re}(c_2) > \operatorname{Re}(a_1) > 0;$$

$$\mathrm{E}_{62}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \frac{\Gamma(c_1)}{\Gamma(c_1-a_3)} \int_0^1\int_0^1 \xi^{a_1-1}(1-\xi)^{a_2-1} \times$$

$$\times \frac{\eta^{a_3-1}(1-\eta)^{c_1-a_3-1}}{(1-x\xi\eta)^{a_1+a_2}} \mathrm{H}_7\left(a_1+a_2; c_2, c_3; \frac{y\xi(1-\xi)}{(1-x\xi\eta)^2}, \frac{z(1-\xi)}{1-x\xi\eta}\right) d\xi d\eta, \tag{163}$$

$$\operatorname{Re}(c_1) > 0, \operatorname{Re}(c_2) > 0, \operatorname{Re}(c_1) > \operatorname{Re}(a_2) > 0;$$

$$\mathrm{E}_{62}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\frac{\Gamma\left(a_1+a_2+a_3\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{a_1-1}(1-\xi)^{a_2-1}\times$$
$$\times\eta^{a_1+a_2-1}(1-\eta)^{a_3-1}\mathrm{E}_{71}\left(a_1+a_2+a_3;c_1,c_2,c_3;x\xi\eta\left(1-\eta\right),y\xi\left(1-\xi\right)\eta^2,z\left(1-\xi\right)\eta\right)d\xi d\eta, \tag{164}$$
$$\mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0,\mathrm{Re}\left(a_3\right)>0;$$

$$\mathrm{E}_{64}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\frac{\Gamma\left(c_1\right)}{\Gamma\left(a_2\right)\Gamma\left(c_1-a_2\right)}\frac{\Gamma\left(c_2\right)}{\Gamma\left(a_3\right)\Gamma\left(c_2-a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{a_2-1}(1-\xi)^{c_1-a_2-1}\times$$
$$\times\frac{\eta^{a_3-1}(1-\eta)^{c_2-a_3-1}}{(1-x\xi-y\eta)^{a_1}}{}_1F_1\left(a_1;c_3;\frac{z}{1-x\xi-y\eta}\right)d\xi d\eta, \tag{165}$$
$$\mathrm{Re}\left(c_1\right)>\mathrm{Re}\left(a_2\right)>0,\mathrm{Re}\left(c_2\right)>\mathrm{Re}\left(a_3\right)>0;$$

$$\mathrm{E}_{64}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\frac{\Gamma\left(c_2\right)}{\Gamma\left(a_3\right)\Gamma\left(c_2-a_3\right)}\int\limits_0^1\int\limits_0^1\xi^{a_1-1}(1-\xi)^{a_2-1}\times$$
$$\times\frac{\eta^{a_3-1}(1-\eta)^{c_2-a_3-1}}{(1-y\xi\eta)^{a_1+a_2}}\mathrm{H}_7\left(a_1+a_2;c_1,c_3;\frac{x\xi\left(1-\xi\right)}{\left(1-y\xi\eta\right)^2},\frac{z\xi}{1-y\xi\eta}\right)d\xi d\eta, \tag{166}$$
$$\mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0,\mathrm{Re}\left(c_2\right)>\mathrm{Re}\left(a_3\right)>0.$$

Proof. The equalities (132) – (166) follow easily from the definition of the associated series, if we use the formula (5) for calculating the beta function. To give an example, we shall prove the integral representation (133). □

Consider a double integral

$$I_2=\mu\int\limits_0^1\int\limits_0^1\xi^{a_1-1}\left(1-\xi\right)^{a_2-1}\frac{\eta^{a_4-1}\left(1-\eta\right)^{c_2-a_4-1}}{\left(1-z\xi\eta\right)^{a_1+a_2}}\Phi_1\left(a_1+a_2,a_3;c_1;\frac{x\left(1-\xi\right)}{1-z\xi\eta},\frac{y\xi}{1-z\xi\eta}\right)d\xi d\eta, \tag{167}$$

where

$$\mu=\frac{\Gamma\left(a_1+a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\frac{\Gamma\left(c_2\right)}{\Gamma\left(a_4\right)\Gamma\left(c_2-a_4\right)}.$$

Using the definition of the Humbert function $\Phi_1$, we rewrite the double integral (167) in the form

$$I_2=\mu\sum_{m,n=0}^{\infty}\frac{\left(a_1+a_2\right)_{m+n}\left(a_3\right)_m}{\left(c_1\right)_{m+n}m!n!}x^my^n\int\limits_0^1\int\limits_0^1\frac{\xi^{a_1+n-1}\left(1-\xi\right)^{a_2+m-1}\eta^{a_4-1}\left(1-\eta\right)^{c_2-a_4-1}}{\left(1-z\xi\eta\right)^{a_1+a_2+m+n}}d\xi d\eta. \tag{168}$$

Applying the Taylor expansion (80) to the integrant in (168), we get

$$I_2=\mu\sum_{m,n,p=0}^{\infty}\frac{\left(a_1+a_2\right)_{m+n}\left(a_3\right)_m\left(a_1+a_2+m+n\right)_p}{\left(c_1\right)_{m+n}m!n!p!}x^my^nz^p\times$$
$$\times\int\limits_0^1\xi^{a_1+n+p-1}\left(1-\xi\right)^{a_2+m-1}d\xi\int\limits_0^1\eta^{a_4+p-1}\left(1-\eta\right)^{c_2-a_4-1}d\eta. \tag{169}$$

Next, applying to (169) the definition of beta function (5), relations (3) and (4), we get the left-hand side of the integral representation (133). □

**Theorem 3.2**. *The following double integral representation formulas are valid:*

$$\mathrm{E}_{23}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(c_1\right)\Gamma\left(c_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_4\right)\Gamma\left(c_2-a_4\right)\Gamma\left(c_1-a_1-a_2\right)}\times$$

$$\times \int_0^1 \int_0^1 \int_0^1 e^{\frac{y\xi\zeta}{1-z\xi\eta}} \xi^{a_1-1}(1-\xi)^{a_2-1}\eta^{a_4-1}(1-\eta)^{c_2-a_4-1}\times$$

$$\times \frac{\zeta^{a_1+a_2-1}(1-\zeta)^{c_1-a_1-a_2-1}}{(1-z\xi\eta)^{a_1+a_2}}\left(1-\frac{x(1-\xi)\zeta}{1-z\xi\eta}\right)^{-a_3} d\xi d\eta d\zeta, \tag{170}$$

$$\mathrm{Re}(a_1)>0, \mathrm{Re}(a_2)>0, \mathrm{Re}(c_2)>\mathrm{Re}(a_4)>0, \mathrm{Re}(c_1)>\mathrm{Re}(a_1+a_2)>0;$$

$$\mathrm{E}_{24}(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}\frac{\Gamma(c_1)}{\Gamma(c_1-a_3-a_4)}\times$$

$$\times \int_0^1 \int_0^1 \int_0^1 \xi^{a_3-1}(1-\xi)^{a_4-1}\eta^{a_1-1}(1-\eta)^{a_2-1}\zeta^{a_3+a_4-1}(1-\zeta)^{c_1-a_3-a_4-1}\times$$

$$\times[1-x\xi(1-\eta)\zeta-y(1-\xi)\eta\zeta]^{-a_1-a_2}{}_1F_1\left(a_1+a_2;c_2;\frac{z\eta}{1-x\xi(1-\eta)\zeta-y(1-\xi)\eta\zeta}\right)d\xi d\eta d\zeta, \tag{171}$$

$$\mathrm{Re}(a_1)>0, \mathrm{Re}(a_2)>0, \mathrm{Re}(a_3)>0, \mathrm{Re}(a_4)>0, \mathrm{Re}(c_1)>\mathrm{Re}(a_3+a_4)>0;$$

$$\mathrm{E}_{24}(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z)=\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_4)\Gamma(c_1-a_2-a_4)\Gamma(c_2-a_1)}\int_0^1 \int_0^1 \int_0^1 e^{\frac{z\zeta}{1-y\eta+y\xi\eta}}\times$$

$$\times\xi^{a_2-1}(1-\xi)^{c_1-a_2-1}\eta^{a_4-1}(1-\eta)^{c_1-a_2-a_4-1}\frac{\zeta^{a_1-1}(1-\zeta)^{c_2-a_1-1}}{(1-x\xi)^{a_3}(1-y\eta+y\xi\eta)^{a_1}}d\xi d\eta d\zeta, \tag{172}$$

$$\mathrm{Re}(c_1)>\mathrm{Re}(a_2)>0, \mathrm{Re}(c_2)>\mathrm{Re}(a_1)>0, \mathrm{Re}(c_1)>\mathrm{Re}(a_2+a_4)>0, \mathrm{Re}(a_4)>0;$$

$$\mathrm{E}_{26}(a_1,a_2,a_3;c_1+c_2,c_3;x,y,z)=\frac{\Gamma(c_1+c_2)\Gamma(c_3)}{\Gamma(a_1)\Gamma(a_3)\Gamma(c_1)\Gamma(c_2-a_3)\Gamma(c_3-a_1)}\int_0^1 \int_0^1 \int_0^1 e^{\frac{z\zeta}{1-y\eta+y\xi\eta}}\times$$

$$\times\xi^{c_1-1}(1-\xi)^{c_2-1}\eta^{a_3-1}(1-\eta)^{c_2-a_3-1}\frac{\zeta^{a_1-1}(1-\zeta)^{c_3-a_1-1}}{[1-y\eta+y\xi\eta]^{a_1}}\,{}_1F_1(a_2;c_1;x\xi)\,d\xi d\eta d\zeta, \tag{173}$$

$$\mathrm{Re}(c_1)>0, \mathrm{Re}(c_2)>\mathrm{Re}(a_3)>0, \mathrm{Re}(c_3)>\mathrm{Re}(a_1)>0;$$

$$\mathrm{E}_{40}(a_1,a_2,a_3;c_1+c_2,c_3;x,y,z)=\frac{\Gamma(c_1+c_2)}{\Gamma(a_2)\Gamma(a_3)\Gamma(c_1-a_2)\Gamma(c_2-a_3)}\int_0^1 \int_0^1 \int_0^1 \xi^{c_1-1}(1-\xi)^{c_2-1}\times$$

$$\times\eta^{a_2-1}(1-\eta)^{c_1-a_2-1}\frac{\zeta^{a_3-1}(1-\zeta)^{c_2-a_3-1}}{[1-x\xi\eta-y\zeta+y\xi\zeta]^{a_1}}\,{}_1F_1\left(a_1;c_3;\frac{z}{1-x\xi\eta-y\zeta+y\xi\zeta}\right)d\xi d\eta d\zeta, \tag{174}$$

$$\mathrm{Re}(c_1)>\mathrm{Re}(a_2)>0, \mathrm{Re}(c_2)>\mathrm{Re}(a_3)>0;$$

$$\mathrm{E}_{41}(a_1,a_2,a_3;c_1+c_2,c_3;x,y,z)=\frac{\Gamma(c_1+c_2)\Gamma(c_3)}{\Gamma(c_2)\Gamma(a_2)\Gamma(a_3)\Gamma(c_1-a_2)\Gamma(c_3-a_3)}\int_0^1 \int_0^1 \int_0^1 \xi^{c_1-1}(1-\xi)^{c_2-1}\times$$

$$\times\eta^{a_2-1}(1-\eta)^{c_1-a_2-1}\frac{\zeta^{a_3-1}(1-\zeta)^{c_3-a_3-1}}{(1-x\xi\eta-z\zeta)^{a_1}}\,{}_1F_1\left(a_1;c_2;\frac{y(1-\xi)}{1-x\xi\eta-z\zeta}\right)d\xi d\eta d\zeta, \tag{175}$$

$$\mathrm{Re}(c_1)>0, \mathrm{Re}(c_2)>0, \mathrm{Re}(c_1)>\mathrm{Re}(a_2)>0, \mathrm{Re}(c_3)>\mathrm{Re}(a_3)>0.$$

Proof. The equalities (170) – (175) are proved in the same way as the proof of the Theorem 3. □

## Conclusion

As is known [15], the list of confluent hypergeometric functions of three variables was compiled recently, however, most of the confluent functions investigated in this paper were first introduced by Jain [11] in 1966, who limited himself to composing systems of partial differential equations corresponding to these functions. At present, all linearly independent solutions are known not only for the systems compiled by Jain, but for all PDE-systems corresponding to 395 confluent hypergeometric functions in three variables [12]. Until now, the scientific community is not aware of any application of the degenerate hypergeometric functions $E_{22}$ – $E_{43}$, $E_{62}$ – $E_{67}$. Therefore, in the future it is necessary to identify such applied problems, the solutions of which would be written through the confluent hypergeometric functions studied in this work.

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