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Article

Self-Similar Solutions of a Multidimensional Degenerate Partial Differential Equation of the Third Order

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Abstract: When studying the boundary value problems' solvability for some partial differential equations encountered in applied mathematics, we frequently need to create systems of partial differential equations and explicitly construct linearly independent solutions explicitly for these systems. Hypergeometric functions frequently serve as solutions that satisfy these systems. In this study, we develop self-similar solutions for a third-order multidimensional degenerate partial differential equation. These solutions are represented using a generalized confluent Kampé de Fériet hypergeometric function of the third order.

Keywords: degenerate partial differential equation; self-similar solution; confluent hypergeometric function; generalized confluent Kampé de Fériet hypergeometric function; hypergeometric-type system

MSC: 33C20; 34A30; 35L25; 35L80

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1. Introduction: Notations and Definitions

The theory of special functions, as a field of mathematical analysis devoted to the study and application of higher transcendental functions, has a long history and rich content due to the penetration and interrelationships with a wide range of topics in the theory of functions, integral and differential equations, and other branches of mathematics. Solving various issues concerning thermal conductivity, lectromagnetic oscillations, aquantum mechanics, eromechanics, dynamics and potential theories gives rise to special functions. They most often appear when solving partial differential equations [1,2].

The significant successes achieved in the theory of the hypergeometric function of one variable have stimulated the development of corresponding theories for functions of two or more variables.

A significant contribution to the theory of two-dimensional hypergeometric functions was conducted by Horn [3], who particularly studied the hypergeometric series of the second order. He discovered that in addition to some functions representable in terms of one variable functions or multiplication of two hypergeometric functions, every depending on one variable, there are 34 different hypergeometric functions of the second order (known as the Horn list). Thus, in the case of two variables, there are 14 complete and 20 confluent hypergeometric functions of the second order. Horn also studied the convergence of hypergeometric functions of two variables [4] (pp. 227–229). In general, Horn concentrated on the study of second-order hypergeometric functions of two independent variables.

Hypergeometric functions of two variables whose order exceeds two are called Kampé de Fériet functions [5] (p. 27).

The purpose of this work is to find self-similar solutions for a multidimensional degenerate third-order partial differential equation.

$$\prod_{j=1}^{n} x_j^{m_j} \cdot \frac{\partial u}{\partial t} - t^l \sum_{k=1}^{n} \left(\prod_{j=1, j\neq k}^{n} x_j^{m_j} \right) \frac{\partial^3 u}{\partial x_k^3} = 0, \;\; l > 0, \;\; m_j > 0, \;\; j = \overline{1, n} \tag{1}$$

in the area $\Omega = \{(\mathbf{x}, t) : x_1 > 0, ..., x_n > 0, t > 0\}$. Further, $\mathbf{x} := (x_1, ..., x_n)$. Note that many degenerate partial differential equations of the third and higher orders appear when solving applied problems [6–8].

When studying natural phenomena and solving problems in physics, engineering, chemistry, biology, and other sciences, it is often challenging to directly establish a relationship between the quantities that describe a particular evolutionary process. However, in many cases, it is possible to relate these quantities (functions) to the rates of their change concerning other (independent) variables. This leads to the formulation of equations that include unknown functions under the derivative sign, known as differential equations.

For instance, the work cited in [9] utilizes both analytical and experimental results to mathematically reveal a deterioration in material stiffness linked to a decrease in the first natural frequency when the excitation frequency is slightly lower than that of an intact structure. Considering the vibration of a homogeneous thin cantilever beam as an oscillatory system with worsening hysteresis behavior, Hasanov and Djuraev [10] derived exact solutions to the fourth-order equation, as follows:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{2a}{t} + c^2 \frac{\partial^4 u(x,t)}{\partial x^4} = 0, \;\; c = \sqrt{\frac{EI}{\rho S}},$$

where ρ is the density, S is the cross section area, EI is the bending stiffness of a beam.

Equation (1) considered above belongs to the type of parabolic equations that arise in modeling processes such as liquid filtration in fractured porous media, groundwater movement with a free surface in multilayer media, and the transfer of moisture, heat, and salts in porous media [11]. Studying these processes requires solving boundary value problems for a third-order equation.

As is known, if the order of degenerate and singular partial differential equations with three variables exceeds two, then any solutions to them are expressed in terms of the Kampé de Fériet hypergeometric function, the order of which is equal to the order of the equation in question. For example, in [12], all eight self-similar solutions of equation are as follows:

$$Lu = x^n y^m u_t - t^k y^m u_{xxx} - t^k x^n u_{yyy} = 0,$$

where m, n, k are positive constant numbers in the domain $\Omega = \{(x, y, t) : \; x > 0, \; y > 0, \; t > 0\}$ and are written out using the Kampé de Fériet hypergeometric function (of two independent variables). The following works are adjacent to this line of research [13–17].

To solve this problem, following the Horn method [3], we define a general hypergeometric function of many variables and compose a system of partial differential equations that satisfies any hypergeometric function of many variables. We introduce the Kampé de Fériet function of many (more than two) variables and determine the linearly independent solutions of the system of differential equations corresponding to this multidimensional Kampé de Fériet function. Finally, we write out self-similar solutions of Equation (1) through a generalized confluent hypergeometric Kampé de Fériet function of the third order.

1.1. Multivariable Hypergeometric Functions

Consider a given multiple power series, as follows:

$$\sum_{|\mathbf{k}|=0}^{\infty} A(\mathbf{k}) x_1^{k_1} x_2^{k_2} x_3^{k_3} \ldots x_n^{k_n}, \tag{2}$$

where the summation is executed according to the index $\mathbf{k} := (k_1, \ldots, k_n)$ with non-negative integer elements $k_j \geq 0$, $i = \overline{1,n}$ for $|\mathbf{k}| := k_1 + \ldots + k_n$.

A multiple power series (2) is a hypergeometric function if the following n relations

$$\frac{A(\mathbf{k} + \mathbf{e}_j)}{A(\mathbf{k})} = f_j(\mathbf{k})$$

are rational functions that depend on $\mathbf{k}$, where $\mathbf{e}_j := (0, \ldots, 0, 1, 0, \ldots 0)$ is a vector consisting of n components, the j-th component of which is equal to one, and the rest are zero.

Let

$$f_j(\mathbf{k}) = \frac{P_j(\mathbf{k})}{Q_j(\mathbf{k})}, \tag{3}$$

where P_j and Q_j are polynomials of $\mathbf{k}$, which have, respectively, degrees p_j and q_j. It is assumed that Q_j has a multiplier $k_j + 1$; P_j and Q_j they do not have common multipliers, except perhaps $k_j + 1$ $(j = \overline{1,n})$.

The largest of the numbers $p_1, \ldots, p_n, q_1, \ldots, q_n$ is called the *order* of the hypergeometric function.

If all the numbers $p_1, \ldots, p_n, q_1, \ldots, q_n$ are the same, i.e., $p_1 = \ldots = p_n = q_1 = \ldots = q_n$, then the hypergeometric function (2) is called *complete*; otherwise, it is *confluent*.

In the case of two variables, there are 14 complete functions for which $p_1 = q_1 = p_2 = q_2 = 2$, and there are 20 confluent functions (Horn's list [3], see also, [4] (pp. 224–227)), which are the limit forms for complete functions and for which $p_1 \leq q_1 = 2$, $p_2 \leq q_2 = 2$; moreover, p_1 and p_2 cannot both be equal to two at the same time.

In the case of three variables, there are 205 complete [5] and 395 confluent [18] second-order hypergeometric functions that are known.

The hypergeometric function of many variables is

$$z(\mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty} A(\mathbf{k}) x_1^{k_1} x_2^{k_2} x_3^{k_3} \ldots x_n^{k_n},$$

where

$$\frac{A(\mathbf{k}_j + 1)}{A(\mathbf{k})} = \frac{P_j(\mathbf{k})}{Q_j(\mathbf{k})}$$

and $P_j(\mathbf{k})$, $P_j(\mathbf{k})$ $(j = \overline{1,n})$ are the same polynomials as in (3) and fulfill a set of linear partial differential equations. This system may be expressed using differential operators, as follows:

$$\delta_j := x_j \frac{\partial}{\partial x_j}, \; j = \overline{1,n} \tag{4}$$

in the form of

$$\left[Q_j(\delta) x_j^{-1} - P_j(\delta)\right] z(\mathbf{x}), \; j = \overline{1,n}. \tag{5}$$

Here, the differential operators $P_j(\delta)$ and $Q_j(\delta)$ were chosen by replacing the elements of the vector $\delta := (\delta_1, \ldots, \delta_n)$ as variables of the polynomials P_j and Q_j, defined in (3).

Horn, investigating the convergence of two dimensional hypergeometric series, established a partial differential equations' systems that they fulfilled [4] (pp. 232–236).

Systems of differential equations and integral representations for all 205 complete second-order hypergeometric functions of the three variables were obtained in [19,20], and relatively few works have been devoted to the study of confluent hypergeometric functions of the three variables [21–23].

1.2. Generalized Kampé de Fériet Function of Many Variables

The Kampé de Fériet functions of two variables have been studied quite well in the mathematical literature. Decomposition formulas (expansions), some summation and analytic continuation formulas and other new results on the double Kampé de Fériet functions, can be found in [24–30].

The next generalization of the Kampé de Fériet functions (two-dimensional case) belongs to Srivastava H.M. and Panda R. [31] (p. 1127, Formula (4.1)), which really defined the extension of the Kampé de Fériet functions from many variables:

$$F^{p:q_1;\dots;q_n}_{l:m_1;\dots;m_n}(\mathbf{x}) \equiv F^{p:q_1;\dots;q_n}_{l:m_1;\dots;m_n}\left[\begin{matrix}(a_p):\left(b'_{q_1}\right);\dots;\left(b^{(n)}_{q_n}\right);\\(\alpha_l):\left(\beta'_{q_1}\right);\dots;\left(\beta^{(n)}_{q_n}\right);\end{matrix} x_1,\dots,x_n\right]=$$

$$\sum_{|\mathbf{k}|=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\left(a_j\right)_{|\mathbf{k}|}\prod\limits_{j=1}^{q_1}\left(b'_j\right)_{k_1}\dots\prod\limits_{j=1}^{q_n}\left(b^{(n)}_j\right)_{k_n}}{\prod\limits_{j=1}^{l}\left(\alpha_j\right)_{|\mathbf{k}|}\prod\limits_{j=1}^{m_1}\left(\beta'_j\right)_{k_1}\dots\prod\limits_{j=1}^{m_n}\left(\beta^{(n)}_j\right)_{k_n}}\frac{x_1^{k_1}}{k_1!}\dots\frac{x_n^{k_n}}{k_n!}, \tag{6}$$

where $(\lambda)_\nu$ is the Pohhammer symbol, i.e.,

$$(\lambda)_0 = 1, (\lambda)_\nu = \lambda(\lambda+1)\cdot\dots\cdot(\lambda+\nu-1) = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)},$$

and $\Gamma(z)$ is known Gamma function.

For convergence of the hypergeometric series (6), the following conditions must be met

$$1+l+m_j-p-q_j \geq 0,\ \ j=\overline{1,n};$$

at equality, i.e., at

$$1+l+m_j-p-q_j = 0,\ \ j=\overline{1,n},$$

it is additionally required that either

$$p>l,\ \ |x_1|^{1/(p-l)}+\dots+|x_n|^{1/(p-l)},$$

or

$$p \leq l,\ \ \max\{|x_1|,\dots,|x_n|\}<1.$$

Along with the confluent Kampé de Fériet functions, there are certain Kampé de Fériet-like hypergeometric matrix functions [32].

Karlsson [33] considered a special case of the Kampé de Fériet hypergeometric function of many variables (6) when

$$q_j = q,\quad m_j = m,\quad j=\overline{1,n},$$

and this special case is designated more simply as $F^{p:q}_{l:m}$.

Introduce the generalized confluent hypergeometric function of Kampé de Fériet as follows:

$$F^{1:0}_{0:2}(a;\mathbf{b},\mathbf{c};\mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty}(a)_{|\mathbf{k}|}\prod_{j=1}^{n}\frac{x_j^{k_j}}{k_j!\left(b_j\right)_{k_j}\left(c_j\right)_{k_j}}, \tag{7}$$

where a, b_j and c_j are real numbers, with $b_j, c_j \neq 0,-1,-2,\dots, j=\overline{1,n}$.

List some properties of the Kampé de Fériet function $F^{1:0}_{0:2}$.

Symmetry in the vector of numerical parameters:

$$F_{0:2}^{1:0}(a;\mathbf{b},\mathbf{c};\mathbf{x}) = F_{0:2}^{1:0}(a;\mathbf{c},\mathbf{b};\mathbf{x}). \tag{8}$$

Symmetry for each numeric parameter:

$$F_{0:2}^{1:0}(a;\mathbf{b},\mathbf{c};\mathbf{x}) = F_{0:2}^{1:0}(a;b_1,...,b_{k-1},c_k,b_{k+1},...,b_n,c_1,...,c_{k_1},b_k,c_{k+1},...,c_n;\mathbf{x}),\ k=\overline{1,n}. \tag{9}$$

Differentiation formula:

$$\frac{\partial^{|\mathbf{k}|}}{\partial x_1^{k_1}...\partial x_n^{k_n}} F_{0:2}^{1:0}(a;\mathbf{b},\mathbf{c};\mathbf{x}) = \frac{(a)_{|\mathbf{k}|}}{(b_1)_{k_1}...(b_n)_{k_n}(c_1)_{k_1}...(c_n)_{k_n}} F_{0:2}^{1:0}(a+|\mathbf{k}|;\mathbf{b}+\mathbf{k},\mathbf{c}+\mathbf{k};\mathbf{x}).$$

2. System of Differential Equations

Following Horn [3], let us compose a system of differential equations of the hypergeometric type for a function of the form (7).

By entering the designation

$$A(\mathbf{k}) = \frac{(a)_{k_1+...+k_n}}{k_1!...k_n!(b_1)_{k_1}....(b_n)_{k_n}(c_1)_{k_1}....(c_n)_{k_n}},$$

make a relationship

$$\frac{A(\mathbf{k}+\mathbf{e}_j)}{A(\mathbf{k})} = \frac{a+k_1+...+k_n}{(1+k_j)(b_j+k_j)(c_j+k_j)},\ j=\overline{1,n}.$$

From here,

$$P_j(\mathbf{k}) = a+k_1+...+k_n,\ j=\overline{1,n},$$

$$Q_j(\mathbf{k}) = (1+k_j)(b_j+k_j)(c_j+k_j),\ j=\overline{1,n}.$$

Now, by substituting $P_j(\mathbf{k})$ and $Q_j(\mathbf{k})$ into the system (5) and using the definition (4) of differential operators δ_1, ..., δ_n, we have

$$\left\{\left[\left(b_j+x_j\frac{\partial}{\partial x_j}\right)\left(c_j+x_j\frac{\partial}{\partial x_j}\right)\left(1+x_j\frac{\partial}{\partial x_j}\right)\frac{1}{x_j}-\left(a+\sum_{k=1}^{n}x_k\frac{\partial}{\partial x_k}\right)\right]z(\mathbf{x})=0,\ j=\overline{1,n}, \right. \tag{10}$$

where $z(\mathbf{x}) = F_{0:2}^{1:0}(a;\mathbf{b},\mathbf{c};\mathbf{x})$. By unwrapping the brackets in the system (10), we get

$$\left\{x_j^2\frac{\partial^3 z}{\partial x_j^3}+(b_j+c_j+1)x_j\frac{\partial^2 z}{\partial x_j^2}+b_jc_j\frac{\partial z}{\partial x_j}-\sum_{k=1}^{n}x_j\frac{\partial z}{\partial x_j}-az=0,\ j=\overline{1,n}. \right. \tag{11}$$

Thus, the hypergeometric function $F_{0:2}^{1:0}(a;\mathbf{b},\mathbf{c};\mathbf{x})$ defined by the equality (7) indeed satisfies the system of differential Equation (11).

Note that the system of differential Equation (11) in the case $n=2$ can be found in [12].

Theorem 1. *The system of differential Equation (11) near the origin has 3^n linearly independent solutions.*

Proof. To obtain the general integral of the system (11), it is enough to note that it does not change its form when replacing

$$z(\mathbf{x}) = \prod_{k=1}^{n} x_j^{\lambda_j}\cdot F_{0:2}^{1:0}(a;\mathbf{b},\mathbf{c};\mathbf{x}), \tag{12}$$

where λ_j represents arbitrary numbers that need to be determined.

Having calculated the necessary derivatives of the function (12) and substituting them into system (11), we obtain

$$\begin{cases} x_j^2 \dfrac{\partial^3 z}{\partial x_j^3} + (3\lambda_j + b_j + c_j + 1) x_j \dfrac{\partial^2 z}{\partial x_j^2} + \left[3\lambda_j(\lambda_j - 1) + 2\lambda_j(b_j + c_j + 1) + b_j c_j - x_j\right] \dfrac{\partial z}{\partial x_j} + \\ \left[\lambda_j(\lambda_j - 1 + b_j)(\lambda_j - 1 + c_j) x_j^{-1} - (\lambda_1 + \ldots + \lambda_n + a)\right] z = 0, \ \ j = \overline{1, n}. \end{cases} \tag{13}$$

From here, we obtain the indicate system

$$\{\lambda_j(\lambda_j - 1 + b_j)(\lambda_j - 1 + c_j) = 0, \ \ j = \overline{1, n}. \tag{14}$$

Thus, we have 3^n solutions.

Indeed, if $\lambda_j \neq 1 - b_j$ and $\lambda_j \neq 1 - c_j$ $(j = \overline{1, n})$, then the system (14) has one (first) trivial solution, as follows:

$$\begin{array}{cccccccccc} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_{n-3} & \lambda_{n-2} & \lambda_{n-1} & \lambda_n \\ 1: & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \end{array}$$

if either $\lambda_j = 1 - b_j$ and $\lambda_j \neq 1 - c_j$, or $\lambda_j \neq 1 - b_j$ and $\lambda_j = 1 - c_j$ $(j = \overline{1, n})$, then the system (14) has $2\,C_n^1$ solutions, as follows:

$$\begin{array}{ccccccccccc} & & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_{n-3} & \lambda_{n-2} & \lambda_{n-1} & \lambda_n \\ C_n^1: & \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. & \begin{array}{c} 1-b_1 \\ 0 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 1-b_2 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1-b_3 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ldots \\ 1-b_n \end{array} \end{array}$$

In a similar way, we will find all solutions of the system (14), as follows:

$$\begin{array}{ccccccccccc} & & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_{n-3} & \lambda_{n-2} & \lambda_{n-1} & \lambda_n \\ C_n^1: & \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. & \begin{array}{c} 1-c_1 \\ 0 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 1-c_2 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1-c_3 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \ldots \\ 1-c_n \end{array} \end{array}$$

$$
\begin{array}{ccccccccccc}
 & & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_{n-3} & \lambda_{n-2} & \lambda_{n-1} & \lambda_n
\end{array}
$$

$$
C_n^2 : \left\{
\begin{array}{ccccccccc}
1-b_1 & 1-b_2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1-b_1 & 0 & 1-b_3 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1-b_1 & 0 & 0 & 1-b_4 & \ldots & 0 & 0 & 0 & 0 \\
1-b_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1-b_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1-b_n \\
0 & 1-b_2 & 1-b_3 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1-b_2 & 0 & 1-b_3 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1-b_2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 1-b_2 & 0 & 0 & \ldots & 1-b_{n-3} & 0 & 0 & 0 \\
0 & 1-b_2 & 0 & 0 & \ldots & 0 & 1-b_{n-2} & 0 & 0 \\
0 & 1-b_2 & 0 & 0 & \ldots & 0 & 0 & 1-b_{n-1} & 0 \\
0 & 1-b_2 & 0 & 0 & \ldots & 0 & 0 & 0 & 1-b_n \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 1-b_{n-3} & 1-b_{n-2} & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 1-b_{n-3} & 0 & 1-b_{n-1} & 0 \\
0 & 0 & 0 & 0 & \ldots & 1-b_{n-3} & 0 & 0 & 1-b_n \\
0 & 0 & 0 & 0 & \ldots & 0 & 1-b_{n-2} & 1-b_{n-1} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1-b_{n-2} & 0 & 1-b_n \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1-b_{n-1} & 1-b_n
\end{array}
\right.
$$

$$
\begin{array}{ccccccccccc}
 & & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_{n-3} & \lambda_{n-2} & \lambda_{n-1} & \lambda_n
\end{array}
$$

$$
C_n^2 : \left\{
\begin{array}{ccccccccc}
1-b_1 & 1-c_2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1-b_1 & 0 & 1-c_3 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1-b_1 & 0 & 0 & 1-c_4 & \ldots & 0 & 0 & 0 & 0 \\
1-b_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1-b_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1-c_n \\
0 & 1-b_2 & 1-c_2 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1-b_2 & 0 & 1-c_3 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1-b_2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 1-b_2 & 0 & 0 & \ldots & 1-c_{n-3} & 0 & 0 & 0 \\
0 & 1-b_2 & 0 & 0 & \ldots & 0 & 1-c_{n-2} & 0 & 0 \\
0 & 1-b_2 & 0 & 0 & \ldots & 0 & 0 & 1-c_{n-1} & 0 \\
0 & 1-b_2 & 0 & 0 & \ldots & 0 & 0 & 0 & 1-c_n \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 1-b_{n-3} & 1-c_{n-2} & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 1-b_{n-3} & 0 & 1-c_{n-1} & 0 \\
0 & 0 & 0 & 0 & \ldots & 1-b_{n-3} & 0 & 0 & 1-b_n \\
0 & 0 & 0 & 0 & \ldots & 0 & 1-b_{n-2} & 1-c_{n-1} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1-b_{n-2} & 0 & 1-c_n \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1-b_{n-1} & 1-c_n
\end{array}
\right.
$$

$$
C_n^2 : \left\{ \begin{array}{ccccccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \dots & \lambda_{n-3} & \lambda_{n-2} & \lambda_{n-1} & \lambda_n \\
1-c_1 & 1-b_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
1-c_1 & 0 & 1-b_3 & 0 & \dots & 0 & 0 & 0 & 0 \\
1-c_1 & 0 & 0 & 1-b_4 & \dots & 0 & 0 & 0 & 0 \\
1-c_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
1-c_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1-b_n \\
0 & 1-c_2 & 1-b_2 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 1-c_2 & 0 & 1-b_3 & \dots & 0 & 0 & 0 & 0 \\
0 & 1-c_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 1-c_2 & 0 & 0 & \dots & 1-b_{n-3} & 0 & 0 & 0 \\
0 & 1-c_2 & 0 & 0 & \dots & 0 & 1-b_{n-2} & 0 & 0 \\
0 & 1-c_2 & 0 & 0 & \dots & 0 & 0 & 1-b_{n-1} & 0 \\
0 & 1-c_2 & 0 & 0 & \dots & 0 & 0 & 0 & 1-b_n \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & 1-c_{n-3} & 1-b_{n-2} & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 1-c_{n-3} & 0 & 1-b_{n-1} & 0 \\
0 & 0 & 0 & 0 & \dots & 1-c_{n-3} & 0 & 0 & 1-b_n \\
0 & 0 & 0 & 0 & \dots & 0 & 1-b_{n-2} & 1-c_{n-1} & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 1-b_{n-2} & 0 & 1-c_n \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1-b_{n-1} & 1-c_n
\end{array} \right.
$$

$$
C_n^2 : \left\{ \begin{array}{ccccccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \dots & \lambda_{n-3} & \lambda_{n-2} & \lambda_{n-1} & \lambda_n \\
1-c_1 & 1-c_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
1-c_1 & 0 & 1-c_3 & 0 & \dots & 0 & 0 & 0 & 0 \\
1-c_1 & 0 & 0 & 1-c_4 & \dots & 0 & 0 & 0 & 0 \\
1-c_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
1-c_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1-c_n \\
0 & 1-c_2 & 1-c_2 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 1-c_2 & 0 & 1-c_3 & \dots & 0 & 0 & 0 & 0 \\
0 & 1-c_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 1-c_2 & 0 & 0 & \dots & 1-c_{n-3} & 0 & 0 & 0 \\
0 & 1-c_2 & 0 & 0 & \dots & 0 & 1-c_{n-2} & 0 & 0 \\
0 & 1-c_2 & 0 & 0 & \dots & 0 & 0 & 1-c_{n-1} & 0 \\
0 & 1-c_2 & 0 & 0 & \dots & 0 & 0 & 0 & 1-c_n \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & 1-c_{n-3} & 1-c_{n-2} & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 1-c_{n-3} & 0 & 1-c_{n-1} & 0 \\
0 & 0 & 0 & 0 & \dots & 1-c_{n-3} & 0 & 0 & 1-c_n \\
0 & 0 & 0 & 0 & \dots & 0 & 1-c_{n-2} & 1-c_{n-1} & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 1-c_{n-2} & 0 & 1-c_n \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1-c_{n-1} & 1-c_n
\end{array} \right.
$$

$$
2^3 C_n^3: \quad \left\{ \begin{array}{ccccccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_{n-3} & \lambda_{n-2} & \lambda_{n-1} & \lambda_n \\
1-b_1 & 1-b_2 & 1-b_3 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 0 & 1-b_{n-2} & 1-b_{n-1} & 1-b_n \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1-c_1 & 1-c_2 & 1-c_3 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 0 & 1-c_{n-2} & 1-c_{n-1} & 1-c_n
\end{array} \right.
$$

....

$$
2^n C_n^n: \quad \left\{ \begin{array}{ccccccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_{n-3} & \lambda_{n-2} & \lambda_{n-1} & \lambda_n \\
1-b_1 & 1-b_2 & 1-b_3 & 1-b_4 & \ldots & 1-b_{n-3} & 1-b_{n-2} & 1-b_{n-1} & 1-b_n \\
1-c_1 & 1-b_2 & 1-b_3 & 1-b_4 & \ldots & 1-b_{n-3} & 1-b_{n-2} & 1-b_{n-1} & 1-b_n \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1-c_1 & 1-c_2 & 1-c_3 & 1-c_4 & \ldots & 1-c_{n-3} & 1-c_{n-2} & 1-c_{n-1} & 1-c_n.
\end{array} \right.
$$

Here, $C_n^m = \dfrac{n!}{m!(n-m!)}$ are binomial coefficients. It is obvious that

$$
1 + 2 \cdot C_n^1 + 2^2 \cdot C_n^2 + \ldots + 2^{n-1} \cdot C_n^{n-1} + 2^n \cdot C_n^n = (1+2)^n = 3^n.
$$

Thus, the system of algebraic Equation (14) has 3^n solutions.

Therefore, the system of hypergeometric Equation (13) has 3^n linearly independent solutions, as follows:

$$
C_n^0 : \{ F_{0:2}^{1:0}(a; \mathbf{b}, \mathbf{c}; \mathbf{x}),
$$

$$
C_n^1: \left\{ \begin{array}{c}
x_1^{1-b_1} F_{0:2}^{1:0}(a+1-b_1; 2-b_1, b_2, \ldots, b_n, c_1-b_1+1, c_2, \ldots, c_n; \mathbf{x}), \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_n^{1-b_n} F_{0:2}^{1:0}(a+1-b_n; b_1, b_2, \ldots, b_{n-1}, 2-b_n, c_1, c_2, \ldots, c_{n-1}, c_n-b_n+1; \mathbf{x}),
\end{array} \right.
$$

$$
C_n^1: \left\{ \begin{array}{c}
x_1^{1-c_1} F_{0:2}^{1:0}(a+1-c_1; b_1-c_1+1, b_2, \ldots, b_n, 2-c_1, c_2, \ldots, c_n; \mathbf{x}), \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_n^{1-c_n} F_{0:2}^{1:0}(a+1-c_n; b_1, b_2, \ldots, b_{n-1}, b_n-c_n+1, c_1, c_2, \ldots, c_{n-1}, 2-c_n; \mathbf{x}),
\end{array} \right.
$$

$$
C_n^2: \left\{ \begin{array}{c}
x_1^{1-b_1} x_2^{1-b_2} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-b_1-b_2; 2-b_1, 2-b_2, b_3, \ldots, b_n, \\ c_1-b_1+1, c_2-b_2+1, c_3, \ldots, c_n; \end{array} \mathbf{x} \right], \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_1^{1-b_1} x_n^{1-b_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-b_1-b_n; 2-b_1, b_2, \ldots, b_{n-1}, 2-b_n, \\ c_1-b_1+1, c_2, \ldots, c_{n-1}, c_n-b_n+1; \end{array} \mathbf{x} \right], \\
x_2^{1-b_2} x_3^{1-b_3} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-b_2-b_3; b_1, 2-b_2, 2-b_3, b_4, \ldots, b_n; \\ c_1, c_2-b_2+1, c_3-b_3+1, c_4, \ldots, c_n; \end{array} \mathbf{x} \right], \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{n-1}^{1-b_{n-1}} x_n^{1-b_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-b_{n-1}-b_n; b_1, \ldots, b_{n-2}, 2-b_{n-1}, 2-b_n, \\ c_1, \ldots, c_{n-2}, c_{n-1}-b_{n-1}+1, c_n-b_n+1; \end{array} \mathbf{x} \right],
\end{array} \right.
$$

$$
C_n^2: \left\{ \begin{array}{c}
x_1^{1-b_1} x_2^{1-c_2} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-b_1-c_2; 2-b_1, b_2-c_2+1, b_3, \ldots, b_n, \\ c_1-b_1+1, 2-c_2, c_3, \ldots, c_n; \end{array} \mathbf{x} \right], \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_1^{1-b_1} x_n^{1-c_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-b_1-c_n; 2-b_1, b_2, \ldots, b_{n-1}, b_n-c_n+1, \\ c_1-b_1+1, c_2, \ldots, c_{n-1}, 2-c_n; \end{array} \mathbf{x} \right], \\
x_2^{1-b_2} x_3^{1-c_3} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-b_2-c_3; b_1, 2-b_2, b_3-c_3+1, b_4, \ldots, b_n; \\ c_1, c_2-b_2+1, 2-c_3, c_4, \ldots, c_n; \end{array} \mathbf{x} \right], \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{n-1}^{1-b_{n-1}} x_n^{1-c_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-b_{n-1}-c_n; b_1, \ldots, b_{n-2}, 2-b_{n-1}, b_n-c_n+1, \\ c_1, \ldots, c_{n-2}, c_{n-1}-b_{n-1}+1, 2-c_n; \end{array} \mathbf{x} \right],
\end{array} \right.
$$

$$
C_n^2 : \left\{ \begin{array}{l}
x_1^{1-c_1} x_2^{1-b_2} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-c_1-b_2; b_1-c_1+1, 2-b_2, b_3, \ldots, b_n, \\ 2-c_1, c_2-b_2+1, c_3, \ldots, c_n; \end{array} \mathbf{x} \right], \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_1^{1-c_1} x_n^{1-b_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-c_1-b_n; b_1-c_1+1, b_2, \ldots, b_{n-1}, 2-b_n, \\ 2-c_1, c_2, \ldots, c_{n-1}, c_n-b_n+1; \end{array} \mathbf{x} \right], \\
x_2^{1-c_2} x_3^{1-b_3} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-c_2-b_3; b_1, b_2-c_2+1, 2-b_3, b_4, \ldots, b_n; \\ c_1, 2-c_2, c_3-b_3+1, c_4, \ldots, c_n; \end{array} \mathbf{x} \right], \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{n-1}^{1-c_{n-1}} x_n^{1-b_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-c_{n-1}-b_n; b_1, \ldots, b_{n-2}, b_{n-1}-c_{n-1}+1, 2-b_n, \\ c_1, \ldots, c_{n-2}, 2-c_{n-1}, c_n-b_n+1; \end{array} \mathbf{x} \right],
\end{array} \right.
$$

$$
C_n^2 : \left\{ \begin{array}{l}
x_1^{1-c_1} x_2^{1-c_2} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-c_1-c_2; b_1-c_1+1, b_2-c_2+1, b_3, \ldots, b_n, \\ 2-c_1, 2-c_2, c_3, \ldots, c_n; \end{array} \mathbf{x} \right], \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_1^{1-c_1} x_n^{1-c_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-c_1-c_n; b_1-c_1+1, b_2, \ldots, b_{n-1}, b_n-c_n+1, \\ 2-c_1, c_2, \ldots, c_{n-1}, 2-c_n; \end{array} \mathbf{x} \right], \\
x_2^{1-c_2} x_3^{1-c_3} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-c_2-c_3; b_1, b_2-c_2+1, b_3-c_3+1, b_4, \ldots, b_n; \\ c_1, 2-c_2, 2-c_3, c_4, \ldots, c_n; \end{array} \mathbf{x} \right], \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{n-1}^{1-c_{n-1}} x_n^{1-c_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-c_{n-1}-c_n; b_1, \ldots, b_{n-2}, b_{n-1}-c_{n-1}+1, b_n-c_n+1, \\ c_1, \ldots, c_{n-2}, 2-c_{n-1}, 2-c_n; \end{array} \mathbf{x} \right],
\end{array} \right.
$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$
2^n \cdot C_n^n : \left\{ \begin{array}{l}
x_1^{1-b_1} \ldots x_n^{1-b_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+n-b_1-\ldots-b_n; 2-b_1, \ldots, 2-b_n, \\ c_1-b_1+1, \ldots, c_2-b_n+1; \end{array} \mathbf{x} \right], \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_1^{1-c_1} x_2^{1-b_2} \ldots x_n^{1-b_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+n-c_1-b_2-\ldots-b_n; b_1-c_1+1, 2-b_2, \ldots, 2-b_n, \\ 2-c_1, c_2-b_2+1, \ldots, c_n-b_n+1; \end{array} \mathbf{x} \right], \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_1^{1-c_1} \ldots x_n^{1-c_n} F_{0:2}^{1:0}\left[\begin{array}{c} a+n-c_1-\ldots-c_n; b_1-c_1+1, \ldots, b_n-c_n+1, \\ 2-c_1, \ldots, 2-c_n; \end{array} \mathbf{x} \right].
\end{array} \right.
$$

Theorem 1 is proved. □

It can be observed that all solutions of the system of hypergeometric Equation (13) are categorized into several groups.

It turns out that by using the symmetry properties (8) and (9) of the Kampé de Fériet function $F_{0:2}^{1:0}$, it is possible to simplify the extended representation of the system solutions (13). Indeed, the first group called C_n^0 contains only one solution, which we denote by z_0, as follows

$$z_0(\mathbf{x}) := F_{0:2}^{1:0}(a; \mathbf{b}, \mathbf{c}; \mathbf{x}); \tag{15}$$

the next two groups called C_n^1 contain n solutions each. However, all $2n$ solutions in these two groups are symmetric with respect to the numerical parameters b_j and c_j $(j = \overline{1,n})$. Therefore, it is sufficient to select one solution from these, specifically the solution that appears first in the first group C_n^1, which we denote by z_1, as follows:

$$z_1(\mathbf{x}) := x_1^{1-b_1} F_{0:2}^{1:0}(a+1-b_1; 2-b_1, b_2, \ldots, b_n, c_1-b_1+1, c_2, \ldots, c_n; \mathbf{x}); \tag{16}$$

Via similar reasoning, we select one solution from the following four groups, called C_n^2, which together contain $2^2 \cdot C_n^2 = 2n(n-1)$ solutions, as follows:

$$z_2(\mathbf{x}) := x_1^{1-b_1} x_2^{1-b_2} F_{0:2}^{1:0}\left[\begin{array}{c} a+2-b_1-b_2; 2-b_1, 2-b_2, b_3, \ldots, b_n, \\ c_1-b_1+1, c_2-b_2+1, c_3, \ldots, c_n; \end{array} \mathbf{x} \right]; \tag{17}$$

Similarly, another solution stands out from the following eight groups called C_n^3, which together contain $2^3 \cdot C_n^3 = 4n(n-1)(n-2)/3$ solutions, as follows:

$$z_3(\mathbf{x}) := x_1^{1-b_1} x_2^{1-b_2} x_3^{1-b_3} F_{0:2}^{1:0}\left[\begin{matrix} a+3-b_1-b_2-b_3; 2-b_1, 2-b_2, 2-b_3, b_4, ..., b_n, \\ c_1-b_1+1, c_2-b_2+1, c_3-b_3+1, c_4, ..., c_n; \end{matrix} \mathbf{x}\right]; \tag{18}$$

The last group, called $2^n \cdot C_n^n$, contains 2^n solutions. The solution that appears first in this group should be singled out, as follows:

$$z_n(\mathbf{x}) := x_1^{1-b_1} ... x_n^{1-b_n} F_{0:2}^{1:0}\left[\begin{matrix} a+n-b_1-...-b_n; c_1-b_1+1, ..., c_n-b_n+1, \\ 2-b_1, ..., 2-b_n; \end{matrix} \mathbf{x}\right]. \tag{19}$$

Combining the formulas of solutions (15)–(19), we conclude that all solutions of the system of hypergeometric Equation (13), which are asymmetric with respect to the numerical parameters b_j and c_j $(j = \overline{1,n})$, are represented as

$$z_k(\mathbf{x}) := \gamma_k \prod_{j=1}^{k} x_j^{1-b_j} \cdot F_{0:2}^{1:0}\left(\begin{matrix} a+k-b_1-...-b_k; 2-b_1, ..., 2-b_k, b_{k+1}, ..., b_n, \\ c_1-b_1+1, ..., c_k-b_k+1, c_{k+1}, ..., c_n; \end{matrix} \mathbf{x}\right), \tag{20}$$

where γ_k is an arbitrary constant; $k = \overline{0,n}$; and the empty product is interpreted as a unit.

3. Self-Similar Solutions

Consider Equation (1) in the area $\Omega = \{(\mathbf{x}, t) : x_1 > 0, ..., x_n > 0, t > 0\}$, the solution of which we will seek in the form of

$$u(\mathbf{x}, t) = P(t)\omega(\xi), \tag{21}$$

where

$$P(t) = \frac{l+1}{nt^{l+1}}, \ l > 0; \quad \xi := (\xi_1, ..., \xi_n), \ \ \xi_j = -\alpha_j^3 P(t) x_j^{1/\alpha_j}; \quad \alpha_j = \frac{1}{m_j+3}, \ m_j > 0. \tag{22}$$

Substituting (21) into Equation (1), we obtain a third-order partial differential equation, as follows:

$$\sum_{j=1}^{n}\left\{\xi_j^2 \frac{\partial^3 \omega}{\partial \xi_j^3} + 3(1-\alpha_j)\xi_j \frac{\partial^2 \omega}{\partial \xi_j^2} + \left[(1-\alpha_j)(1-2\alpha_j) - n\xi_j\right]\frac{\partial \omega}{\partial \xi_j}\right\} - n\omega = 0,$$

which is equivalent to the following system of hypergeometric differential equations:

$$\left\{\xi_j^2 \frac{\partial^3 \omega}{\partial \xi_j^3} + \left[(1-\alpha_j) + (1-2\alpha_j) + 1\right]\xi_j \frac{\partial^2 \omega}{\partial \xi_j^2} + (1-\alpha_j)(1-2\alpha_j)\omega_\xi - \sum_{k=1}^{n} \xi_k \frac{\partial \omega}{\partial \xi_j} - \omega = 0, j = \overline{1,n}. \right. \tag{23}$$

Now comparing the latter system (23) with system (11) and taking into account (21), by virtue of (20), it is easy to write out self-similar solutions of a degenerate partial differential Equation (1) in the following form:

$$u_0(\mathbf{x}, t) = \gamma_0 P \cdot F_{0:2}^{1:0}\left[\begin{matrix} 1; \\ 1-2\alpha_1, ..., 1-2\alpha_n, 1-\alpha_1, ..., 1-\alpha_n; \end{matrix} \xi\right], \tag{24}$$

$$u_k(\mathbf{x}, t) = \gamma_k P^{1+2\alpha_1+...+2\alpha_k} x_1^2 ... x_k^2 \times$$

$$F_{0:2}^{1:0}\left[\begin{matrix} 1+2\alpha_1+...+2\alpha_k; \\ 1+2\alpha_1, ..., 1+2\alpha_k, 1-2\alpha_{k+1}, ..., 1-2\alpha_n, 1+\alpha_1, ..., 1+\alpha_k, 1-\alpha_{k+1}, ..., 1-\alpha_n; \end{matrix} \xi\right], \tag{25}$$

where $k = \overline{1,n}$, and $P = P(t)$, ξ, α_j $(j = \overline{1,n})$ are defined in (22).

Thus, the following theorem is proved.

Theorem 2. *Self-similar solutions of a degenerate differential Equation (1) are represented by Formulas (24) and (25).*

4. Conclusions

In this work, self-similar solutions were constructed for a degenerate third-order differential equation with n independent variables (1). The solution of the equation was sought as a product of the functions P(t) and $\omega(\xi)$ (21). Thus, solving the original Equation (1) was reduced to solving a third-order hypergeometric differential Equation (23). Special solutions of Equation (23) are the n so-called self-similar solutions, written in forms (24) and (25). The solution involves the function $F^{1:0}_{0:2}$, which is a generalized confluent hypergeometric Kampé de Fériet function. A theorem was proven that, near the origin, the degenerate partial differential Equation (1) has 3^n solutions.

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