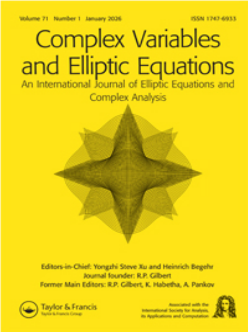

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Fundamental solutions of a multidimensional degenerate elliptic equation

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ABSTRACT

Fundamental solutions of a two-dimensional degenerate elliptic equation have recently become known which are expressed via the Appell hypergeometric function. The main result of the present paper is the construction of fundamental solutions for a class of multidimensional degenerate elliptic equations. These fundamental solutions are directly connected with multiple hypergeometric function of Lauricella and their properties. In this paper, new properties of the Lauricella hypergeometric function are established, which allow finding fundamental solutions of the considered equation in explicit forms. In addition, the properties of the constructed fundamental solutions, which are necessary in the future for solving boundary value problems for a multidimensional degenerate elliptic equation, are investigated.

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1. Introduction

Special functions are used for solving many problems of mathematical physics [1, 2]. These include the Gauss hypergeometric series, Bessel and Hermite functions, Lauricella hypergeometric functions, etc. The Hermite functions are actively applied in algorithms and information systems that are used in medical diagnostics [3]. The Bessel functions are used in solving a number of problems of hydrodynamics, radiophysics, and thermal conductivity [4, Part 2]. Some functions that are used in astronomy can be arranged in hypergeometric series [5, Chapter 3]. Multidimensional hypergeometric functions are used in the superstrings theory [6].

The study of boundary value problems for degenerate equations is one of the important directions of the modern theory of partial differential equations. In the formulation and construction of local and nonlocal boundary value problems, the main role is played by fundamental solutions.

Existence and properties of fundamental solutions for the Tricomi equation

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$$yu_{xx} + u_{yy} = 0$$

have been studied by several authors. An exposition of the known and new results on the fundamental solutions for the Tricomi operator together with references to the original literature are to be found in the triad of the works by Barros-Neto and Gelfand [7–9] which is the standard work on the subject. These results are extended, with small modifications, to the generalizations of the Tricomi equation:

$$y^m u_{xx} + u_{yy} = 0$$

and

$$y^m \sum_{j=1}^{n} u_{x_j x_j} + u_{yy} = 0,$$

where m is a positive real number (for details, see [10]). Two explicit fundamental solutions for the Tricomi equations are later used by many researchers to study various problems for a mixed-type equation [11]. In this direction, more interesting results are found in [12].

Fundamental solutions of the elliptic equations with two singular coefficients are expressed by the second Appell function F_2, and when the dimension of the equation and the number of singular coefficients exceed two – by the first Lauricella hypergeometric function $F_A^{(n)}$ with three and more variables. Fundamental solutions of the elliptic equation with two lines of degeneration

$$y^m u_{xx} + x^n u_{yy} = 0$$

are written out in terms of the Appell function F_2 in explicit forms [13] and found successful application in potential theory [14]. The fundamental solutions of the three-dimensional singular elliptic equation [15] are also applied to the solution of boundary value problems in the first octant of the unit ball [16, 17].

In the work [18], all 16 fundamental solutions of the four-dimensional degenerate elliptic equation

$$y^m z^k t^l u_{xx} + x^n z^k t^l u_{yy} + x^n y^m t^l u_{zz} + x^n y^m z^k u_{tt} = 0, \quad m > 0, n > 0, k > 0, l > 0$$

in the domain $R_+^4 := \{(x, y, z, t) : x > 0, y > 0, z > 0, t > 0\}$ are constructed in the explicit forms.

Recently, the fundamental solutions for a degenerate elliptic equation

$$y^{m+1} \left[xu_{xx} + pu_x\right] + x^{n+1} \left[yu_{yy} + qu_y\right] = 0, \quad \frac{n+2p}{n+2} > 0, \ \frac{m+2q}{m+2} > 0 \qquad (1)$$

in a domain $R_+^2 = \{(x, y) : x > 0, \ y > 0\}$ were built and expressed by Appell's hypergeometric function F_2 of two variables [19].

We also note several works in which fundamental solutions of the Helmholtz equation with one, two, three or more singular coefficients are constructed in explicit forms (see, respectively, [20–23]). In the work [24] the application of hypergeometric functions to the construction of particular solutions higher-order partial differential equations is discussed.

The purpose of the present paper is to construct fundamental solutions in explicit forms for the multidimensional analogue of Equation (1) which are expressed through the multiple Lauricella hypergeometric function $F_A^{(n)}$ (see Section 4). Therefore, first we will list the well-known (Section 2) and new (Section 3) properties of the Lauricella function $F_A^{(n)}$.

2. Multiple hypergeometric functions

The Gauss hypergeometric function can be represented by the following series [25, p. 56, Equation 2.1(2)]

$$F(a,b;c;z) \equiv F\left[\begin{array}{l} a,b; \\ c; \end{array} z\right] = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, \quad |z| < 1, \tag{2}$$

where $(\lambda)_n$ is a Pochhammer symbol: $(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1)$, $n = 1,2,\ldots$; $(\lambda)_0 = 1$.

If $Rec > Reb > 0$, we have Euler's formula [25, p. 56, Equation 2.1(10)]

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t. \tag{3}$$

We observe that Euler's integral (3) transforms into an integral of the same type if we put $s = 1-t$. From this we obtain a transformation formula given by Euler

$$F(a,b;c;z) = (1-z)^{-b} F\left(c-a,b;c;\frac{z}{z-1}\right). \tag{4}$$

The two functions $F(a,b;c;z)$ and $F(a',b';c';z)$ are said to be contiguous when one of the three differences $a'-a$, $b'-b$, $c'-c$ is equal to ± 1, the other two being zero. Three hypergeometric functions, contiguous two by two, are linked by a linear equation. There are fifteen relations of Gauss between contiguous functions [25, p. 56, Equation 2.8(31)–(45)].

Three functions $F(a,b;c;z)$, $F(a',b';c';z)$, $F(a'',b'';c'';z)$ such that the differences $a'-a$, $b'-b$, $c'-c$, $a''-a$, $b''-b$, $c''-c$ take one of the three values $-1, 0, 1$ are still linked by a linear equation; there are $\frac{26\cdot 25}{1\cdot 2}$ of these relations; here is an example [26, p. 3]

$$F(a+1,b;c;z) - F(a,b;c;z) = \frac{b}{c} zF(a+1,b+1;c+1;z). \tag{5}$$

The great success of the theory of hypergeometric functions in one variable has stimulated the development of the corresponding theory in two or more variables. Appell [27] has defined in 1880 four functions F_1 to F_4, which are all analogues to Gauss' $F(a,b;c;x)$. For instance, the second Appell function F_2 has a form

$$F_2\left[\begin{array}{l} a,b_1,b_2; \\ c_1,c_2; \end{array} x,y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| + |y| < 1. \tag{6}$$

The following relation between contiguous hypergeometric Appell functions

$$F_2\left[\begin{array}{l} a+1,b_1,b_2; \\ c_1,c_2; \end{array} x,y\right] - F_2\left[\begin{array}{l} a,b_1,b_2; \\ c_1,c_2; \end{array} x,y\right]$$

$$= \frac{b_1}{c_1} x F_2 \left[\begin{matrix} a, b_1+1, b_2; \\ c_1+1, c_2; \end{matrix} \; x, y \right] + \frac{b_2}{c_2} y F_2 \left[\begin{matrix} a, b_1, b_2+1; \\ c_1, c_2+1; \end{matrix} \; x, y \right] \tag{7}$$

is known [26, p. 21].

Burchnall and Chaundy [28] gave an expansion of double hypergeometric function F_2 in series of simpler hypergeometric functions:

$$F_2 \left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} \; x, y \right] = \sum_{r=0}^{\infty} \frac{(a)_r (b_1)_r (b_2)_r}{r! (c_1)_r (c_2)_r} \times$$
$$\times x^r y^r F(a+r, b_1+r; c_1+r; x) F\left(a+r, b_2+r; c_2+r; y\right). \tag{8}$$

Lauricella hypergeometric function [29]

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) \equiv F_A^{(n)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \; \mathbf{x} \right]$$
$$= \sum_{|\mathbf{k}|=0}^{\infty} (a)_{|\mathbf{k}|} \prod_{j=1}^{n} \left[\frac{(b_j)_{k_j}}{(c_j)_{k_j}} \frac{x_j^{k_j}}{k_j!} \right], \quad \sum_{j=1}^{n} |x_j| < 1, \; n \in \mathrm{N} \tag{9}$$

is a natural generalization of the classical Gauss hypergeometric function (2) and the Appell function (6) to the case of many complex variables and their corresponding complex parameters. Hereinafter

$$\mathbf{b} := (b_1, \dots, b_n), \mathbf{c} := (c_1, \dots, c_n), \; \mathbf{x} := (x_1, \dots, x_n); \mathrm{N} = \{1, 2, \dots, \};$$
$$|\mathbf{b}| := b_1 + \dots + b_n, \quad |\mathbf{c}| := c_1 + \dots + c_n;$$
$$\mathbf{k} := (k_1, \dots, k_n), |\mathbf{k}| := k_1 + \dots + k_n, \quad k_1 \geq 0, \dots, k_n \geq 0.$$

The integral representation [26, p. 115, Equation (5)]

$$F_A^{(n)}(a, \; \mathbf{b}; \mathbf{c}; \mathbf{x}) = \prod_{j=1}^{n} \left[\frac{\Gamma(c_j)}{\Gamma(b_j) \Gamma(c_j - b_j)} \right] \int_0^1 \cdots \int_0^1 \prod_{j=1}^{n} \left[\xi_j^{b_j - 1} (1 - \xi_j)^{c_j - b_j - 1} \right]$$
$$\times \left(1 - \sum_{j=1}^{n} \xi_j x_j \right)^{-a} d\xi_1 \cdots d\xi_n, \quad Re\left(b_j\right) > 0, \; Re\left(c_j - b_j\right) > 0, \quad j = \overline{1, n} \tag{10}$$

is easily obtained from the series (9) by using Euler's integral of the first kind for the beta-function. One- and two-dimensional analogues of the integral representation (10) are found in [25] (see, Equation 2.1(10) and 5.8(2), respectively).

Hasanov and Srivastava [30] proved that for all $n \in \mathrm{N} \backslash \{1\}$ is true a recurrence formula

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x})$$
$$= \sum_{|\mathbf{k}'|=0}^{\infty} \frac{(a)_{|\mathbf{k}'|} (b_1)_{|\mathbf{k}'|} (b_2)_{k_2} \cdots (b_n)_{k_n}}{k_2! \cdots k_n! (c_1)_{|\mathbf{k}'|} (c_2)_{k_2} \cdots (c_n)_{k_n}} \times$$
$$\times x_1^{|\mathbf{k}'|} x_2^{k_2} \cdots x_n^{k_n} F\left(a + |\mathbf{k}'|, b_1 + |\mathbf{k}'|; c_1 + |\mathbf{k}'|; x_1\right)$$

$$\times\, F_A^{(n-1)}\left(a+|\mathbf{k}'|, b_2+k_2,\ldots,b_n+k_n; c_2+k_2,\ldots, c_n+k_n; x_2,\ldots,x_n\right), \tag{11}$$

where $|\mathbf{k}'| := k_2+\cdots+k_n$.

Laurricella function $F = F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x})$ verifies a system of n second-order linear partial differential equations [26, p. 117]:

$$\begin{cases} x_k(1-x_k)\dfrac{\partial^2 F}{\partial x_k^2} - x_k \sum_{j=1,j\neq k}^{n} x_j \dfrac{\partial^2 F}{\partial x_k \partial x_j} + \left[c_k - (a+b_k+1)\,x_k\right]\dfrac{\partial F}{\partial x_k} \\ -b_k \sum_{j=1,j\neq k}^{n} x_j \dfrac{\partial F}{\partial x_j} - ab_k F = 0, \quad k=\overline{1,n}. \end{cases} \tag{12}$$

This system contains as special cases: for $n=1$ the hypergeometric equation of Gauss; for $n=2$, the system of equations of Appell (see, Equation 2.1(1) and 5.9(10) in [25]).

Appell and Kampé de Fériet [26] have demonstrated a series of propositions concerning system of the form which lead to the following result: the integrand of the system (12) verified by the function $F_A^{(n)}$ depends linearly on 2^n arbitrary constants.

To obtain the general integral of system (12), it is sufficient to note that it does not change form when the change is made to

$$F = x_1^{\lambda_1}\cdots x_n^{\lambda_n}\tilde{F},$$

where λ_1, λ_2,…, λ_n are having some suitable values:

$$\begin{array}{cl} & \begin{array}{cccccc} \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_{n-1} & \lambda_n \end{array} \\ 1 & \{\begin{array}{cccccc} 0 & 0 & ,0 & \ldots & 0 & 0 \end{array} \\ n & \left\{\begin{array}{cccccc} 1-c_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1-c_2 & 0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1-c_n & 0 \end{array}\right. \\ \dfrac{n(n-1)}{1\cdot 2} & \left\{\begin{array}{cccccc} 1-c_1 & 1-c_2 & 0 & \cdots & 0 & 0 \\ 0 & 1-c_2 & 1-c_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 1-c_{n-1} & 1-c_n \end{array}\right. \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1 & \{\begin{array}{cccccc} 1-c_1 & 1-c_2 & 1-c_3 & \cdots & 1-c_{n-1} & 1-c_n. \end{array} \end{array}$$

We deduce the 2^n particular integrals [26, p. 118]:

$$1\{F_A^{(n)}\left[\begin{array}{l} a, b_1,\ldots,b_n; \\ c_1,\ldots,c_n; \end{array} \mathbf{x}\right],$$

$$n\begin{cases} x_1^{1-c_1} F_A^{(n)}\left[\begin{array}{l} a+1-c_1, b_1+1-c_1, b_2, b_3,\ldots,b_n; \\ 2-c_1, c_2, c_3,\ldots,c_n; \end{array} \mathbf{x}\right], \\ x_2^{1-c_2} F_A^{(n)}\left[\begin{array}{l} a+1-c_2, b_1, b_2+1-c_2, b_3,\ldots,b_n; \\ c_1, 2-c_2, c_3,\ldots,c_n; \end{array} \mathbf{x}\right], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n^{1-c_n} F_A^{(n)}\left[\begin{array}{l} a+1-c_n, b_1,\ldots,b_{n-1}, b_n+1-c_n; \\ c_1,\ldots,c_{n-1}, 2-c_n; \end{array} \mathbf{x}\right], \end{cases}$$

$$\frac{n(n-1)}{1\cdot 2}\left\{\begin{array}{l} x_1^{1-c_1}x_2^{1-c_2}F_A^{(n)}\left[\begin{array}{ll} a+2-c_1-c_2, b_1+1-c_1, & \\ b_2+1-c_2, b_3,\ldots,b_n; & \mathbf{x} \\ 2-c_1, 2-c_2, c_3,\ldots,c_n; & \end{array}\right], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1^{1-c_1}x_n^{1-c_n}F_A^{(n)}\left[\begin{array}{ll} a+2-c_1-c_n, b_1+1-c_1, b_2,\ldots, & \\ b_{n-1}, b_n+1-c_n; & \mathbf{x} \\ 2-c_1, c_2,\ldots,c_{n-1}, 2-c_n; & \end{array}\right], \\ x_2^{1-c_2}x_3^{1-c_3}F_A^{(n)}\left[\begin{array}{ll} a+2-c_2-c_3, b_1, b_2+1-c_2, & \\ b_3+1-c_3, b_4,\ldots,b_n; & \mathbf{x} \\ c_1, 2-c_2, 2-c_3, c_4,\ldots,c_n; & \end{array}\right], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-1}^{1-c_{n-1}}x_n^{1-c_n}F_A^{(n)}\left[\begin{array}{ll} a+2-c_{n-1}-c_n, b_1,\ldots,b_{n-2}, & \\ b_{n-1}+1-c_{n-1}, b_n+1-c_n; & \mathbf{x} \\ c_1,\ldots,c_{n-2}, 2-c_{n-1}, 2-c_n; & \end{array}\right], \end{array}\right.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots..$$

$$1\left\{x_1^{1-c_1}\cdots x_n^{1-c_n}F_A^{(n)}\left[\begin{array}{ll} a+n-c_1-\cdots-c_n, b_1+1-c_1,\ldots,b_n+1-c_n; & \mathbf{x} \\ 2-c_1,\ldots,2-c_n; & \end{array}\right].\right.$$

When none of the numbers $\lambda_1, \lambda_2,\ldots, \lambda_n$ is equal to an integer, we obtain the general integral of the system (12), by multiplying these 2^n particular integrals by arbitrary constants, then by adding them together.

By virtue of symmetry of the function $F_A^{(n)}$ with respect to the parameters $b_1,\ldots, b_n$, $c_1,\ldots, c_n$ and variables $x_1,\ldots,x_n$, it is possible to group the above linearly independent solutions of the system of hypergeometric Equations (12). As a result, a number of solutions of the system (12), which are necessary to further studies, will decrease. Thus, all solutions of the system (12) are expressed by the formula:

$$\omega_k(\mathbf{x}) = \prod_{j=1}^{k}\left[x_j^{1-c_j}\right]F_A^{(n)}\left[\begin{array}{ll} a+k-\sum_{j=1}^{k}c_j, b_1+1-c_1,\ldots, & \\ b_k+1-c_k, b_{k+1},\ldots,b_n; & \mathbf{x} \\ 2-c_1,\ldots,2-c_k, c_{k+1},\ldots,c_n; & \end{array}\right], \quad k=\overline{0,n}. \tag{13}$$

The factor $\prod_{j=1}^{k}$ in (13) means 1 if $k=0$, also the sum $\sum_{j=1}^{k}=0$ if $k=0$.

3. New properties of Lauricella function

Theorem 1: . *The following relation between contiguous Lauricella functions:*

$$F_A^{(n)}(a+1,\mathbf{b};\mathbf{c};\mathbf{x}) - F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = \sum_{j=1}^{n}\frac{b_j}{c_j}x_jF_A^{(n)}\left(a+1,\mathbf{b}+\mathbf{e}_j;\mathbf{c}+\mathbf{e}_j;\mathbf{x}\right) \tag{14}$$

is valid, where $\mathbf{e}_j := (0,\ldots,1,\ldots,0)$ *denote the vectors with jth component equal to 1 and the others equal to 0.*

Proof: In cases $n=1$ and $n=2$ the relations (5) and (7) are known.

Let us prove the relation (14) for any natural numbers n. By virtue of the definition (9), we have

$$
\begin{aligned}
&F_A^{(n)}\left(a+1, \mathbf{b}; \mathbf{c}; \mathbf{x}\right) - F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; \mathbf{x}\right) \\
&= \sum_{|\mathbf{k}|=0}^{\infty} \left[(a+1)_{|\mathbf{k}|} - (a)_{|\mathbf{k}|}\right] \prod_{i=1}^{n} \left[\frac{(b_i)_{k_i}}{(c_i)_{k_i}} \frac{x_i^{k_i}}{k_i!}\right] \\
&= \frac{1}{a} \sum_{|\mathbf{k}|=0}^{\infty} |\mathbf{k}|\, (a)_{|\mathbf{k}|} \prod_{i=1}^{n} \left[\frac{(b_i)_{k_i}}{(c_i)_{k_i}} \frac{x_i^{k_i}}{k_i!}\right].
\end{aligned}
$$

Then applying the following easily-derivable identity

$$
\sum_{|\mathbf{k}|=0}^{\infty} |\mathbf{k}|\, (a)_{|\mathbf{k}|} \prod_{i=1}^{n} \left[\frac{(b_i)_{k_i}}{(c_i)_{k_i}} \frac{x_i^{k_i}}{k_i!}\right] = \sum_{|\mathbf{k}|=0}^{\infty} (a)_{|\mathbf{k}|+1} \sum_{j=1}^{n} \frac{(b_j)_{k_j+1}}{(c_j)_{k_j+1}} \frac{x_j^{k_j+1}}{k_j!} \prod_{i=1,\, i\neq j}^{n} \left[\frac{(b_i)_{k_i}}{(c_i)_{k_i}} \frac{x_i^{k_i}}{k_i!}\right]
$$

and elementary properties of the Pochhammer symbol:

$$
(a)_{|\mathbf{k}|+1} = a(a+1)_{|\mathbf{k}|}, \quad (b_j)_{k_j+1} = b_j(b_j+1)_{k_j}, (c_j)_{k_j+1} = c_j(c_j+1)_{k_j},
$$

we get the assertion (14) of Theorem 1. ■

Remark 1: In cases $n = 2$ and $n = 3$ the contiguous relations (7) and (14) with applications to the theory of the boundary value problems for the singular elliptic equations are found respectively in [16] and [17].

Remark 2: The contiguous relation (14) first occurs in [31], where thanks to this relation, the solution of the generalized Holmgren problem for the multidimensional singular elliptic equation is found in explicit form.

Let's determine some necessary notation:

$$
A_l(k,n) = \sum_{i=l}^{k+1} \sum_{j=i}^{n} m_{i,j}, B_l(k,n) = \sum_{i=l}^{k} m_{i,k} + \sum_{i=k+1}^{n} m_{k+1,i};
$$

$$
A_2(0,0) = B_2(0,0) = 0; \quad |\mathbf{m}_n| := \sum_{i=2}^{n} \sum_{j=i}^{n} m_{i,j}, \quad M_n! := \prod_{i=2}^{n} \prod_{j=i}^{n} (m_{i,j})!,
$$

where $k, n \in \mathrm{N}, k \leq n; l \in \mathrm{N} \setminus \{1\};\ m_{i,j} \in \mathrm{N} \cap \{0\}(2 \leq i \leq j \leq n); \mathrm{N} = \{1, 2, \ldots, \}$. Hereinafter, $\sum_{i=1}^{m}, \sum_{i=2}^{n}$ denote zero if $m = 0$ or $n = 1$.

Theorem 2: *The following expansion formula holds for all $n \in \mathrm{N}$*

$$
F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; \mathbf{x}\right) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A_2(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B_2(k,n)}}{(c_k)_{B_2(k,n)}}
$$

$$\times \prod_{k=1}^{n} \left\{ x_k^{B_2(k,n)} F \left[\begin{matrix} a + A_2(k,n), b_k + B_2(k,n); \\ c_k + B_2(k,n); \end{matrix} \; x_k \right] \right\}, \tag{15}$$

where $F(a, b; c; z)$ is the Gaussian hypergeometric function defined in (2).

Proof: We carry out the proof by the method of mathematical induction. In the case $n = 1$ the equality (15) is obvious ($A_2(1,1) = B_2(1,1) = 0$).

Let $n = 2$. Since $A_2(1,2) = A_2(2,2) = B_2(1,2) = B_2(2,2) = m_{2,2} := r$, we obtain the formula (8).

So the formula (15) works at $n = 1$ and $n = 2$.

Now we assume that for $n = s$ the equality (15) holds:

$$F_A^{(s)}(a, b_1, \ldots, b_s; c_1, \ldots, c_s; x_1, \ldots, x_s) = \sum_{|\mathbf{m}_s|=0}^{\infty} \frac{(a)_{A_2(s,s)}}{M_s!}$$
$$\times \prod_{k=1}^{s} \frac{(b_k)_{B_2(k,s)}}{(c_k)_{B_2(k,s)}} x_k^{B_2(k,s)} F \left[\begin{matrix} a + A_2(k,s), b_k + B_2(k,s); \\ c_k + B_2(k,s); \end{matrix} \; x_k \right]. \tag{16}$$

Let $n = s + 1$. We prove that the following formula

$$F_A^{(s+1)}(a, b_1, \ldots, b_{s+1}; c_1, \ldots, c_{s+1}; x_1, \ldots, x_{s+1})$$
$$= \sum_{|\mathbf{m}_{s+1}|=0}^{\infty} \frac{(a)_{A_2(s+1,s+1)}}{M_{s+1}!} \prod_{k=1}^{s+1} \frac{(b_k)_{B_2(k,s+1)}}{(c_k)_{B_2(k,s+1)}}$$
$$\times \prod_{k=1}^{s+1} x_k^{B_2(k,s+1)} F \left[\begin{matrix} a + A_2(k,s+1), b_k + B_2(k,s+1); \\ c_k + B_2(k,s+1); \end{matrix} \; x_k \right] \tag{17}$$

is valid.

We rewrite the Hasanov-Srivastava's formula (11) in the form

$$F_A^{(s+1)}(a, b_1, \ldots, b_{s+1}; c_1, \ldots, c_{s+1}; x_1, \ldots, x_{s+1})$$
$$= \sum_{m_{2,2},\ldots,m_{2,s+1}=0}^{\infty} \frac{(a)_{A_2(1,s+1)}(b_1)_{B_2(1,s+1)}(b_2)_{m_{2,2}} \cdots (b_{s+1})_{m_{2,s+1}}}{m_{2,2}! \cdots m_{2,s+1}!(c_1)_{B_2(1,s+1)}(c_2)_{m_{2,2}} \cdots (c_{s+1})_{m_{2,s+1}}}$$
$$\times x_1^{B_2(1,s+1)} x_2^{m_{2,2}} \cdots x_{s+1}^{m_{2,s+1}} F \left[\begin{matrix} a + A_2(1,s+1), b_1 + M_2(1,s+1); \\ c_1 + B_2(1,s+1); \end{matrix} \; x_1 \right]$$
$$\times F_A^{(s)} \left[\begin{matrix} a + A_2(1,s+1), b_2 + m_{2,2}, \ldots, b_{s+1} + m_{2,s+1}; \\ c_2 + m_{2,2}, \ldots, c_{s+1} + m_{2,s+1}; \end{matrix} \; x_2, \ldots, x_{s+1} \right]. \tag{18}$$

By virtue of the formula (16) we have

$$F_A^{(s)} \left[\begin{matrix} a + A_2(1,s+1), b_2 + m_{2,2}, \ldots, b_{s+1} + m_{2,s+1}; \\ c_2 + m_{2,2}, \ldots, c_{s+1} + m_{2,s+1}; \end{matrix} \; x_2, \ldots, x_{s+1} \right]$$

$$= \sum_{m_{i,j}=0_{(3\le i\le j\le s+1)}}^{\infty} \frac{(a+A_2(1,s+1))_{A_3(s+1,s+1)}}{m_{ij}!} \prod_{k=2}^{s+1} \frac{(b_k+m_{2,k})_{B_3(k,s+1)}}{(c_k+m_{2,k})_{B_3(k,s+1)}} x_k^{B_3(k,s+1)}$$

$$\times \prod_{k=2}^{s+1} F\left[\begin{matrix} a+A_2(1,s+1)+A_3(k,s+1), b_k+m_{2,k}+B_3(k,s+1); \\ c_k+m_{2,k}+B_3(k,s+1); \end{matrix} \; x_k\right]. \quad (19)$$

Substituting from (19) into (18) we obtain

$$F_A^{(s+1)}(a, b_1, \ldots, b_{s+1}; c_1, \ldots, c_{s+1}; x_1, \ldots, x_{s+1})$$

$$= \sum_{m_{i,j}=0_{(2\le i\le j\le s+1)}}^{\infty} \frac{(a)_{A_2(1,s+1)+A_3(s+1,s+1)}}{m_{ij}!} \prod_{k=1}^{s+1} \frac{(b_k)_{m_{2,k}+B_3(k,s+1)}}{(c_k)_{m_{2,k}+B_3(k,s+1)}} x_k^{m_{2,k}+B_3(k,s+1)}$$

$$\times \prod_{k=1}^{s+1} F\left[\begin{matrix} a+A_2(1,s+1)+A_3(k,s+1), b_k+m_{2,k}+B_3(k,s+1); \\ c_k+m_{2,k}+B_3(k,s+1); \end{matrix} \; x_k\right].$$

Further, by virtue of the following obvious equalities

$$A_2(1,s+1)+A_3(k,s+1) = A_2(k,s+1), \quad 1 \le k \le s+1, s \in \mathrm{N},$$
$$m_{2,k}+B_3(k,s+1) = B_2(k,s+1), 1 \le k \le s+1, s \in \mathrm{N},$$

we finally find the equality (17). Q.E.D. ■

Theorem 3: *Let a, b_k, c_k be real numbers, where $c_k \neq 0, -1, -2, \ldots$ and $a > |\mathbf{b}| > 0$ and $c_k > b_k$. Then for $n = 1, 2, \ldots$, the following limit correlation is true*

$$\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon^{|\mathbf{b}|}} F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; 1-\frac{f_1(\varepsilon)}{\varepsilon}, \cdots, 1-\frac{f_n(\varepsilon)}{\varepsilon}\right)\right]$$
$$= \frac{\Gamma(a-|\mathbf{b}|)}{\Gamma(a)} \prod_{k=1}^{n}\left[\frac{\left|f_k(0)\right|^{-b_k}\Gamma(c_k)}{\Gamma(c_k-b_k)}\right],$$

where $f_k(\varepsilon)$ are arbitrary functions with $f_k(0) \neq 0$ ($k = \overline{1,n}$).

Proof: By virtue of the decomposition formula (15) we obtain

$$F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; 1-\frac{f_1(\varepsilon)}{\varepsilon}, \ldots, 1-\frac{f_n(\varepsilon)}{\varepsilon}\right) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A_2(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B_2(k,n)}}{(c_k)_{B_2(k,n)}}$$

$$\times \prod_{k=1}^{n}\left(1-\frac{f_k(\varepsilon)}{\varepsilon}\right)^{B_2(k,n)} F\left[\begin{matrix} a+A_2(k,n), b_k+B_2(k,n); \\ c_k+B_2(k,n); \end{matrix} \; 1-\frac{f_k(\varepsilon)}{\varepsilon}\right]. \quad (20)$$

Applying now the familiar autotransformation formula (4) to each hypergeometric function included in the sum (20), we get

$$F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; 1-\frac{f_1(\varepsilon)}{\varepsilon}, \ldots, 1-\frac{f_n(\varepsilon)}{\varepsilon}\right)$$

$$= \varepsilon^{|\mathbf{b}|} \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A_2(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B_2(k,n)}}{(c_k)_{B_2(k,n)} \left[f_k(\varepsilon)\right]^{b_k}} \left(\frac{\varepsilon}{f_k(\varepsilon)} - 1 \right)^{B_2(k,n)}$$

$$\times \prod_{k=1}^{n} F \left[\begin{matrix} c_k - a + B_2(k,n) - A_2(k,n), b_k + B_2(k,n); \\ c_k + B_2(k,n); \end{matrix} \; 1 - \frac{\varepsilon}{f_k(\varepsilon)} \right].$$

It should be noted here that the sum $B_2(1,n) + B_2(2,n) + \cdots + B_2(n,n)$ has the parity property, which plays an important role in the calculation of some values of hypergeometric functions. In fact, by virtue of equality

$$\sum_{k=2}^{n} \sum_{i=2}^{k} m_{i,k} = \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} m_{k+1,i}$$

we obtain

$$\sum_{k=1}^{n} B_2(k,n) = 2 \sum_{k=2}^{n} \sum_{i=2}^{k} m_{i,k} = 2 \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} m_{k+1,i}$$

which helps to calculate the limit

$$\lim_{\varepsilon \to 0} \prod_{k=1}^{n} \left(\frac{\varepsilon}{f_k(\varepsilon)} - 1 \right)^{B_2(k,n)} = \prod_{k=1}^{n} (-1)^{B_2(k,n)} = 1.$$

Now we calculate the limit

$$\lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon^{|\mathbf{b}|}} F_A^{(n)} \left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{f_1(\varepsilon)}{\varepsilon}, \ldots, 1 - \frac{f_n(\varepsilon)}{\varepsilon} \right) \right] = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A_2(n,n)}}{M_n!}$$

$$\times \prod_{k=1}^{n} \left[f_k(0)\right]^{-b_k} \frac{(b_k)_{B_2(k,n)}}{(c_k)_{B_2(k,n)}} F \left[\begin{matrix} c_k - a + B_2(k,n) - A_2(k,n), b_k + B_2(k,n); \\ c_k + B_2(k,n); \end{matrix} \; 1 \right].$$

Applying the summation formula [25, p. 56, Equation 2.1(14)]

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad Rec > Reb > 0, \; Re(c-a-b) > 0.$$

to the Gauss hypergeometric functions in the last sum and taking into account the following previously proven equality [32, Equation (4.11)]

$$\sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A_2(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B_2(k,n)} \, (a-b_k)_{A_2(k,n)-B_2(k,n)}}{(a)_{A_2(k,n)}} = \frac{\Gamma\left(a - |\mathbf{b}|\right)}{\Gamma(a)} \prod_{k=1}^{n} \frac{\Gamma(a)}{\Gamma\left(a - b_k\right)},$$

we obtain the assertion of Theorem 3. Q.E.D. ∎

Let us introduce the notation: $|\mathbf{x}| := x_1 + \cdots + x_n$.

Theorem 4: *. Let be $x_1 > 0,\dots, x_n > 0$. If $c_1 > b_1 > 0,\dots, c_n > b_n > 0$ and $a + |\mathbf{b}| = |\mathbf{c}|$, then the Lauricella hypergeometric function $F_A^{(n)}$ has a logarithmic singularity*

$$F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; \mathbf{x}\right) \sim -\frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma\left(c_j\right)}{\Gamma\left(b_j\right)} x_j^{b_j - c_j} \right] \ln(1 - |\mathbf{x}|)$$

as $|\mathbf{x}| \to 1 - 0$. The same argument leads to the more general result, that if $c_1 > b_1 > 0,\dots, c_n > b_n > 0$ and $a + |\mathbf{b}| > |\mathbf{c}|$, then

$$F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; \mathbf{x}\right) \sim \frac{\Gamma\left(a + |\mathbf{b}| - |\mathbf{c}|\right)}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma\left(c_j\right)}{\Gamma\left(b_j\right)} x_j^{b_j - c_j} \right] (1 - |\mathbf{x}|)^{|\mathbf{c}| - |\mathbf{b}| - a} \tag{21}$$

as $|\mathbf{x}| \to 1 - 0$.

Note that in case $n = 2$ Theorem 4 is proven for the Appell function F_2 in [13].

Proof: Let us prove the statement of the theorem on logarithmic singularity. We assume that the following conditions are met: $c_j > b_j > 0$ and $a + |\mathbf{b}| = |\mathbf{c}|$. It is known [29], that the hypergeometric function Lauricella $F_A^{(n)}(a, \mathbf{b};\ \mathbf{c};\ \mathbf{x})$ converges absolutely, if $x_1 > 0,\dots, x_n > 0$ and $|\mathbf{x}| < 1$.

It is clear that the point M with coordinates $x_j = a_j^2/A$ $(j = \overline{1, n})$ lies on the hyperplane $|\mathbf{x}| = 1$, where $A := a_1^2 + \cdots + a_n^2, \quad a_1 > 0, \ \dots a_n > 0$. Let us show that each point of this hyperplane is a point of logarithmic singularity.

For brevity, we denote the Lauricell hypergeometric function by $F_A^{(n)}$. Then, if $b_i > 0, \quad i = \overline{1, n}$, then the integral representation (10) for $F_A^{(n)}$ can be easily transformed to the form

$$\prod_{j=1}^{n} \left[\frac{\Gamma\left(b_j\right) \Gamma\left(c_j - b_j\right)}{\Gamma\left(c_j\right)} \right] F_A^{(n)}\left(a, \mathbf{b};\mathbf{c};\mathbf{x}\right)$$
$$= \int_0^1 \cdots \int_0^1 \left(1 - \sum_{j=1}^{n} t_j x_j\right)^{-a} \prod_{j=1}^{n} \left[t_j^{b_j - 1} \left(1 - t_j\right)^{c_j - b_j - 1} \mathrm{d}t_j \right] = I,$$

meaningful only if $x_1 > 0, \dots, x_n > 0, |\mathbf{x}| < 1$. Now we write $x_j = \xi_j r$, where $\xi_j > 0, \xi_1 + \cdots + \xi_n = 1, 0 < r < 1$, and investigate the behaviour of I for $r \to 1 - 0$. We obtain

$$I = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \int_0^1 \cdots \int_0^1 \left(t_1 x_1 + \cdots + t_n x_n\right)^k \prod_{j=1}^{n} \left[t_j^{b_j - 1} \left(1 - t_j\right)^{c_j - b_j - 1} \mathrm{d}t_j \right]$$
$$= a_0 + \sum_{k=1}^{\infty} \frac{(a)_k}{k!} E_k r^k,$$

where

$$a_0 = \prod_{j=1}^{n} \left[\frac{\Gamma\left(b_j\right)\Gamma\left(c_j - b_j\right)}{\Gamma\left(c_j\right)} \right]$$

and

$$\begin{aligned} E_k &= \int_0^1 \cdots \int_0^1 \left(t_1 x_1 + \cdots + t_n x_n\right)^k \prod_{j=1}^{n} \left[t_j^{b_j - 1} \left(1 - t_j\right)^{c_j - b_j - 1} \mathrm{d}t_j \right] \\ &= \int_0^1 \cdots \int_0^1 \left(1 - t_1 x_1 - \cdots - t_n x_n\right)^k \prod_{j=1}^{n} \left[t_j^{c_j - b_j - 1} \left(1 - t_j\right)^{b_j - 1} \mathrm{d}t_j \right]. \end{aligned}$$

Now let's put $t_1 = \theta_1/(k\xi_1), \quad \ldots, t_n = \theta_n/(k\xi_n)$. Then, by virtue of $|\mathbf{c}| - |\mathbf{b}| = a$, we get

$$E_k = \frac{1}{k^{|c|-|b|}} \prod_{j=1}^{n} \frac{1}{\xi_j^{c_j - b_j}} \int_0^{k\xi_1} \cdots \int_0^{k\xi_n} \left(1 - \frac{\Theta}{k}\right)^k \prod_{j=1}^{n} \left[\theta_j^{c_j - b_j - 1} \left(1 - \frac{\theta_j}{k\xi_j}\right)^{b_j - 1} d\theta_j \right],$$

where $\Theta := \theta_1 + \cdots + \theta_n$. Passing to the limit as $k \to \infty$, we obtain

$$\begin{aligned} k^a E_k &\to \prod_{j=1}^{n} \frac{1}{\xi_j^{c_j - b_j}} \int_0^{\infty} \cdots \int_0^{\infty} e^{-\Theta} \prod_{j=1}^{n} \left[\theta_j^{c_j - b_j - 1} d\theta_j \right] \\ &= \prod_{j=1}^{n} \left[\frac{1}{\xi_j^{c_j - b_j}} \int_0^{\infty} e^{-\theta_j} \theta_j^{c_j - b_j - 1} d\theta_j \right] = \prod_{j=1}^{n} \left[\frac{\Gamma\left(c_j - b_j\right)}{\xi_j^{c_j - b_j}} \right]. \end{aligned}$$

Using the well-known asymptotic formula for the gamma function[25, p. 62, Equation (4)]:

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta} \left[1 + \frac{1}{2z} \left(\alpha - \beta\right)\left(\alpha + \beta - 1\right) + \mathrm{O}\left(\frac{1}{z^2}\right) \right], \quad z \to \infty,$$

we have

$$\begin{aligned} \frac{(a)_k}{k!} E_k &= \frac{E_k}{\Gamma\left(a\right)} \frac{\Gamma\left(a + k\right)}{\Gamma(1 + k)} \sim \frac{E_k}{\Gamma\left(a\right)} k^{a-1} \\ &= \frac{k^{a-1}}{\Gamma\left(a\right) k^a} \prod_{j=1}^{n} \left[\frac{\Gamma\left(c_j - b_j\right)}{\xi_j^{c_j - b_j}} \right] = \frac{1}{k\Gamma\left(a\right)} \prod_{j=1}^{n} \left[\frac{\Gamma\left(c_j - b_j\right)}{\xi_j^{c_j - b_j}} \right] \end{aligned}$$

as $k \to \infty$. ■

Further, in the process of proving Theorem 4, the following lemma given by Hobson [33] is very important.

Lemma 1: *If $\sum_{n=0}^{\infty} a_n x^n$, $\sum_{n=0}^{\infty} b_n x^n$ both converge within the interval $(-1, 1)$, a_n being positive and such that $\sum_{n=0}^{\infty} a_n$ is divergent, and if b_n/a_n oscillates between the limits U and L, then the upper and lower limits of $\sum_{n=0}^{\infty} b_n x^n / \sum_{n=0}^{\infty} a_n x^n$, as x converges to 0,* *are in the interval* (L, U). *In particular, if* b_n/a_n *converges to a definite limit, as* $n \sim \infty$, $\sum_{n=0}^{\infty} b_n x^n / \sum_{n=0}^{\infty} a_n x^n$ *converges to the same limit, as x converges to 1. If* b_n/a_n *diverges to* ∞, *so also does* $\sum_{n=0}^{\infty} b_n x^n / \sum_{n=0}^{\infty} a_n x^n$, *as* $x \sim 1$.

By virtue of Lemma 1, we have

$$\lim_{r\to 1-0} \frac{\sum_{k=1}^{\infty} \frac{(a)_k}{k!} E_k r^k}{\sum_{k=1}^{\infty} \frac{1}{k} r^k} = \lim_{k\to\infty} \frac{\frac{(a)_k}{k!} E_k}{\frac{1}{k}} = \frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)}{\xi_j^{c_j - b_j}} \right].$$

Hence, taking into account the well-known expansion of the logarithmic function

$$\ln(1-r) = -\sum_{k=1}^{\infty} \frac{1}{k} r^k,$$

we obtain

$$\begin{aligned} I &\sim -\frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)}{\xi_j^{c_j - b_j}} \right] \ln(1-r) \\ &= -\frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)\, r^{c_j - b_j}}{x_j^{c_j - b_j}} \right] \ln(1 - |\mathbf{x}|) \\ &\sim -\frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)}{x_j^{c_j - b_j}} \right] \ln(1 - |\mathbf{x}|) \end{aligned}$$

as $r \to 1 - 0$.

Therefore, if $c_j > b_j > 0$, $a + |\mathbf{b}| = |\mathbf{c}|$ and x_j are positive $(j = \overline{1, n})$, then

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) \sim -\frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j)}{\Gamma(b_j)\, x_j^{c_j - b_j}} \right] \ln(1 - |\mathbf{x}|),$$

as $|\mathbf{x}| \to 1 - 0$.

The second statement of Theorem 4 is proved similarly. Q.E.D.

4. Fundamental solutions of a multidimensional degenerate elliptic equation

We consider the equation

$$\sum_{k=1}^{n} \prod_{j=1, j\neq k}^{n} \left[x_j^{m_j+1} \right] \left(x_k \frac{\partial^2 u}{\partial x_k^2} + p_k \frac{\partial u}{\partial x_k} \right) = 0, \quad 0 < p_k < 1,\ m_k > -2p_k \tag{22}$$

in the domain $R_+^n = \{(x_1, \ldots, x_n) : x_1 > 0, \ldots, x_n > 0\}$.

Let $x := (x_1, \ldots, x_n)$ be any point and $\xi := (\xi_1, \ldots, \xi_n)$ be any fixed point of R_+^n.

Let x and ξ be two points of the domain R_+^n. We are looking for a solution of Equation (22) in the form

$$u = P\omega\left(\sigma\right), \tag{23}$$

where ω is a new unknown function,

$$P = r^{-2\beta}, \quad \beta = \frac{n-2}{2} + \sum_{k=1}^{n} \alpha_k,\ \alpha_k = \frac{m_k + 2p_k}{2\left(m_k+2\right)};$$

$$\sigma = \left(\sigma_1, \ldots, \sigma_n\right), \quad \sigma_k = \frac{r^2 - r_k^2}{r^2},\ r^2 = \sum_{j=1}^{n} \frac{4}{\left(m_j+2\right)^2}\left(x_j^{\frac{m_j+2}{2}} - \xi_j^{\frac{m_j+2}{2}}\right)^2,$$

$$r_k^2 = \frac{4}{(m_k+2)^2}\left(x_k^{\frac{m_k+2}{2}} + \xi_k^{\frac{m_k+2}{2}}\right)^2 + \sum_{j=1_{j\neq k}}^{n} \frac{4}{\left(m_j+2\right)^2}\left(x_j^{\frac{m_j+2}{2}} - \xi_j^{\frac{m_j+2}{2}}\right)^2, k = \overline{1,n}.$$

We calculate all necessary derivatives and substitute them into Equation (22):

$$\sum_{k=1}^{n} A_k \frac{\partial^2\omega}{\partial\sigma_k^2} + \sum_{k=1}^{n}\sum_{l=k+1}^{n} B_{k,l}\frac{\partial^2\omega}{\partial\sigma_k\partial\sigma_l} + \sum_{k=1}^{n} C_k\frac{\partial\omega}{\partial\sigma_k} + D\omega = 0, \tag{24}$$

where

$$A_k = PX\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}\left[x_i^{m_i}\right]\left(\frac{\partial\sigma_k}{\partial x_j}\right)^2,$$

$$B_{k,l} = 2PX\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}\left[x_i^{m_i}\right]\frac{\partial\sigma_k}{\partial x_j}\frac{\partial\sigma_l}{\partial x_j},$$

$$C_k = PX\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}\left[x_i^{m_i}\right]\frac{\partial^2\sigma_k}{\partial x_j^2}$$

$$+ 2X\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}\left[x_i^{m_i}\right]\frac{\partial P}{\partial x_j}\frac{\partial\sigma_k}{\partial x_j} + P\sum_{j=1}^{n} p_j\prod_{i=1,i\neq j}^{n}\left[x_i^{m_i+1}\right]\frac{\partial\sigma_k}{\partial x_j},$$

$$D = X\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}\left[x_i^{m_i}\right]\frac{\partial^2 P}{\partial x_j^2} + \sum_{j=1}^{n} p_j\prod_{i=1,i\neq j}^{n}\left[x_i^{m_i+1}\right]\frac{\partial P}{\partial x_j}, \quad X = \prod_{j=1}^{n}[x_j].$$

It is not difficult to calculate the derivatives ($k, j = \overline{1,n}$):

$$\frac{\partial\sigma_k}{\partial x_k} = -\frac{8}{\left(m_k+2\right)r^2}x_k^{\frac{m_k}{2}}\xi_k^{\frac{m_k+2}{2}} - \frac{4}{\left(m_k+2\right)r^2}x_k^{\frac{m_k}{2}}\left(x_k^{\frac{m_k+2}{2}} - \xi_k^{\frac{m_k+2}{2}}\right)\sigma_k, \tag{25}$$

$$\frac{\partial\sigma_k}{\partial x_j} = -\frac{4}{\left(m_j+2\right)r^2}x_j^{\frac{m_j}{2}}\left(x_j^{\frac{m_j+2}{2}} - \xi_j^{\frac{m_j+2}{2}}\right)\sigma_k, \quad j\neq k, \tag{26}$$

$$\frac{\partial^2\sigma_k}{\partial x_k^2} = -\frac{4m_k}{(m_k+2)\,r^2}x_k^{\frac{m_k-2}{2}}\xi_k^{\frac{m_k+2}{2}} - \frac{4\,(2m_k+3)}{(m_k+2)\,r^2}x_k^{m_k}\sigma_k$$

$$+\frac{2\,(3m_k+4)}{(m_k+2)\,r^2}x_k^{\frac{m_k-2}{2}}\xi_k^{\frac{m_k+2}{2}}\sigma_k + \frac{32}{(m_k+2)^2\,r^4}x_k^{m_k}\left(x_k^{\frac{m_k+2}{2}} - \xi_k^{\frac{m_k+2}{2}}\right)^2\sigma_k, \quad (27)$$

$$\frac{\partial^2\sigma_k}{\partial x_j^2} = \frac{2m_j}{\left(m_j+2\right)r^2}x_j^{\frac{m_j-2}{2}}\xi_j^{\frac{m_j+2}{2}}\sigma_k - \frac{4\left(m_j+1\right)}{\left(m_j+2\right)r^2}x_j^{m_j}\sigma_k$$

$$+\frac{32}{\left(m_j+2\right)^2 r^4}x_j^{m_j}\left(x_j^{\frac{m_j+2}{2}} - \xi_j^{\frac{m_j+2}{2}}\right)^2\sigma_k, \quad j\neq k, \quad (28)$$

$$\frac{\partial P}{\partial x_i} = -\frac{4\beta P}{\left(m_j+2\right)r^2}x_j^{\frac{m_j}{2}}\left(x_j^{\frac{m_j+2}{2}} - \xi_j^{\frac{m_j+2}{2}}\right), \quad (29)$$

$$\frac{\partial^2 P}{\partial x_i^2} = -\frac{16\beta(\beta+1)P}{\left(m_j+2\right)^2 r^4}x_j^{m_j}\left(x_j^{\frac{m_j+2}{2}} - \xi_j^{\frac{m_j+2}{2}}\right)^2$$

$$-\frac{4\beta\left(m_j+1\right)P}{\left(m_j+2\right)r^2}x_j^{m_j} + \frac{2\beta m_j P}{\left(m_j+2\right)r^2}x_j^{\frac{m_j-2}{2}}\xi_j^{\frac{m_j+2}{2}}, \quad (30)$$

After some simple calculations, by virtue of (25)–(30), we find ($k=\overline{1,n}$)

$$A_k = -\frac{4x^{m+1}P(r)}{r^2}\left(\frac{\xi_k}{x_k}\right)^{\frac{m_k+2}{2}}\sigma_k\,(1-\sigma_k)\,; \quad (31)$$

$$B_{k,l} = \frac{4x^{m+1}P(r)}{r^2}\left[\left(\frac{\xi_k}{x_k}\right)^{\frac{m_k+2}{2}} + \left(\frac{\xi_l}{x_l}\right)^{\frac{m_l+2}{2}}\right]\sigma_k\sigma_l,\ k<l,\ l=\overline{1,n}; \quad (32)$$

$$C_k = -\frac{4x^{m+1}P(r)}{r^2}\left\{[2\alpha_k-(1+\beta)\sigma_k]\left(\frac{\xi_k}{x_k}\right)^{\frac{m_k+2}{2}} - \sigma_k\sum_{j=1}^{n}\alpha_j\left(\frac{\xi_j}{x_j}\right)^{\frac{m_j+2}{2}}\right\}; \quad (33)$$

$$D = \frac{4\beta x^{m+1}P(r)}{r^2}\sum_{i=1}^{n}\alpha_i\left(\frac{\xi_j}{x_j}\right)^{\frac{m_j+2}{2}}. \quad (34)$$

Substituting Equations (31)–(34) into Equation (24), we get the system of hypergeometric equations:

$$\left\{\begin{array}{l}\sigma_k\,(1-\sigma_k)\,\dfrac{\partial^2\omega}{\partial\sigma_k^2} - \sigma_k\displaystyle\sum_{j=1,j\neq k}^{n}\sigma_j\frac{\partial^2\omega}{\partial\sigma_j\partial\sigma_k} + [2\alpha_k-(1+\alpha_k+\beta)\,\sigma_k]\,\frac{\partial\omega}{\partial\sigma_k}\\ \qquad -\alpha_k\sum_{j=1,j\neq k}^{n}\sigma_j\dfrac{\partial\omega}{\partial\sigma_j} - \alpha_k\beta\omega = 0,\ k=\overline{1,n},\end{array}\right. \quad (35)$$

Comparing now the system (35) with the system (12), it is easy to determine the solutions of the system (35):

$$1:\{\,F_A^{(n)}\left[\begin{array}{l}\beta,\alpha_1,\ldots,\alpha_n;\\ 2\alpha_1,\ldots,2\alpha_n;\end{array}\ \sigma\right],$$

$$n:\left\{\begin{array}{l} \sigma_1^{1-2\alpha_1}F_A^{(n)}\left[\begin{array}{l}\beta+1-2\alpha_1,1-\alpha_1,\alpha_2,\dots,\alpha_n;\\ 2-2\alpha_1,2\alpha_2,\dots,2\alpha_n;\end{array}\sigma\right],\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \sigma_n^{1-2\alpha_n}F_A^{(n)}\left[\begin{array}{l}\beta+1-2\alpha_n,\alpha_1,\dots,\alpha_{n-1},1-\alpha_n;\\ 2\alpha_1,\dots,2\alpha_{n-1},2-2\alpha_n;\end{array}\sigma\right],\end{array}\right.$$

$$\frac{n(n-1)}{1\cdot 2}:\left\{\begin{array}{l} \sigma_1^{1-2\alpha_1}\sigma_2^{1-2\alpha_2}F_A^{(n)}\left[\begin{array}{l}\beta+2-2\alpha_1-2\alpha_2,\\ 1-\alpha_1,1-\alpha_2,\alpha_3,\dots,\alpha_n;\\ 2-2\alpha_1,2-2\alpha_2,2\alpha_3,\dots,2\alpha_n;\end{array}\sigma\right],\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \sigma_1^{1-2\alpha_1}\sigma_n^{1-2\alpha_n}F_A^{(n)}\left[\begin{array}{l}\beta+2-2\alpha_1-2\alpha_n,\\ 1-\alpha_1,\alpha_2,\dots,\alpha_{n-1},1-\alpha_n;\\ 2-2\alpha_1,2\alpha_2,\dots,2\alpha_{n-1},2-2\alpha_n;\end{array}\sigma\right],\\ \sigma_2^{1-2\alpha_2}\sigma_3^{1-2\alpha_3}F_A^{(n)}\left[\begin{array}{l}\beta+2-2\alpha_2-2\alpha_3,\\ \alpha_1,1-\alpha_2,1-\alpha_3,\alpha_4,\dots,\alpha_n;\\ 2\alpha_1,2-2\alpha_2,2-2\alpha_3,2\alpha_4,\dots,2\alpha_n;\end{array}\sigma\right],\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \sigma_{n-1}^{1-2\alpha_{n-1}}\sigma_n^{1-2\alpha_n}F_A^{(n)}\left[\begin{array}{l}\beta+2-2\alpha_{n-1}-2\alpha_n,\\ \alpha_1,\dots,\alpha_{n-2},1-\alpha_{n-1},1-\alpha_n;\\ 2\alpha_1,\dots,2\alpha_{n-2},2-2\alpha_{n-1},2-2\alpha_n;\end{array}\sigma\right],\end{array}\right.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$1:\left\{\sigma_1^{1-2\alpha_1}\cdots\sigma_n^{1-2\alpha_n}F_A^{(n)}\left[\begin{array}{l}\beta+n-2\alpha_1-\cdots-2\alpha_n,1-\alpha_1,\dots,1-\alpha_n;\\ 2-2\alpha_1,\dots,2-2\alpha_n;\end{array}\sigma\right],\right.$$

and substituting these solutions in (23), by virtue of (13), we obtain the fundamental solutions of Equation (22) in the form

$$\begin{aligned} q_k(x;\xi) &= \gamma_k r^{-2\beta_k}\prod_{j=1}^{k}\left[\left(x_j\xi_j\right)^{1-p_j}\right] \\ &\quad\times F_A^{(n)}\left[\begin{array}{l}\beta_k,1-\alpha_1,\dots,1-\alpha_k,\alpha_{k+1},\dots,\alpha_n;\\ 2-2\alpha_1,\dots,2-2\alpha_k,2\alpha_{k+1},\dots,2\alpha_n;\end{array}\sigma_1,\dots,\sigma_n\right], k=\overline{0,n}, \end{aligned} \tag{36}$$

where γ_k are constants, which are determined when solving boundary value problems for Equation (22);

$$\beta_k=\frac{n-2}{2}+k-\sum_{j=1}^{k}\alpha_j+\sum_{j=k+1}^{n}\alpha_j,\quad k=\overline{0,n};\sigma_j=1-\frac{r_j^2}{r^2},\quad j=\overline{1,n}.$$

Lemma 2: *The fundamental solutions $q_k(x;\xi)$ of Equation (22) defined in (36) have a following properties:*

$$\left.\frac{\partial q_0(x;\xi)}{\partial x_j}\right|_{x_j=0}=0,\quad j=\overline{1,n}; \tag{37}$$

$$\left.q_k(x;\xi)\right|_{x_j=0}=0,\quad j=\overline{1,k},k=\overline{1,n-1}; \tag{38}$$

$$\left.\frac{\partial q_k(x;\xi)}{\partial x_j}\right|_{x_j=0} = 0, \quad j=\overline{k+1,n},\ k=\overline{1,n-1}; \tag{39}$$

$$q_n(x;\xi)\big|_{x_j=0} = 0, \quad j=\overline{1,n}. \tag{40}$$

Proof: The fulfilment of the equalities (38) and (40) follows directly from the definition of the fundamental solution (36).

To prove the equalities (37) and (39) Theorem 1 on the relation between contiguous hypergeometric Lauricella functions is used. Indeed, let us consider the fundamental solution

$$q_0(x;\xi) = \gamma_0 r^{-2\beta_0} F_A^{(n)}\left[\begin{matrix}\beta_0,\alpha_1,\ldots,\alpha_n; \\ 2\alpha_1,\ldots,2\alpha_n;\end{matrix}\ \sigma_1,\ldots,\sigma_n\right], \quad \beta_0 = \frac{n-2}{2} + \sum_{j=1}^{n}\alpha_j$$

and calculate the derivative with respect to $x_1,\ldots,x_n$:

$$\frac{\partial q_0}{\partial x_j} = -\beta_0\gamma_0 r^{-2\beta_0-2}\frac{\partial r^2}{\partial x_j}F_A^{(n)} + \gamma_0 r^{-2\beta_0}\sum_{k=1}^{n}\frac{\partial F_A^{(n)}}{\partial \sigma_k}\frac{\partial \sigma_k}{\partial x_j}.$$

Now using the ready derivatives (25), (26) and

$$\frac{\partial}{\partial \sigma_k}F_A^{(n)}\left[\begin{matrix}a,b_1,\ldots,b_n; \\ c_1,\ldots,c_n;\end{matrix}\ \sigma\right] = \frac{ab_k}{c_k}F_A^{(n)}\left[\begin{matrix}a+1,b_1,\ldots,b_{k-1},b_k+1,b_{k+1},\ldots,b_n; \\ c_1,\ldots,c_{k-1},c_k+1,c_{k+1},\ldots,c_n;\end{matrix}\ \sigma\right],$$

and also a relation between contiguous Lauricella functions (14) (see Theorem 1), we get

$$\frac{\partial q_0}{\partial x_j} = -\frac{4\beta_0\gamma_0}{m_j+2}r^{-2\beta_0-2}x^{\frac{m_j}{2}}\left\{x^{\frac{m_j+2}{2}}F_A^{(n)}\left[\begin{matrix}\beta_0+1,\alpha_1,\ldots,\alpha_n; \\ 2\alpha_1,\ldots,2\alpha_n;\end{matrix}\ \sigma\right] - \xi^{\frac{m_j+2}{2}}\tilde{F}_A^{(n)}\right\},$$

where

$$\begin{aligned}\tilde{F}_A^{(n)} &= F_A^{(n)}\left[\begin{matrix}\beta_0+1,\alpha_1,\ldots,\alpha_n; \\ 2\alpha_1,\ldots,2\alpha_n;\end{matrix}\ \sigma\right] \\ &- F_A^{(n)}\left[\begin{matrix}\beta_0+1,\alpha_1,\ldots,\alpha_{j-1},\alpha_j+1,\alpha_{j+1},\ldots,\alpha_n; \\ 2\alpha_1,\ldots,2\alpha_{j-1},2\alpha_j+1,2\alpha_{j+1},\ldots,2\alpha_n;\end{matrix}\ \sigma\right].\end{aligned}$$

Since $\lim_{x_j\to 0}\sigma_j = 0$, then $\lim_{x_j\to 0}\tilde{F}_A^{(n)} = 0$. Hence, the equality (37) is valid.

The equality (39) is proved in a similar way. Q.E.D. ■

Lemma 3: *In the case $n > 2$ the fundamental solutions $q_k(x;\xi)(k=\overline{1,n})$ of Equation (22) have a singularity of order $\dfrac{1}{r^{n-2}}$ as $r \to 0$.*

Proof: Using the transformation for the Lauricella function $F_A^{(n)}$ [26, p. 116, Equation (9)]

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\sigma_1,\dots,\sigma_n\right)=(1-|\sigma|)^{-a}F_A^{(n)}\left(a,\mathbf{c}-\mathbf{b};\mathbf{c};\frac{\sigma_1}{|\sigma|-1},\dots,\frac{\sigma_n}{|\sigma|-1}\right),$$

where $|\sigma|:=\sigma_1+\cdots+\sigma_n$, we rewritten the fundamental solution $q_k(x;\xi)$ of Equation (22) defined in (36) in the form

$$q_k(x;\xi)=\gamma_k R^{-2\beta_k}\prod_{j=1}^{k}\left[\left(x_j\xi_j\right)^{1-p_j}\right]$$
$$\times F_A^{(n)}\left[\begin{matrix}\beta_k,1-\alpha_1,\dots,1-\alpha_k,\alpha_{k+1},\dots,\alpha_n; & \frac{\tilde{\sigma}_1}{R^2},\dots,\frac{\tilde{\sigma}_n}{R^2}\\ 2-2\alpha_1,\dots,2-2\alpha_k,2\alpha_{k+1},\dots,2\alpha_n; & \end{matrix}\right], \tag{41}$$

where

$$R^2=\sum_{j=1}^{n}\frac{4}{\left(m_j+2\right)^2}\left(x_j^{\frac{m_j+2}{2}}+\xi_j^{\frac{m_j+2}{2}}\right)^2;\quad \tilde{\sigma}_k=\frac{16}{(m_k+2)^2}x_k^{\frac{m_k+2}{2}}\xi_k^{\frac{m_k+2}{2}},\ k=\overline{1,n}.$$

It is easy to calculate the limit of the sum of variables of the Lauricella function in (41):

$$\sum_{j=1}^{n}\frac{\tilde{\sigma}_j}{R^2}=1-\frac{r^2}{R^2}\to 1-0,\quad r\to 0.$$

Since in the representation (41) the sum of the upper parameters of the Lauricella function $F_A^{(n)}$ is greater than the sum of the lower parameters, then one can use assertion (21) of Theorem 4 to determine the behaviour of the fundamental solution $q_k(x;\xi)$:

$$q_k(x;\xi)\sim\gamma_k R^{-2\beta_k}\left(\frac{R}{r}\right)^{n-2}\prod_{j=1}^{k}\left[\frac{\Gamma\left(2-2\alpha_j\right)}{\Gamma(1-\alpha_j)}\left(x_j\xi_j\right)^{1-p_j}\left(\frac{\tilde{\sigma}_j}{R^2}\right)^{\alpha_j-1}\right]$$
$$\times\prod_{j=k+1}^{n}\left[\frac{\Gamma\left(2\alpha_j\right)}{\Gamma(\alpha_j)}\left(\frac{\tilde{\sigma}_j}{R^2}\right)^{-\alpha_j}\right],$$

i.e.

$$q_k(x;\xi)\sim\frac{\tilde{\gamma}_k}{r^{n-2}}\prod_{j=1}^{n}\left(x_j\xi_j\right)^{-\left(m_j+2p_j\right)/4}, \tag{42}$$

where

$$\tilde{\gamma}_k=\prod_{j=1}^{k}\left[\frac{\Gamma\left(2-2\alpha_j\right)}{\Gamma(1-\alpha_j)}\left(\frac{4}{m_j+2}\right)^{2\alpha_j-2}\right]\prod_{j=k+1}^{n}\left[\frac{\Gamma\left(2\alpha_j\right)}{\Gamma(\alpha_j)}\left(\frac{4}{m_j+2}\right)^{-2\alpha_j}\right].$$

Since $x_j>0$, $\xi_j>0$ and $m_j+2p_j>0$ $(j=\overline{1,n})$, the last relation (42) shows that the fundamental solutions $q_k(x;\xi)$ $(k=\overline{0,n})$ for $n>2$ have a singularity of order $\frac{1}{r^{n-2}}$ as $r\to 0$. Q.E.D. ■

Thus, the main result of this Section can be formulated as follows

Remark 3: In paper [19] it is proved that in case $n = 2$ the fundamental solutions of Equation (22) have a logarithmic singularity $\ln \frac{1}{r}$ as $r \to 0$.

Theorem 5: . *The functions defined in (36) are the fundamental solutions of Equation (22) and satisfy the boundary conditions (37)–(40).*

Proof: The proof of Theorem 5 follows from Lemmas 2 and 3. ■

5. Conclusion

Note that using the constructed fundamental solutions (36), one can find simple and double layer potentials, volume potentials, and also Green's functions associated with Equation (22), which are used in solving boundary value problems.

By the above method, taking into account new properties of the Lauricella's hypergeometric function established in Section 3, fundamental solutions of the multidimensional degenerate elliptic equation

$$\sum_{k=1}^{n} \prod_{j=1,j\neq k}^{n} \left[x_j^{m_j}\right] \frac{\partial^2 u}{\partial x_k^2} = 0, \quad m_j > 0,\ x_j > 0,\ j = \overline{1,n}$$

can also be constructed.

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