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Some Systems of PDE Associated with the Multiple Confluent Hypergeometric Functions and Their Applications

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Abstract—In this article, new functions, which satisfy certain systems of partial differential equations, are introduced. As an application, the self-similar and fundamental solutions of the second order multidimensional systems of partial differential equations with singular coefficients are constructed in explicit forms.

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1. INTRODUCTION

Hypergeometric functions occur naturally in a wide variety of problems in applied mathematics, statistics, operations research, theoretical physics, and engineering sciences. For instance, Srivastava and Kashyap [1] presented a number of interesting applications of hypergeometric functions in one and more variables in queuing theory and related stochastic processes. The work of Niukkanen [2] on the multiple hypergeometric functions is motivated by various physical and quantum chemical applications of such functions. Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations, which are solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [3] (see, also [4–8]).

The success of the theory of hypergeometric functions in one variable has stimulated the development of a corresponding theory in two and more variables. Appell [9] has defined, in 1880, four functions F_1 to F_4 [10, pp. 22, 23], which are all analogous to Gauss' $F(a,b;c;z)$ (see, Equation (2)). Seven confluent forms of the four Appell series were defined by Humbert [11], and he denoted these confluent hypergeometric series in two variables by Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1 and Ξ_2. A great merit in the further development of the theory of the hypergeometric series in two variables belongs to Horn [12], who gave a general definition and order classification of double hypergeometric series. He has investigated the convergence of hypergeometric series of two variables and established the systems of partial differential equations which they satisfy. Horn studied such second-order convergent hypergeometric series that either are not expressed in terms of series of one variable or are not products of two hypergeometric

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series in one variable. There are essentially 34 such convergent hypergeometric series of the second order (14 complete and 20 merging).

Lauricella ([13], p. 114) further generalized the four Applell series F_1, F_2, F_3, F_4 to series in n variables and defined his multiple hypergeometric series denoted by $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$. Moreover, he introduced 14 complete hypergeometric series in three variables of the second order. Saran [14] initiated a systematic study of these ten triple hypergeometric series of Lauricella's set. Other triple hypergeometric series studied in the literature include, in addition to the three-variable analogues of several multiple hypergeometric series, the series introduced by Dhawan [15], Samar [16], and Exton [17]. An expansion of the results on the hypergeometric functions in three variables together with references to the original literature are to be found in the monograph [10] by Srivastava and Karlsson. This work also contains an extensive bibliography of all relevant papers up to 1985. For instance, these authors presented a table of 205 distinct complete triple Gaussian series together with their sources, if known.

Hasanov and Ruzhansky [18] constructed Euler-type integral representations for 205 complete triple hypergeometric series in three variables. Recently, these authors [19] compiled a system of partial differential equations, which satisfy the indicated 205 hypergeometric functions and found all linearly-independent solutions at the origin, if exist. In [20], some analytic continuation formulas for the hypergeometric functions in three variables of second order are obtained. In [21], some exact solutions for a partial system of second-order hypergeometric equations are obtained and some decomposition formulas are constructed.

Despite many applications of confluent forms of hypergeometric functions of more than two variables, they have been relatively little studied. In the works of Exton [22] and Jain [23], some functions were studied that are confluent forms of the complete hypergeometric functions in three variables. 395 confluent hypergeometric functions in three variables have recently become known [24]; thus the set of all possible second-order confluent hypergeometric functions in three variables is completed.

When the number of the variables exceeds two or three, some confluent forms of the multiple Lauricella functions are defined in [10, pp. 34–36].

The plan of this paper is as follows. In Section 2, we briefly give some preliminary information, which will be used later and define new multiple confluent hypergeometric functions. Thus, we try to expand the class of such functions. In Section 3, we find a domain of the definition of the multiple confluent hypergeometric functions. In Section 4, we write the systems of partial differential equations, which satisfy the confluent hypergeometric functions in many variables, defined in Section 2. In Section 5, we find all linearly-independent solutions of the systems of the hypergeometric type, complied in Section 4, at the origin, if exist. In Sections 6 and 7, we obtain in explicit forms self-similar and fundamental solutions of the second order multidimensional partial differential equations with singular coefficients.

2. PRELIMINARIES AND NEW DEFINITIONS

A function

$$F(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k,\quad c\neq 0,-1,-2,...$$

is known as the Gaussian hypergeometric function, where $(\lambda)_n$ is a Pochhammer symbol defined by

$$(\lambda)_n=\lambda(\lambda+1)...(\lambda+n-1),\quad n=1,2,...;\quad (\lambda)_0\equiv 1.$$

Lauricella further generalized the four Appell series F_1,..., F_4 to series in n variables and defined his multiple hypergeometric series as follows [25, p. 114]: in the domain $\{\mathbf{x}:\ |x_1|+...+|x_n|<1\}$

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x})=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|}(b_1)_{k_1}...(b_n)_{k_n}}{(c_1)_{k_1}...(c_n)_{k_n}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!};\tag{1}$$

in the domain $\{\mathbf{x}:\ |x_1|<1,\ ...,\ |x_n|<1\}$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x})=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a_1)_{k_1}...(a_n)_{k_n}(b_1)_{k_1}...(b_n)_{k_n}}{(c)_{|\mathbf{k}|}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!},\tag{2}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|}(b_1)_{k_1}\cdots(b_n)_{k_n}}{(c)_{|\mathbf{k}|}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!};\tag{3}$$

in the domain $\left\{\mathbf{x}:\ \sqrt{|x_1|}<1,...,\sqrt{|x_n|}<1\right\}$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|}(b)_{|\mathbf{k}|}}{(c_1)_{k_1}\cdots(c_n)_{k_n}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!}.\tag{4}$$

Here and further the following notation are used

$$\mathbf{a}:=\left(a_1,...,a_n\right),\quad \mathbf{b}:=\left(b_1,...,b_n\right),\quad \mathbf{c}:=\left(c_1,...,c_n\right),\quad \mathbf{x}:=\left(x_1,...,x_n\right);\tag{5}$$

$$\mathbf{k}:=\left(k_1,...,k_n\right);\quad |\mathbf{k}|:=k_1+...+k_n,\quad k_1\geq 0,...,k_n\geq 0.\tag{6}$$

In all definitions (1)–(4), as usual, the denominator parameters c, c_1, ..., c_n are neither zero nor a negative integer. Lauricella [13] (see, also [25]) gave several elementary properties of these series including, for instance, integral representations of the Eulerian type, transformations and reducible cases, and the systems of partial differential equations associated with them. Currently, multiple hypergeometric Lauricella's functions are used to find explicit solutions to many applied problems [26–28].

Among the important confluent forms of Lauricella series are the confluent series $\Phi_2^{(n)}$ and $\Psi_2^{(n)}$ in n variables, defined by [29, p. 446, Eq. (7.2)]

$$\Phi_2^{(n)}\left[\mathbf{b};c;\mathbf{x}\right]=\sum_{|\mathbf{k}|=\mathbf{0}}^{\infty}\frac{(b_1)_{k_1}\cdots(b_n)_{k_n}}{(c)_{|\mathbf{k}|}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!}\tag{7}$$

and [30, p. 429] (see, also [25, p. 134, Eq. (34)])

$$\Psi_2^{(n)}\left[a;\mathbf{c};\mathbf{x}\right]=\sum_{|\mathbf{k}|=\mathbf{0}}^{\infty}\frac{(a)_{|\mathbf{k}|}}{(c_1)_{k_1}\cdots(c_n)_{k_n}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!},\tag{8}$$

so that

$$\begin{aligned}\Phi_2^{(n)}\left[\mathbf{b};c;\quad \mathbf{x}\right]&=\lim_{|a|\to\infty}F_D^{(n)}\left[a,\mathbf{b};c;\quad \frac{x_1}{a},...,\frac{x_n}{a}\right]\\&=\lim_{\min\{|a_1|,...,|a_n|\}\to\infty}F_B^{(n)}\left[\mathbf{a},\mathbf{b};c;\quad \frac{x_1}{a_1},...,\frac{x_n}{a_n}\right]\end{aligned}\tag{9}$$

and

$$\Psi_2^{(n)}\left[a;\mathbf{c};\mathbf{x}\right]=\lim_{|b|\to\infty}F_C^{(n)}\left[a,b;\ \mathbf{c};\quad \frac{x_1}{b},...,\frac{x_n}{b}\right]=\lim_{\min\{|b_1|,...,|b_n|\}\to\infty}F_A^{(n)}\left[a,\mathbf{b},\mathbf{c};\quad \frac{x_1}{b_1},...,\frac{x_n}{b_n}\right].$$

Some other confluent forms of Lauricella series have appeared in the literature. These include the confluent series $\Phi_D^{(n)}$ introduced by Srivastava and Exton [31], and the confluent series $\Xi_1^{(n)}$ and $\Phi_3^{(n)}$ used by Exton [32]. By definition, we have [31, p. 373, Eq. (12)]

$$\Psi_D^{(n)}\left[a,b_1,...,b_{n-1};c;\mathbf{x}\right]=\sum_{|\mathbf{k}|=\mathbf{0}}^{\infty}\frac{(a)_{|\mathbf{k}|}(b_1)_{k_1}\cdots(b_{n-1})_{k_{n-1}}}{(c)_{|\mathbf{k}|}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!},\tag{10}$$

so that

$$\Phi_D^{(n)}\left[a,b_1,...,b_{n-1};c;\quad \mathbf{x}\right]=\lim_{|b_n|\to\infty}F_D^{(n)}\left[a,b_1,...,b_n;c;\quad x_1,...,x_{n-1},\frac{x_n}{b_n}\right];$$

and [32, p. 43, Eqs. (2.1.1.4) and (2.1.1.5)]

$$\Xi_1^{(n)}\left[a_1,...,a_n,b_1,...,b_{n-1};c;\mathbf{x}\right]=\sum_{|\mathbf{k}|=\mathbf{0}}^{\infty}\frac{(a_1)_{k_1}...(a_n)_{k_n}\,(b_1)_{k_1}\cdots(b_{n-1})_{k_{n-1}}}{(c)_{|\mathbf{k}|}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!}$$

$$=\lim_{|b_n|\to\infty} F_B^{(n)}\left[a_1,...,a_n,b_1,...,b_n;c;\quad x_1,...,x_{n-1},\frac{x_n}{b_n}\right], \tag{11}$$

$$\Phi_3^{(n)}\left[b_1,...,b_{n-1};c;\mathbf{x}\right]=\sum_{|\mathbf{k}|=\mathbf{0}}^{\infty}\frac{(b_1)_{k_1}...(b_{n-1})_{k_{n-1}}}{(c)_{|\mathbf{k}|}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!}$$

$$=\lim_{|b_n|\to\infty}\Phi_2^{(n)}\left[b_1,...,b_n;c;\quad x_1,...,x_{n-1},\frac{x_n}{b_n}\right]. \tag{12}$$

Making use of the relationship (9), it is easy to observe that $\Phi_3^{(n)}$ is a confluent form of Lauricella series $F_D^{(n)}$ as well as $F_B^{(n)}$.

Introduce a following confluent hypergeometric functions

$$\Psi_{1,r}^{(n)}\left[a,b_1,...,b_r;\mathbf{c};\mathbf{x}\right]=\sum_{|\mathbf{k}|=\mathbf{0}}^{\infty}\frac{(a)_{|\mathbf{k}|}(b_1)_{k_1}...(b_r)_{k_r}}{(c_1)_{k_1}...(c_n)_{k_n}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!}, \tag{13}$$

$$\Psi_{1,r}^{(n)}\left[a,b_1,...,b_r;\mathbf{c};\mathbf{x}\right]=\lim_{\min\{|b_{r+1}|,...,|b_n|\}\to\infty}F_A^{(n)}\left[a,\mathbf{b},\mathbf{c};\quad\frac{x_{r+1}}{b_{r+1}},...,\frac{x_n}{b_n}\right],$$

$$\Xi_{1,r}^{(n)}\left[\mathbf{a},b_1,...,b_r;c;\mathbf{x}\right]=\sum_{|\mathbf{k}|=\mathbf{0}}^{\infty}\frac{(a_1)_{k_1}...(a_n)_{k_n}(b_1)_{k_1}...(b_r)_{k_r}}{(c)_{|\mathbf{k}|}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!}$$

$$=\lim_{\min\{|b_{r+1}|,...,|b_n|\}\to\infty}F_B^{(n)}\left[\mathbf{a},\mathbf{b};c;\quad x_1,...,x_r,\frac{x_{r+1}}{b_{r+1}},...,\frac{x_n}{b_n}\right], \tag{14}$$

$$\Xi_{2,r}^{(n)}\left[a_1,...,a_r,b_1,...,b_r;c;\mathbf{x}\right]=\sum_{|\mathbf{k}|=\mathbf{0}}^{\infty}\frac{(a_1)_{k_1}...(a_r)_{k_r}(b_1)_{k_1}...(b_r)_{k_r}}{(c)_{|\mathbf{k}|}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!}$$

$$=\lim_{\min\{|a_{r+1}|,...,|a_n|\}\to\infty}\Xi_{1,r}^{(n)}\left[\mathbf{a},b_1,...,b_r;c;\quad x_1,...,x_r,\frac{x_{r+1}}{a_{r+1}},...,\frac{x_n}{a_n}\right], \tag{15}$$

where $1\le r\le n-1$.

An interesting unification (and generalization) of Lauricella's multiple series $F_A^{(n)}$ and $F_B^{(n)}$ and Horn's double series H_2 was considered by Erdelyi [33], who defined his general series in the form [33, p. 13, Eq. (28)]

$$H_{n+p,n}\left(a,\mathbf{b},\mathbf{d},\mathbf{g};\mathbf{c};\mathbf{x};\mathbf{y}\right)=\sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|-|\mathbf{l}|}(b_1)_{k_1}...(b_n)_{k_n}}{(c_1)_{k_1}...(c_n)_{k_n}}$$

$$\times(d_1)_{l_1}...(d_p)_{l_p}(g_1)_{l_1}...(g_p)_{l_p}\frac{x_1^{k_1}}{k_1!}...\frac{x_n^{k_n}}{k_n!}\frac{y_1^{l_1}}{l_1!}...\frac{y_p^{l_p}}{l_p!}, \tag{16}$$

where n and p are integers, $0\le p\le n$. Here and further, in addition to the notations (5) and (6), the following notation are used

$$\mathbf{d}:=(d_1,...,d_p),\quad \mathbf{g}:=(g_1,...,g_p),\quad \mathbf{y}:=(y_1,...,y_p);$$

$$\mathbf{l}:=(l_1,...,l_p);\quad |\mathbf{l}|:=l_1+...+l_p,\quad l_1\ge 0,...,l_p\ge 0.$$

Evidently, we have

$$H_{n,n}=F_A^{(n)},\quad H_{n,0}=F_B^{(n)}.$$

From the hypergeometric series (16) one can determine the following confluent hypergeometric function in $n+p$ variables

$$\mathrm{H}_A^{(n,p)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x};\mathbf{y}\right)=\lim_{\varepsilon\to 0}H_{n+p,n}\left(a,\mathbf{b},\underbrace{\frac{1}{\varepsilon},...,\frac{1}{\varepsilon}}_{2p};\mathbf{c};\mathbf{x},\varepsilon^2\mathbf{y}\right).$$

When defining the confluent hypergeometric function $\mathrm{H}_A^{(n,p)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x};\mathbf{y}\right)$, we used the equality (see, for instance, [10, p. 124]) $\lim\limits_{\varepsilon\to 0}\left[(1/\varepsilon)_q\,\varepsilon^q\right]=1$, where q is a nonnegative integer. So, the confluent hypergeometric function $\mathrm{H}_A^{(n,p)}$ has a form

$$\mathrm{H}_A^{(n,p)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x};\mathbf{y}\right)=\sum_{|\mathbf{k}|,|\mathbf{l}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|-|\mathbf{l}|}(b_1)_{k_1}...(b_n)_{k_n}}{(c_1)_{k_1}...(c_n)_{k_n}}\frac{x_1^{k_1}}{k_1!}...\frac{x_n^{k_n}}{k_n!}\frac{y_1^{l_1}}{l_1!}...\frac{y_p^{l_p}}{l_p!}. \tag{17}$$

Note that well-known Humbert functions Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1 and Ξ_2 are two dimensional analogous of the multiple hypergeometric functions defined in (7), (8), (10)–(15), and some particular cases of the function $\mathrm{H}_A^{(n,p)}$ are found in [34–36].

3. CONVERGENCE OF THE MULTIPLE CONFLUENT SERIES

Horn [12] gave the general definition of the double hypergeometric series, following which it is not difficult to define the hypergeometric series with the two and more variables. A power series

$$u\left(\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}A\left(\mathbf{k}\right)x_1^{k_1}...x_n^{k_n} \tag{18}$$

is said to be hypergeometric if the ratio of any two adjacent coefficients is a rational function of the components of the summation index $\mathbf{k}$, that is, for all $j=1,...,n$

$$\frac{A\left(\mathbf{k}+\mathbf{e}_j\right)}{A\left(\mathbf{k}\right)}=f_j\left(\mathbf{k}\right),$$

where $\mathbf{e}_j:=(0,...,1,...,0)$ denote the vectors with jth component equal to 1 and the others equal to 0.

The positive quantities r_1, ..., r_n are called the associated radii of convergence of the multiple power series (18), if the power series is absolutely convergent for $|x_1|<r_1$, ..., $|x_n|<r_n$, and divergent when $|x_1|>r_1$, ..., $|x_n|>r_n$. We put $\max r_1=R_1$,..., $\max r_n=R_n$. Investigating the convergence of (18), we define

$$\Upsilon_j\left(\mu_1,...,\mu_n\right)=\lim v_j\left(\mu_1 t,...,\mu_n t\right),\quad t\to\infty,\quad j=\overline{1,n}.$$

It is easy to show that

$$R_j=\left|\Upsilon_j\left(\mathbf{e}_j\right)\right|^{-1},\quad j=\overline{1,n}.$$

The domains of convergence of the Lauricella's series are represented in (1)–(4). In the case of multiple confluent series one or more, or perhaps all of Υ_j vanishes identically. The region of convergence simplifies considerably and all inequalities, which may be necessary to secure convergence, are recorded as follows

$$\Phi_D^{(n)},\ \Xi_1^{(n)}:\left\{(x_1,...,x_n):\ |x_1|<1,...,|x_{n-1}|<1,\ |x_n|<\infty\right\};$$

$$\Phi_2^{(n)},\ \Phi_3^{(n)},\ \Psi_2^{(n)}:\left\{(x_1,...,x_n):\ |x_1|<\infty,...,|x_n|<\infty\right\};$$

$$\Xi_{1,r}^{(n)},\ \Xi_{2,r}^{(n)}:\left\{(x_1,...,x_n):\ |x_1|<1,...,|x_r|<1,\ |x_{r+1}|<\infty,...,|x_n|<\infty\right\};$$

$$\Psi_{1,r}^{(n)}:\left\{(x_1,...,x_n):\ |x_1|+...+|x_r|<1,\ |x_{r+1}|<\infty,...,|x_n|<\infty\right\};$$

$$\mathrm{H}_A^{(n,p)}:\left\{(x_1,...,x_n,\ y_1,...,y_p):\ |x_1|+...+|x_n|<1,\ |y_1|<\infty,...,|y_p|<\infty\right\}.$$

4. SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

Horn [12] has established the systems of partial differential equations which the double confluent Humbert functions satisfy (see, also [37, p. 235]). The work [23] contains systems of PDE corresponding to 38 triple confluent hypergeometric functions and, in the work [36] a three-dimensional analogue of the function (17) and the corresponding system are used to search for fundamental solutions of the two-dimensional Helmholtz equation with a singular coefficient.

The power series (18), where

$$\frac{A(\mathbf{k}+\mathbf{e}_j)}{A(\mathbf{k})}=\frac{P_j(\mathbf{k})}{Q_j(\mathbf{k})} \tag{19}$$

and $P_j(\mathbf{x})$, $Q_j(\mathbf{x})$ are some polynomials in the n variables $(x_1,...,x_n)=:\mathbf{x}$, satisfies a system of linear partial differential equations, which can be written in terms of the differential operators

$$\theta_j:=x_j\frac{\partial}{\partial x_j},\quad j=\overline{1,n}$$

as

$$Q_j(\theta)\left(x_j^{-1}u(\mathbf{x})\right)=P_j(\theta)u(\mathbf{x}),\quad j=\overline{1,n}, \tag{20}$$

where the differential operators $P_j(\theta)$ and $Q_j(\theta)$ are obtained by substituting the components of the vector $\theta:=(\theta_1,...,\theta_n)$, as the arguments of the polynomials P_j and Q_j in (19).

It is easy to see that the coefficients of the confluent series $\Phi_2^{(n)}$ defined in (7), which are given by

$$A(\mathbf{k})=\frac{(b_1)_{k_1}...(b_n)_{k_n}}{(c)_{|\mathbf{k}|}k_1!...k_n!},$$

satisfy the relations (19) for

$$P_j(\mathbf{k})=b_j+k_j,\quad Q_j(\mathbf{k})=(c+|\mathbf{k}|)(1+k_j).$$

So, confluent series $\Phi_2^{(n)}$ belongs to the family of Horn hypergeometric series. The system (20) corresponding to P_j and Q_j has the form

$$\left(c+\sum_{m=1}^{n}\theta_m\right)(1+\theta_j)\left(x_j^{-1}\Phi_2^{(n)}(\mathbf{x})\right)=(b_j+\theta_j)\,\Phi_2^{(n)}(\mathbf{x}),\quad j=\overline{1,n}.$$

Setting $\theta_s=x_s\dfrac{\partial}{\partial x_s}$, removing parentheses, and bearing in mind that

$$\left(1+x_j\frac{\partial}{\partial x_j}\right)\left(x_j^{-1}\Phi_2^{(n)}(\mathbf{x})\right)=\frac{\partial}{\partial x_j}\Phi_2^{(n)}(\mathbf{x}),$$

we arrive at

$$\sum_{j=1}^{n}x_j\frac{\partial^2\Phi_2^{(n)}}{\partial x_k\partial x_j}+(c-x_k)\frac{\partial\Phi_2^{(n)}}{\partial x_k}-b_k\Phi_2^{(n)}=0,\quad k=\overline{1,n},$$

which, therefore, is a system in the class of Horn hypergeometric systems.

We note that the property (19) holds also for the other confluent series $\Psi_2^{(n)}$, $\Phi_D^{(n)}$, $\Xi_1^{(n)}$, $\Phi_3^{(n)}$, $\Psi_{1,r}^{(n)}$, $\Xi_{1,r}^{(n)}$, $\Xi_{2,r}^{(n)}$, $\mathrm{H}_A^{(n,p)}$. Similarly, these confluent hypergeometric functions satisfy the following systems of partial differential equations

$$x_k\frac{\partial^2\Psi_2^{(n)}}{\partial x_k^2}+(c_k-x_k)\frac{\partial\Psi_2^{(n)}}{\partial x_k}-\sum_{\substack{j=1\\ j\neq k}}^{n}x_j\frac{\partial\Psi_2^{(n)}}{\partial x_j}-a\Psi_2^{(n)}=0,\quad k=\overline{1,n}; \tag{21}$$

$$\sum_{j=1}^{n} x_j (1-x_k) \frac{\partial^2 \Phi_D^{(n)}}{\partial x_k \partial x_j} + [c - (a + b_k + 1) x_k] \frac{\partial \Phi_D^{(n)}}{\partial x_k}$$

$$- b_k \sum_{\substack{j=1 \\ j\neq k}}^{n} x_j \frac{\partial \Phi_D^{(n)}}{\partial x_j} - ab_k \Phi_D^{(n)} = 0, \quad k = \overline{1, n-1},$$

$$\sum_{j=1}^{n} x_j \frac{\partial^2 \Phi_D^{(n)}}{\partial x_j \partial x_n} - \sum_{j=1}^{n-1} x_j \frac{\partial \Phi_D^{(n)}}{\partial x_j} + (c - x_n) \frac{\partial \Phi_D^{(n)}}{\partial x_n} - a\Phi_D^{(n)} = 0;$$

$$x_k (1-x_k) \frac{\partial^2 \Xi_1^{(n)}}{\partial {x_k}^2} + \sum_{\substack{j=1 \\ j\neq k}}^{n} x_j \frac{\partial^2 \Xi_1^{(n)}}{\partial x_k \partial x_j} + [c - (a_k + b_k + 1) x_k] \frac{\partial u}{\partial x_k} - a_k b_k \Xi_1^{(n)} = 0, \quad k = \overline{1, n-1},$$

$$\sum_{j=1}^{n} x_j \frac{\partial^2 \Xi_1^{(n)}}{\partial x_j \partial x_n} + (c - x_n) \frac{\partial \Xi_1^{(n)}}{\partial x_n} - a_n \Xi_1^{(n)} = 0;$$

$$\sum_{j=1}^{n} x_j \frac{\partial^2 \Phi_3^{(n)}}{\partial x_k \partial x_j} + (c - x_k) \frac{\partial \Phi_3^{(n)}}{\partial x_k} - a_k \Phi_3^{(n)} = 0, \quad k = \overline{1, n-1},$$

$$\sum_{j=1}^{n} x_j \frac{\partial^2 \Phi_3^{(n)}}{\partial x_j \partial x_n} + c \frac{\partial \Phi_3^{(n)}}{\partial x_n} - \Phi_3^{(n)} = 0;$$

$$x_k (1-x_k) \frac{\partial^2 \Psi_{1,r}^{(n)}}{\partial {x_k}^2} - x_k \sum_{\substack{j=1 \\ j\neq k}}^{n} x_j \frac{\partial^2 \Psi_{1,r}^{(n)}}{\partial x_k \partial x_j} + [c_k - (a + b_k + 1) x_k] \frac{\partial \Psi_{1,r}^{(n)}}{\partial x_k}$$

$$- b_k \sum_{\substack{j=1 \\ j\neq k}}^{n} x_j \frac{\partial \Psi_{1,r}^{(n)}}{\partial x_j} - ab_k \Psi_{1,r}^{(n)} = 0, \quad k = \overline{1, r},$$

$$x_n \frac{\partial^2 \Psi_{1,r}^{(n)}}{\partial x_n^2} - \sum_{j=1}^{n-1} x_j \frac{\partial \Psi_{1,r}^{(n)}}{\partial x_j} + (c_n - x_n) \frac{\partial \Psi_{1,r}^{(n)}}{\partial x_n} - a\Psi_{1,r}^{(n)} = 0, \quad k = \overline{r+1, n};$$

$$x_k (1-x_k) \frac{\partial^2 \Xi_{1,r}^{(n)}}{\partial x_k^2} + \sum_{\substack{j=1 \\ j\neq k}}^{n} x_j \frac{\partial^2 \Xi_{1,r}^{(n)}}{\partial x_k \partial x_j} + [c - (a_k + b_k + 1) x_k] \frac{\partial \Xi_{1,r}^{(n)}}{\partial x_k} - a_k b_k \Xi_{1,r}^{(n)} = 0, \quad k = \overline{1, r},$$

$$\sum_{j=1}^{n} x_j \frac{\partial^2 \Xi_{1,r}^{(n)}}{\partial x_k \partial x_j} + (c - x_k) \frac{\partial \Xi_{1,r}^{(n)}}{\partial x_k} - a_k \Xi_{1,r}^{(n)} = 0, \quad k = \overline{r+1, n};$$

$$x_k (1-x_k) \frac{\partial^2 \Xi_{2,r}^{(n)}}{\partial x_k^2} + \sum_{\substack{j=1 \\ j\neq k}}^{n} x_j \frac{\partial^2 \Xi_{2,r}^{(n)}}{\partial x_k \partial x_j} + [c - (a_k + b_k + 1) x_k] \frac{\partial \Xi_{2,r}^{(n)}}{\partial x_k} - a_k b_k \Xi_{2,r}^{(n)} = 0, \quad k = \overline{1, r},$$

$$\sum_{j=1}^{n} x_j \frac{\partial^2 \Xi_{2,r}^{(n)}}{\partial x_k \partial x_j} + c \frac{\partial \Xi_{2,r}^{(n)}}{\partial x_k} - \Xi_{2,r}^{(n)} = 0, \quad k = \overline{r+1, n};$$

$$x_k (1-x_k) \frac{\partial^2 \mathrm{H}_A^{(n,p)}}{\partial x_k^2} - x_k \sum_{\substack{j=1 \\ j\neq k}}^{n} x_j \frac{\partial^2 \mathrm{H}_A^{(n,p)}}{\partial x_k \partial x_j} + x_k \sum_{j=1}^{p} y_j \frac{\partial^2 \mathrm{H}_A^{(n,p)}}{\partial x_k \partial y_j} - b_k \sum_{\substack{j=1 \\ j\neq k}}^{n} x_j \frac{\partial \mathrm{H}_A^{(n,p)}}{\partial x_j}$$

$$+\left[c_k-(a+b_k+1)\,x_k\right]\frac{\partial \mathrm{H}_A^{(n,p)}}{\partial x_k}+b_k\sum_{j=1}^{p}y_j\frac{\partial \mathrm{H}_A^{(n,p)}}{\partial y_j}-ab_k\mathrm{H}_A^{(n,p)}=0,\quad k=\overline{1,n},$$

$$\sum_{l=1}^{p}y_l\frac{\partial^2 \mathrm{H}_A^{(n,p)}}{\partial y_l\partial y_j}-\sum_{l=1}^{n}x_l\frac{\partial^2 \mathrm{H}_A^{(n,p)}}{\partial x_l\partial y_j}+(1-a)\frac{\partial \mathrm{H}_A^{(n,p)}}{\partial y_j}+\mathrm{H}_A^{(n,p)}=0,\quad j=\overline{1,p}. \tag{22}$$

Note that two-dimensional analogues of these systems are found in the monograph [37, p. 235, Eqs. (23)–(29)].

5. LINEARLY INDEPENDENT SOLUTIONS

In order to find the linearly independent solutions, for instance, of system (21), we will search the solutions in the form

$$u=\prod_{j=1}^{n}x_j^{\lambda_j}w\left(x_1,...,x_n\right), \tag{23}$$

where w is unknown function and λ_j $(j=\overline{1,n})$ are constants, which are to be determined. So, substituting (23) into the system (21), we get

$$x_k\frac{\partial^2 w}{\partial x_k^2}+\left(c_k+2\lambda_k-x_k\right)\frac{\partial w}{\partial x_k}-\sum_{\substack{j=1\\ j\neq k}}^{n}x_j\frac{\partial w}{\partial x_j}-\left(a+\lambda_k\right)w+\frac{\lambda_k\left(\lambda_k-1+c_k\right)}{x_k}w=0,$$
$$k=\overline{1,n}. \tag{24}$$

We note that system in (24) is analogical to system (21), therefore, we require that the conditions

$$\lambda_k\left(\lambda_k-1+c_k\right)=0,\quad k=\overline{1,n} \tag{25}$$

should be satisfied. It is evident that the system (25) has the following solutions

$$\begin{array}{ccccccc} & \lambda_1 & \lambda_2 & \lambda_3 & ... & \lambda_{n-1} & \lambda_n \\ 1: & \{0 & 0 & 0 & ... & 0 & 0 \end{array}$$

$$C_n^1:\left\{\begin{array}{cccccc} 1-c_1 & 0 & 0\ ... & 0 & 0 \\ 0 & 1-c_2 & 0\ ... & 0 & 0 \\ ... & ... & ...\ ... & ... & ... \\ 0 & 0 & 0\ ... & 1-c_n & 0 \end{array}\right.$$

$$C_n^2:\left\{\begin{array}{cccccc} 1-c_1 & 1-c_2 & 0 & ... & 0 & 0 \\ 0 & 1-c_2 & 1-c_3 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & \\ 0 & 0 & 0 & ... & 1-c_{n-1} & 1-c_n \end{array}\right.$$

$$...$$

$$1:\ \{1-c_1\ \ 1-c_2\ \ 1-c_3\ \ ...\ \ 1-c_{n-1}\ \ 1-c_n,$$

where $C_n^k = \dfrac{n!}{k!(n-k)!}$ are binomial coefficients. Thus, the system (25) has the number

$$1 + C_n^1 + C_n^2 + C_n^3 + ... + C_n^{n-2} + C_n^{n-1} + 1 = 2^n$$

of solutions. Finally, substituting all values of λ_1,..., λ_n into (24), we find the following 2^n linearly independent solutions of the system (21) at the origin:

$$1 : \{ \Psi_2^{(n)} [a; c_1, ..., c_n; \mathbf{x}] ,$$

$$C_n^1 : \begin{cases} x_1^{1-c_1} \Psi_2^{(n)} [a+1-c_1; 2-c_1, c_2, ..., c_n; \mathbf{x}], \\ ... \\ x_n^{1-c_n} \Psi_2^{(n)} [a+1-c_n; c_1, ..., c_{n-1}, 2-c_n; \mathbf{x}], \end{cases}$$

$$C_n^2 : \begin{cases} x_1^{1-c_1} x_2^{1-c_2} \Psi_2^{(n)} [a+2-c_1-c_2; 2-c_1, 2-c_2, c_3, ..., c_n; \mathbf{x}] , \\ ... \\ x_1^{1-c_1} x_n^{1-c_n} \Psi_2^{(n)} [a+2-c_1-c_n; 2-c_1, c_2, ..., c_{n-1}, 2-c_n; \mathbf{x}] , \\ x_2^{1-c_2} x_3^{1-c_3} \Psi_2^{(n)} [a+2-c_2-c_3; c_1, 2-c_2, 2-c_3, c_4, ..., c_n; \mathbf{x}] , \\ ... \\ x_{n-1}^{1-c_{n-1}} x_n^{1-c_n} \Psi_2^{(n)} [a+2-c_{n-1}-c_n; c_1, ..., c_{n-2}, 2-c_{n-1}, 2-c_n; \mathbf{x}] , \end{cases}$$

$$...$$

$$1 : \left\{ x_1^{1-c_1} ... x_n^{1-c_n} \Psi_2^{(n)} [a+1-c_1-...-c_n; 2-c_1, ..., 2-c_n; \mathbf{x}] \right. .$$

When none of the numbers c_1, c_2, ..., c_n is equal to a negative integer, we obtain the general solution of the system (21) by multiplying these 2^n partial solutions to arbitrary constants and taking their sum.

It is easy to see that in the first group there is one solution $\left(C_n^0 = 1\right)$, in the second group there are $C_n^1 = n$ solutions, the third group consists of $C_n^2 = n(n-1)/2$ solutions, etc. So the system of hypergeometric equations (21) really has 2^n solutions.

However, within each group, the functions included in this group are symmetrical with respect to the numerical parameters. Therefore, for further purposes, it is enough to select one solution from each group, or more precisely, the solution that comes first in each group. So $n+1$ linearly independent solutions to the system of equations (21) will be identified

$$\omega_0(\mathbf{x}) = C_0 \, \Psi_2^{(n)} [a; c_1, ..., c_n; \mathbf{x}] , \tag{26}$$

$$\omega_1(\mathbf{x}) = C_1 \, x_1^{1-c_1} \Psi_2^{(n)} [a+1-c_1; 2-c_1, c_2, ..., c_n; \mathbf{x}] , \tag{27}$$

$$\omega_2(\mathbf{x}) = C_2 \, x_1^{1-c_1} x_2^{1-c_2} \Psi_2^{(n)} [a+2-c_1-c_2; 2-c_1, 2-c_2, c_3, ..., c_n; \mathbf{x}] , \tag{28}$$

$$...$$

$$\omega_n(\mathbf{x}) = C_n \, x_1^{1-c_1} ... x_n^{1-c_n} \Psi_2^{(n)} [a+1-c_1-...-c_n; 2-c_1, ..., 2-c_n; \mathbf{x}] , \tag{29}$$

x which are mutually asymmetric with respect to the numerical parameters, where $C_j \, (j = \overline{0, n})$ are arbitrary constants.

Similarly, the system (22) has $n+1$ linearly independent solutions at the origin

$$\omega_0(\mathbf{x}; \mathbf{y}) = C_0 \, \mathrm{H}_A^{(n,p)} \left[\begin{matrix} a, b_1, ..., b_n; \\ c_1, ..., c_n; \end{matrix} \mathbf{x}; \mathbf{y} \right] , \tag{30}$$

$$\omega_1(\mathbf{x};\mathbf{y}) = C_1\, x_1^{1-c_1} \mathrm{H}_A^{(n,p)} \left[\begin{matrix} a+1-c_1, b_1+1-c_1, b_2, ..., b_n; \\ 2-c_1, c_2, ..., c_n; \end{matrix} \mathbf{x};\mathbf{y} \right], \tag{31}$$

$$\omega_2(\mathbf{x};\mathbf{y}) = C_2\, x_1^{1-c_1} x_2^{1-c_2} \mathrm{H}_A^{(n,p)} \left[\begin{matrix} a+2-c_1-c_2, b_1+1-c_1, b_2+1-c_2, b_3, ..., b_n; \\ 2-c_1, 2-c_2, c_3, ..., c_n; \end{matrix} \mathbf{x};\mathbf{y} \right], \tag{32}$$

$$...$$

$$\omega_n(\mathbf{x};\mathbf{y}) = C_n\, x_1^{1-c_1} ... x_n^{1-c_n} \mathrm{H}_A^{(n,p)} \left[\begin{matrix} a+n-c_1-...-c_n, b_1+1-c_1, ..., b_n+1-c_n; \\ 2-c_1, ..., 2-c_n; \end{matrix} \mathbf{x};\mathbf{y} \right], \tag{33}$$

which are mutually asymmetric with respect to the numerical parameters, where C_j $(j=\overline{0,n})$ are arbitrary constants.

6. SELF-SIMILAR SOLUTIONS OF THE SECOND ORDER PARTIAL DIFFERENTIAL EQUATION WITH n LINES OF DEGENERACY

In the domain $\Omega = \left\{x_1, ..., x_n, t) : x_1 > 0, ..., x_n > 0, t > 0\right\}$, we consider the equation

$$u_t - \sum_{k=1}^{n} \left(\frac{\partial^2 u}{\partial x_k^2} + \frac{2\alpha_k}{x_k} \frac{\partial u}{\partial x_k} \right) = 0, \quad \alpha_k = \text{const}. \tag{34}$$

The solution to equation (34) is sought in the form

$$u(\mathbf{x}, t) = t^{-1/2} \omega\left(\xi\right), \tag{35}$$

where

$$\mathbf{x} := (x_1, ..., x_n); \quad \xi := (\xi_1, ..., \xi_n), \quad \xi_k := -\frac{x_k^2}{4nt}, \quad k = \overline{1,n}, \tag{36}$$

and $\omega(\xi)$ is an unknown function. Substituting (35) into (34), we have second order partial differential equation

$$\sum_{k=1}^{n} \left[\xi_k \frac{\partial^2 \omega}{\partial \xi_k^2} + \left(\alpha_k + \frac{1}{2} - n\xi_k \right) \frac{\partial \omega}{\partial x_k} \right] - \frac{n}{2} \omega = 0,$$

which is equivalent to the following system of PDE of the hypergeometric type

$$\xi_k \frac{\partial^2 \omega}{\partial \xi_k^2} + (\alpha_k - \xi_k) \frac{\partial \omega}{\partial \xi_k} - \sum_{\substack{j=1 \\ j \neq k}}^{n} \xi_j \frac{\partial \omega}{\partial \xi_j} - \frac{1}{2} \omega = 0, \quad k = \overline{1,n}. \tag{37}$$

Comparing now the system (37) with the system (21) and taking into account an expressions (26)–(29) of the solutions of the system (21), we get a linearly independent self-similar solutions of the degenerate equation (34) at the origin:

$$u_0(\mathbf{x}, t) = \mu_0 \frac{1}{t^{1/2}} \Psi_2^{(n)} \left[\frac{1}{2}; \frac{1+2\alpha_1}{2}, \frac{1+2\alpha_2}{2}, \frac{1+2\alpha_3}{2}, \frac{1+2\alpha_4}{2}, ..., \frac{1+2\alpha_n}{2}; \xi \right],$$

$$u_1(\mathbf{x}, t) = \mu_1 \frac{x_1^{1-2\alpha_1}}{t^{1-\alpha_1}} \Psi_2^{(n)} \left[1-\alpha_1; \frac{3-2\alpha_1}{2}, \frac{1+2\alpha_2}{2}, \frac{1+2\alpha_3}{2}, \frac{1+2\alpha_4}{2}, ..., \frac{1+2\alpha_n}{2}; \xi \right],$$

$$u_2(\mathbf{x}, t) = \mu_2 \frac{x_1^{1-2\alpha_1} x_2^{1-2\alpha_2}}{t^{3/2-\alpha_1-\alpha_2}} \Psi_2^{(n)} \left[\frac{3-2\alpha_1-2\alpha_2}{2}; \frac{3-2\alpha_1}{2}, \frac{3-2\alpha_2}{2}, \frac{1+2\alpha_3}{2}, ..., \frac{1+2\alpha_n}{2}; \xi \right],$$

...

$$u_n(\mathbf{x},t) = \mu_n t^{\alpha-\frac{n+1}{2}} \prod_{j=1}^{n} x_j^{(1-2\alpha_j)/2} \Psi_2^{(n)} \left[\frac{n+1-2\alpha}{2}; \frac{3-2\alpha_1}{2}, ..., \frac{3-2\alpha_n}{2}; \xi \right],$$

where μ_j $(j=\overline{0,n})$ are arbitrary constants, and $\mathbf{x}$ and ξ are defined in (36); $\alpha := \alpha_1 + ... + \alpha_n$.

Note that in the case $n = 2$ self-similar solutions of equation (34) at the origin are found in [38].

7. FUNDAMENTAL SOLUTIONS OF MULTIDIMENSIONAL HELMHOLTZ EQUATION WITH n SINGULAR COEFFICIENTS

Let us consider the generalized singular Helmholtz equation with several parameters

$$\sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} + (\lambda_1 + ... + \lambda_p)\, u = 0 \tag{38}$$

in the domain $R_m^{n+} := \{(x_1, ..., x_m) : x_1 > 0, ..., x_n > 0\}$, where m is a dimension of Euclidean space R_n, n is a number of the singular coefficients of equation; $m \geq 2, 0 \leq n \leq m$; $\alpha := (\alpha_1, ..., \alpha_n)$ and $\lambda := (\lambda_1, ..., \lambda_p)$ are real numbers, $0 < 2\alpha_j < 1$, $j = 1, ..., n$.

The solution to equation (38) is sought in the form

$$u(x;\xi;\lambda) = P(r) w(\sigma, \eta), \tag{39}$$

where

$$P(r) = r^{-2\beta_0}, \quad \beta_0 = \frac{m-2}{2} + \alpha_1 + ... + \alpha_n;$$

$$r = |x-\xi| = \sqrt{\sum_{i=1}^{m} (x_i - \xi_i)^2}, \quad x := (x_1, ..., x_m) \in R_m^{n+}, \quad \xi := (\xi_1, ..., \xi_m) \in R_m^{n+}.$$

Here $\omega(\sigma, \eta)$ is an unknown function and depends on the variables

$$\sigma := (\sigma_1, ..., \sigma_n), \quad \sigma_k = -\frac{4x_k\xi_k}{r^2}; \quad \eta := (\eta_1, ..., \eta_p), \quad \eta_l = \frac{1}{4}\lambda_l r^2.$$

Substituting (39) into (38), we have a system of second order partial differential equation

$$\begin{aligned} &\sigma_i (1-\sigma_i) \frac{\partial^2 \omega}{\partial \sigma_i^2} - \sigma_i \sum_{j=1, j\neq i}^{n} \sigma_j \frac{\partial^2 \omega}{\partial \sigma_i \partial \sigma_j} + \sigma_i \sum_{j=1}^{p} \eta_j \frac{\partial^2 \omega}{\partial \sigma_k \partial \eta_j} + [2\alpha_i - (\alpha + \alpha_i + 1)\sigma_i] \frac{\partial \omega}{\partial \sigma_i} \\ &\qquad - \alpha_i \sum_{j=1, j\neq i}^{n} \sigma_j \frac{\partial \omega}{\partial \sigma_j} + \alpha_i \sum_{j=1}^{p} \eta_j \frac{\partial \omega}{\partial \eta_j} - \alpha\alpha_i \omega = 0, \quad i = \overline{1,n}, \\ &\sum_{l=1}^{p} \eta_l \frac{\partial^2 \omega}{\partial \eta_l \partial \eta_j} - \sum_{l=1}^{n} \sigma_l \frac{\partial^2 \omega}{\partial \sigma_l \partial \eta_j} + (1-\alpha) \frac{\partial \omega}{\partial \eta_j} + \omega = 0, \quad j = \overline{1,p}. \end{aligned} \tag{40}$$

Comparing now the system (40) with the system (22) and taking into account an expressions (30)–(33) of the solutions of the system (22), we get a linearly independent fundamental solutions of the degenerate equation (38) at the origin:

$$q_0(x;\xi;\lambda) = \gamma_0 r^{-2\beta_0} \mathrm{H}_A^{(n,p)} \left[\begin{matrix} \beta_0, \alpha_1, ..., \alpha_n; \\ 2\alpha_1, ..., 2\alpha_n; \end{matrix} \sigma; \eta \right],$$

$$q_k(x;\xi;\lambda) = \frac{\gamma_k}{r^{2\beta_k}} \prod_{i=1}^{k} (x_i\xi_i)^{1-2\alpha_i} \mathrm{H}_A^{(n,p)} \left[\begin{matrix} \beta_k, 1-\alpha_1, ..., 1-\alpha_k, \alpha_{k+1}, ..., \alpha_n; \\ 2-2\alpha_1, ..., 2-2\alpha_k, 2\alpha_{k+1}, ..., 2\alpha_n; \end{matrix} \sigma; \eta \right],$$

$$k = \overline{1, n-1},$$

$$q_n(x;\xi;\lambda) = \gamma_n r^{-2\beta_n} \prod_{i=1}^{n} (x_i \xi_i)^{1-2\alpha_i} \, \mathrm{H}_A^{(n,p)} \left[\begin{matrix} \beta_n, 1-\alpha_1, ..., 1-\alpha_n; \\ 2-2\alpha_1, ..., 2-2\alpha_n; \end{matrix} \sigma; \eta \right],$$

where

$$\beta_0 = \frac{m-2}{2} + \sum_{i=1}^{n} \alpha_i, \quad \gamma_0 = 2^{2\beta_0 - m} \frac{\Gamma(\beta_0)}{\pi^{m/2}} \prod_{i=1}^{n} \frac{\Gamma(\alpha_i)}{\Gamma(2\alpha_i)};$$

$$\beta_k = \frac{m-2}{2} + k - \sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \alpha_i, \quad k = \overline{1,\, n-1},$$

$$\gamma_k = 2^{2\beta_k - m} \frac{\Gamma(\beta\alpha_k)}{\pi^{m/2}} \prod_{i=1}^{k} \frac{\Gamma(1-\alpha_i)}{\Gamma(2-2\alpha_i)} \prod_{i=k+1}^{n} \frac{\Gamma(\alpha_i)}{\Gamma(2\alpha_i)}, k = \overline{1,\, n-1};$$

$$\beta_n = \frac{m-2}{2} + n - \sum_{i=1}^{n} \alpha_i, \quad \gamma_n = 2^{2\alpha_n - m} \frac{\Gamma(\beta_n)}{\pi^{m/2}} \prod_{i=1}^{n} \frac{\Gamma(1-\alpha_i)}{\Gamma(2-2\alpha_i)}.$$

Note that the fundamental solutions of equation (38) in particular case ($p = 1$) are found in [35] and expressed by $\mathrm{H}_A^{(n,1)}$.

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The authors of this work declare that they have no conflicts of interest.

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