Chelyabinsk Physical and Mathematical Journal. 2025. Vol. 10, iss. 3. P. 513–526.

DOI: 10.47475/2500-0101-2025-10-3-513-526

EXPANSIONS OF KAMPÉ DE FÉRIET HYPERGEOMETRIC FUNCTIONS

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In 1940–1941, Burchnall and Chaundy, using the inverse pairs of symbolic operators introduced, systematically presented a number of expansion formulas for second-order Appell and Humbert hypergeometric functions in series of simpler hypergeometric functions. In 2006, Hasanov and Srivastava generalized Burchnall–Chaundy operators and found expansions for a class of multiple hypergeometric functions, however, the application of these operators is still limited to second-order hypergeometric functions. In this paper, we obtain expansions for the Kampé de Fériet functions of arbitrary orders.

Keywords: *Appell functions, Humbert functions, Burchnall–Chaundy operators, Kampé de Fériet functions, expansions of hypergeometric functions.*

1. Introduction

Hypergeometric functions naturally occur in a wide variety of problems in applied mathematics, statistics, operations research, theoretical physics, and engineering sciences. In scientific literature it is found a number of interesting applications of hypergeometric functions in one and more variables in queuing theory and related stochastic processes [1], in physical and quantum chemical problems [2]. Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [3].

The great success of the theory of hypergeometric functions of a single variable has stimulated the development of a corresponding theory in two or more variables. Appell has defined, in 1880, four series, F_1 to F_4 which are all analogous to Gauss' $F(a,b;c;z)$. Picard has pointed out that one of these series is intimately related to a function studied by Pochhammer in 1870, and Picard and Goursat also constructed a theory of Appell's series which is analogous to Riemann's theory of Gauss' hypergeometric series. P. Humbert has studied confluent hypergeometric series in two variables. An expansion of the results of the French school together with references to the original literature are to be found in the monograph by Appell and Kampé de Fériet [4], which is the standart

The research is supported by the Intercontinental Research Center "Analysis and PDE" (Ghent University, Belgium), grant G.0H94.18N and by the Special Research Fund of Methusalem programme (Ghent University), grant 01M01021.

work on the subject. This work also contains an extensive bibliography of all relevant papers up to 1926.

A great merit in the further development of the theory of the hypergeometric series in two variables belongs to Horn, who gave a general definition and order classification of double hypergeometric series. He has investigated the convergence of hypergeometric series of two variables and established the systems of partial differential equations which they satisfy. Horn investigated in particular hypergeometric series of order two and found that, apart from certain series which are either expressible in terms of one variable or are products of two hypergeometric series, each in one variable, there are essentially 34 distinct convergent series of order two.

The idea of extending the number of parameters in the Gauss function seems to have occurred for the first time, in the work of Clausen [5]. He introduced a series with three numerator parameters and two denominator parameters. Over the next hundred years the well-known set of special summation theorems associated with the names of Saalschutz [6], Dixon [7] and Dougall [8] were developed. Modern notation ${}_pF_q$ of the generalized hypergeometric functions was due to Barnes [9].

Just as the Gaussian series $F(a, b; c; z)$ was generalized to ${}_pF_q$ by increasing the numbers of the numerator and denominator parameters, the four Appell series were unified and generalized by Kampé de Fériet [10] who defined a general hypergeometric series in two variables (see, [4, p. 150, eq. (29)]). Since then, a lot of work has been devoted to the study of the Kampé de Fériet functions [11–20].

For a given multiple hypergeometric function, it is useful to fund an expansion formulas which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. Using the symbolic method Burchnall and Chaundy [21; 22] obtained more than 15 pairs of expansions for 11 double hypergeometric functions. The well-known Poole's formula [23] played an important role in the studies of Burchnall and Chaundy, but this one formula was not enough for the expansion of all functions from the Horn's list [24, pp. 224–227]. In a recent work [25], the authors managed to find expansions for some double hypergeometric functions of order 2. There are decomposition formulas for the second order hypergeometric functions when the dimension exceeds two [26–28].

At present expansion formulas have been established for some hypergeometric functions of three or more variables. In 2006–2007 Hasanov and Srivastava [29; 30] introduced multidimensional analogous of the Burchnall–Chaundy operators and found expansion formulas for many triple hypergeometric functions, and they proved recurrent formulas when the dimension of the second-order hypergeometric functions exceeds three. However, due to the recurrence, additional difficulties may arise in the applications of these expansion formulas. Recently expansion formulas free from recurrence for multiple hypergeometric Lauricella functions are obtained and used to find explicit solutions to boundary value problems for a multidimensional elliptic equation with several singular coefficients [31; 32].

Thus, research on the expansion of hypergeometric functions in the cases of two and many variables has been limited to second-order functions. This work is devoted to the expansion of the Kampé de Fériet function in the form of an infinite sum of the product of generalized hypergeometric functions of arbitrary orders. Note, at present some applications of the Kampé de Fériet functions to the solving the boundary value problems for singular partial differential equations [33; 34], and integral representations for the Kampé de Fériet functions [35] are known.

2. Preliminaries

The celebrated Gauss hypergeometric function,

$$F(a,b;c;z) \equiv F\left[\begin{array}{c} a,b; \\ c; \end{array} z\right] := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},\ c \neq 0,-1,-2,\ldots, \tag{1}$$

is contained in the generalized hypergeometric function ${}_pF_q$ involving p numerator parameters, $a_1,\ldots,a_p$, and q denominator parameters, $b_1,\ldots,b_q$, as special case. Here, the expression $(a)_k$, called the Pochhammer symbol, is defined in terms of the gamma function $\Gamma(s)$ by

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}.$$

For an integer $k \geqslant 0$ it is a product of the form

$$(a)_0 = 1,\ (a)_k = a(a+1)\ldots(a+k-1),\ k = 1,2,\ldots$$

Following the standard notations and conventions, we define the generalized hypergeometric function ${}_pF_q$ here as follows [24, p. 182] :

$${}_pF_q\left[\begin{array}{c} (a_p)\,; \\ (b_q)\,; \end{array} z\right] \equiv {}_pF_q\left[(a_p)\,;(b_q)\,;z\right] := \sum_{k=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(a_j)_k}{\prod\limits_{j=1}^{q}(b_j)_k} \frac{z^k}{k!}.$$

Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z, the numerator parameters a_1, …, a_p, and the denominator parameters b_1, …, b_q take on complex values, provided that $b_j \neq 0,-1,-2,\ldots;\ j = 1,\ \ldots,\ q$.

Gauss' series (1) in the present notation is

$${}_2F_1(a,b;c;z) \equiv F\left[\begin{array}{c} a,b; \\ c; \end{array} z\right] := F(a,b;c;z).$$

If we set $z=1$ in (1), we obtain the well-known summation theorem of the $F(a,b;c;z)$ series of unit argument, due to Gauss

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \mathrm{Re}(c-a-b) > 0. \tag{2}$$

The great success of the theory of hypergeometric series in one variable has stimulated the development of a corresponding theory in two and more variables. Appell has defined, in 1880, four series [36]

$$F_1\left(a,b,b';c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_{m+n} m!n!} x^m y^n, \tag{3}$$

$$F_2\left(a,b,b';c,c';x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_m\left(c'\right)_n m!n!} x^m y^n, \tag{4}$$

$$F_3\left(a,a',b,b';c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m\left(a'\right)_n(b)_m\left(b'\right)_n}{(c)_{m+n} m!n!} x^m y^n, \tag{5}$$

$$F_4\left(a,b;c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m\left(c'\right)_n m!n!}x^m y^n, \tag{6}$$

which are all analogous to Gauss' $F(a,b;c;z)$. Here, in all definitions (3)–(6), as usual, the denominator parameters c and c' are neither zero nor a negative integer.

The standard work on the theory of Appell series is the monograph by Appell and Kampé de Fériet [4], which contains an extensive bibliography of all relevant papers up to 1926 (by, for example, L. Pochhammer, J. Horn, É. Picard, and É. Gursat). See [24, pp. 222–45] for a review of the subsequent work on the subject; see also Bailey [37], Exton [38], Slater [39] and Srivastava and Karlsson [40].

Seven confluent forms of the four Appell series were defined by Humbert [41], and he denoted these confluent hypergeometric series in two variables by

$$\Phi_1\left(\alpha,\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}\left(\beta\right)_m}{(\gamma)_{m+n}m!n!}x^m y^n,\ \ |x|<1, \tag{7}$$

$$\Phi_2\left(\beta,\beta';\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m\left(\beta'\right)_n}{(\gamma)_{m+n}m!n!}x^m y^n, \tag{8}$$

$$\Phi_3\left(\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n, \tag{9}$$

$$\Psi_1\left(\alpha,\beta;\gamma,\gamma';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}\left(\beta\right)_m}{(\gamma)_m(\gamma')_n m!n!}x^m y^n,\ \ |x|<1, \tag{10}$$

$$\Psi_2\left(\alpha;\gamma,\gamma';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m\left(\gamma'\right)_n m!n!}x^m y^n, \tag{11}$$

$$\Xi_1\left(\alpha,\alpha',\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m\left(\alpha'\right)_n(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n,\ \ |x|<1, \tag{12}$$

$$\Xi_2\left(\alpha,\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m\left(\beta\right)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n,\ \ |x|<1, \tag{13}$$

where the denominator parameters γ and γ' are neither zero nor a negative integer. A hypergeometric functions defined in (7) to (13) are called *Humbert functions*.

Just as the Gaussian series $F(a,b;c;z)$ was generalized to ${}_pF_q$ by increasing the numbers of the numerator and denominator parameters, the four Appell series were unified and generalized by Kampé de Fériet [10] who defined a general hypergeometric series in two variables (see, [4, p. 150, eq. (29)]). The notation introduced by Kampé de Fériet for his double hypergeometric series of of superior order was subsequently abbreviated by Burchnall and Chaundy [22, p. 112]. Srivastava and Panda [12] (see, also [40, Section 3.1]) gave the definition of the more general double hypergeometric series (than the one defined by Kampé de Fériet) in a slightly modified notation

$$F^{p:q,k}_{l:r,s}\left[\begin{matrix}(a_p):\ (b_q)\ ;\ (c_k)\ ;\\ (\alpha_l):(\beta_r);(\gamma_s);\end{matrix} x,y\right] = F^{p:q,k}_{l:r,s}\left[\begin{matrix}a_1,\dots,a_p:\ b_1,\dots,b_q\,;\ c_1,\dots,c_k;\\ \alpha_1,\dots,\alpha_l:\beta_1,\dots,\beta_r;\gamma_1,\dots,\gamma_s;\end{matrix} x,y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{m+n}\prod_{j=1}^{q}(b_j)_m\prod_{j=1}^{k}(c_j)_n}{\prod_{j=1}^{l}(\alpha_j)_{m+n}\prod_{j=1}^{r}(\beta_j)_m\prod_{j=1}^{s}(\gamma_j)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \quad (14)$$

where $p,q,k,l,r,s\in\mathbb{N}\cup\{0\}$, and for convergence,

$$\text{(i)}\quad p+q<l+r+1,\quad p+k<l+s+1,\quad |x|<\infty,\quad |y|<\infty,$$

or

$$\text{(ii)}\quad p+q=l+r+1,\quad p+k=l+s+1,\quad \text{and}$$

$$\begin{cases} |x|^{1/(p-l)}+|y|^{1/(p-l)}<1, & \text{if } p>l,\\ \max\{|x|,\ |y|\}<1, & \text{if } p\leqslant l.\end{cases}$$

Although the double hypergeometric series defined by (14) reduces to Kampé de Fériet series in the *special* case:

$$q=k \ \text{ and } \ r=s,$$

yet it is usually referred to in the literature as the Kampé de Fériet series.

In this paper we consider the following four Kampé de Fériet hypergeometric series

$$F^{1:q,k}_{0:r,s}\left[\begin{matrix}a:(b_q);(c_k);\\ -:(\beta_r);(\gamma_s);\end{matrix} x,y\right],\quad F^{0:\,q\,,k}_{1:r,s}\left[\begin{matrix}-:(b_q);(c_k);\\ \alpha:(\beta_r);(\gamma_s);\end{matrix} x,y\right],$$

$$F^{1:q,k}_{1:r,s}\left[\begin{matrix}a:(b_q);(c_k);\\ \alpha:(\beta_r);(\gamma_s);\end{matrix} x,y\right],\quad F^{2:\,q,\,k}_{0:r,s}\left[\begin{matrix}a_1,a_2:\ (b_q)\ ;\ (c_k)\ ;\\ -:(\beta_r);(\gamma_s);\end{matrix} x,y\right],$$

all of which are analogous to Appell's and Humbert's functions.

3. Symbolic forms and expansions of Burchnall and Chaundy

For a given multiple hypergeometric function, it is useful to fund an expansion formulas which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables.

Burchnall and Chaundy [21] introduce the operators

$$\nabla(h)=\frac{\Gamma(h)\,\Gamma(\delta+\delta'+h)}{\Gamma(\delta+h)\,\Gamma(\delta'+h)}$$

and

$$\Delta(h)=\frac{\Gamma(\delta+h)\,\Gamma(\delta'+h)}{\Gamma(h)\,\Gamma(\delta+\delta'+h)},$$

where

$$\delta=x\frac{\partial}{\partial x},\quad \delta'=y\frac{\partial}{\partial y}.$$

Then $\nabla(h)\,(h)_m(h)_n x^m y^n = (h)_{m+n} x^m y^n$, and so, if $(h)_m(h)_n$ occurs in the numerator of the coefficient of $x^m y^n$, it is changed into $(h)_{m+n}$ by the operator $\nabla(h)$. The operator $\Delta(h)$ effects a similar change in the denominator. Now Appell's series can be obtained from the product of two simple hypergeometric series by changes of this character [4, p. 13] (see, also [37, p. 76]) and we can therefore write symbolically

$$F_2\,(a,b,b';c,c';x,y) = \nabla(a)F(a,b;c;x)F(a,b';c';y), \tag{15}$$

$$F_3\,(a,a',b,b';c;x,y) = \Delta(c)F(a,b;c;x)F(a',b';c;y),$$

$$F_4\,(a,b;c,c';x,y) = \nabla(a)\nabla(b)F(a,b;c;x)F(a,b';c';y),$$

$$F_1\,(a,b,b';c;x,y) = \nabla(a)\Delta(c)F(a,b;c;x)F(a,b';c;y),$$

thus factorizing Appell's functions by means of the operators ∇ and Δ; we also obtain transformations of Appell's functions as

$$F_1\,(a,b,b';c;x,y) = \nabla(a)F_3\,(a,a,b,b';c;x,y)\,,$$

$$F_1\,(a,b,b';c;x,y) = \Delta(c)F_2\,(a,b,b';c,c;x,y)\,,$$

$$F_4\,(a,b;c,c';x,y) = \nabla(b)F_2\,(a,b,b;c,c';x,y)\,,$$

and some others.

These symbolic forms are used to obtain a large number of expansions of Appell's functions in terms of each other, of Appell's functions in terms of products of ordinary hypergeometric functions, or vice versa. To give an example, by Gauss' formula (2) for $F(a,b;c;1)$, we have symbolically

$$\nabla(h) = \frac{(-\delta)_r(-\delta')_r}{(h)_r r!}. \tag{16}$$

Note, that for every analytic function $f(x)$ the following Poole's formula [23]

$$(-\delta)_r\,\{f(x)\} = (-1)^r x^r \frac{d^r}{dx^r}\{f(x)\}$$

is valid. Now,

$$(-\delta)_r F(a,b;c;x) = (-1)^r \frac{(a)_r(b)_r}{(c)_r} x^r F(a+r,b+r;c+r;x),$$

and hence (15) suggests the expansion

$$F_2(a,b,b';c,c';x,y) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r\,(b')_r}{r!(c)_r\,(c')_r} \times$$
$$\times x^r y^r F(a+r,b+r;c+r;x)F\,(a+r,b'+r;c'+r;y)\,, \tag{17}$$

By inversion of (15) in the form

$$F(a,b;c;x)F(a,b';c';y) = \Delta(a)F_2(a,b,b';c,c';x,y)$$

and corresponding expansion of $\Delta(a)$, the companion to (17),

$$F(a,b;c;x)F\left(a,b';c';y\right)=\sum_{r=0}^{\infty}(-1)^r\frac{(a)_r(b)_r\left(b'\right)_r}{r!(c)_r\left(c'\right)_r}\times$$
$$\times x^r y^r F_2\left(a+r,b+r,b'+r;c+r,c'+r;x,y\right)$$

is obtained. These expansions can be proved without symbolic methods by comparing coefficients of equal powers of x and y on both sides.

By these methods Burchnall and Chaundy obtained 15 pairs of expansions involving Appell's functions and ordinary hypergeometric functions, as well as a further considerable number of expansions involving hypergeometric series of higher order, and Humbert's confluent hypergeometric series, Φ, Ψ, and Ξ.

4. Expansions of the higher order Kampé de Fériet series

Following the Burchnall-Chaundy method outlined in the previous section, we can write symbolically

$$F^{1:q,k}_{0:m,n}\left[\begin{matrix}a:(b_q);(c_k);\\-:(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\nabla(a)\,{}_{q+1}F_m\left[\begin{matrix}a,(b_q);\\(\beta_m);\end{matrix}\;x\right]\,{}_{k+1}F_n\left[\begin{matrix}a,(c_k);\\(\gamma_n);\end{matrix}\;y\right],\tag{18}$$

$$F^{0:q,k}_{1:m,n}\left[\begin{matrix}-:(b_q);(c_k);\\\alpha:(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\Delta(\alpha)\,{}_qF_{m+1}\left[\begin{matrix}(b_q);\\\alpha,(\beta_m);\end{matrix}\;x\right]\,{}_kF_{n+1}\left[\begin{matrix}(c_k);\\\alpha,(\gamma_n);\end{matrix}\;y\right],\tag{19}$$

$$F^{1:q,k}_{1:m,n}\left[\begin{matrix}a:(b_q);(c_k);\\\alpha:(\beta_m);(\gamma_n);\end{matrix}x,y\right]=$$
$$=\nabla(a)\Delta(\alpha){}_{q+1}F_{m+1}\left[\begin{matrix}a,(b_q);\\\alpha,(\beta_m);\end{matrix}\;x\right]\,{}_{k+1}F_{n+1}\left[\begin{matrix}a,(c_k);\\\alpha,(\gamma_n);\end{matrix}\;y\right],\tag{20}$$

$$F^{1:q,k}_{1:m,n}\left[\begin{matrix}a:(b_q);(c_k);\\\alpha:(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\nabla(a)F^{0:q+1,k+1}_{1:m,n}\left[\begin{matrix}-:a,(b_q);a,(c_k);\\\alpha:(\beta_m);(\gamma_n);\end{matrix}x,y\right],\tag{21}$$

$$F^{1:q,k}_{1:m,n}\left[\begin{matrix}a:(b_q);(c_k);\\\alpha:(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\Delta(\alpha)F^{1:q,k}_{0:m+1,n+1}\left[\begin{matrix}a:(b_q);(c_k);\\-:\alpha,(\beta_m);\alpha,(\gamma_n);\end{matrix}x,y\right],\tag{22}$$

$$F^{2:q,k}_{0:m,n}\left[\begin{matrix}a_1,a_2:\;(b_q)\;;\;(c_k);\\-:(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\nabla\left(a_2\right)F^{1:q+1,k+1}_{0:m,n}\left[\begin{matrix}a_1:a_2,(b_q);a_2,(c_k);\\-:(\beta_m);(\gamma_n);\end{matrix}x,y\right].\tag{23}$$

We note further that on a generalized hypergeometric series we have

$$\frac{1}{(\delta+\alpha)_r}{}_{q+1}F_{m+1}\left[\begin{matrix}a,(b_q);\\\alpha,(\beta_m);\end{matrix}\;x\right]=\frac{1}{(\alpha)_r}{}_{q+1}F_{m+1}\left[\begin{matrix}a,(b_q);\\\alpha+r,(\beta_m);\end{matrix}\;x\right]\tag{24}$$

with similar formulae for the double series, provided that we replace δ by δ', $\delta+\delta'$ when the parameter c occurs with suffix n, $m+n$ instead of m.

Again, in the coefficient of $x^m y^n$, $1/\left(-a-\delta-\delta'+1\right)_r$ changes $(a)_{m+n}$ into

$$(-1)^r\frac{(a)_r(a-r)_{m+n}}{(a-r)_{2r}},$$

so that, for instance,

$$\frac{1}{(-a-\delta-\delta'+1)_r} F^{1:q,k}_{1:m,n}\left[\begin{matrix} a:(b_q);(c_k);\\ \alpha:(\beta_m);(\gamma_n);\end{matrix} x,y\right]= \\ =(-1)^r\frac{(a)_r}{(a-r)_{2r}} F^{1:q,k}_{1:m,n}\left[\begin{matrix} a-r:(b_q);(c_k);\\ \alpha:(\beta_m);(\gamma_n);\end{matrix} x,y\right]. \tag{25}$$

Finally, we note that

$$(-\delta)_{r\,q+1}F_{m+1}\left[\begin{matrix} a,(b_q);\\ \alpha,(\beta_m);\end{matrix}\ x\right]=(-1)^r\frac{(a)_r\,(b_q)_r}{(\alpha)_r\,(\beta_m)_r}x^r{}_{q+1}F_{m+1}\left[\begin{matrix} a+r,(b_q+r);\\ \alpha+r,(\beta_m+r);\end{matrix}\ x\right], \tag{26}$$

the corresponding results for the double series being sufficiently shown by the formula

$$(-\delta)_r\,(-\delta')_r F^{1:q,k}_{1:m,n}\left[\begin{matrix} a:(b_q);(c_k);\\ \alpha:(\beta_m);(\gamma_n);\end{matrix} x,y\right]= \\ =\frac{(a)_{2r}\,(b_q)_r\,(c_k)_r}{(\alpha)_{2r}\,(\beta_m)_r\,(\gamma_n)_r}x^ry^r F^{1:q,k}_{1:m,n}\left[\begin{matrix} a+2r:(b_q+r);(c_k+r);\\ \alpha+2r:(\beta_m+r);(\gamma_n+r);\end{matrix} x,y\right], \tag{27}$$

where $2r$ replaces r in association with the "double" parameters a, α, i.e. parameters which have the suffix $m+n$ in the defining series. Combining (25), (27) we find that

$$\frac{(-\delta)_r\,(-\delta')_r}{(-a-\delta-\delta'+1)_r} F^{1:q,k}_{1:m,n}\left[\begin{matrix} a:(b_q);(c_k);\\ \alpha:(\beta_m);(\gamma_n);\end{matrix} x,y\right]= \\ =(-1)^r\frac{(a)_r\,(b_q)_r\,(c_k)_r}{(\alpha)_{2r}\,(\beta_m)_r\,(\gamma_n)_r}x^ry^r F^{1:q,k}_{1:m,n}\left[\begin{matrix} a:(b_q+r);(c_k+r);\\ \alpha+2r:(\beta_m+r);(\gamma_n+r);\end{matrix} x,y\right]. \tag{28}$$

Here, in all equations (26)–(28), and further until the end of this work, we accept the following notation

$$(\lambda)_0:=1,\ (\lambda)_r:=\lambda(\lambda+1)...(\lambda+r-1),\quad r=1,2,...;$$

$$(\lambda_\nu)_0=1,\ (\lambda_\nu)_r:=(\lambda_1)_r\,(\lambda_2)_r\,...\,(\lambda_\nu)_r\,,\ \ r,\ \nu=1,2,...;$$

$$(\lambda_q):=(\lambda_1,...,\lambda_q)\,,\ \ (\lambda_\nu+r):=(\lambda_1+r,\ ...,\ \lambda_\nu+r)\,,\ \ r,\ \nu=1,2,\ldots.$$

We quote as lemmas the known identities [21]

$$\frac{\Gamma(h)\Gamma(m+n+h)}{\Gamma(m+h)\Gamma(n+h)}=\sum_{r=0}^{\infty}\frac{(-m)_r(-n)_r}{r!(h)_r}, \tag{29}$$

$$\frac{\Gamma(m+h)\Gamma(n+h)}{\Gamma(h)\Gamma(m+n+h)} = \sum_{r=0}^{\infty}\frac{(-m)_r(-n)_r}{r!(-h-m-n+1)_r}= \tag{30}$$

$$= \sum_{r=0}^{\infty}(-)^r\frac{(h)_{2r}(-m)_r(-n)_r}{r!(h+r-1)_r(m+h)_r(n+h)_r}, \tag{31}$$

$$\frac{\Gamma(h)\Gamma(m+n+h)\Gamma(m+k)\Gamma(n+k)}{\Gamma(m+h)\Gamma(n+h)\Gamma(k)\Gamma(m+n+k)}= \\ =\sum_{r=0}^{\infty}\frac{(k-h)_r(k)_{2r}(-m)_r(-n)_r}{r!(k+r-1)_r(m+k)_r(n+k)_r(h)_r}= \tag{32}$$

$$=\sum_{r=0}^{\infty}\frac{(h-k)_r(-m)_r(-n)_r}{r!(h)_r(-k-m-n+1)_r}. \tag{33}$$

We illustrate our method by first considering a particular example. In (29) replace m, n by δ, δ', so that we have a symbolic form (16). Thus in (18) we have

$$F^{1:q,k}_{0:m,n}\left[\begin{matrix} a:(b_q);(c_k); \\ -:(\beta_m);(\gamma_n); \end{matrix} x,y\right]=$$
$$=\sum_{r=0}^{\infty}\frac{(-\delta)_r(-\delta')_r}{r!(a)_r}\,{}_{q+1}F_m\left[\begin{matrix} a,(b_q); \\ (\beta_m); \end{matrix}\ x\right]\,{}_{k+1}F_n\left[\begin{matrix} a,(c_k); \\ (\gamma_n); \end{matrix}\ y\right]=$$
$$=\sum_{r=0}^{\infty}\frac{(a)_r\,(b_q)_r\,(c_k)_r}{r!\,(\beta_m)_r\,(\gamma_n)_r}x^ry^r\,{}_{q+1}F_m\left[\begin{matrix} a+r,(b_q+r); \\ (\beta_m+r); \end{matrix}\ x\right]\,{}_{k+1}F_n\left[\begin{matrix} a+r,(c_k+r); \\ (\gamma_n+r); \end{matrix}\ y\right]$$

in virtue of (26), which gives $F^{1:q,k}_{0:m,n}$ expanded in a series of products of generalized hypergeometric functions. So far the argument has been purely symbolic, but we can make a more convincing use of our material. Write $\mathbf{C}_{m,n}S(x,y)$ to denote the coefficient of x^my^n in a double series $S(x,y)$ with an analogous symbol for the simple series. Then, if the expansion above is to be valid, we must have, on equating corresponding coefficients on the two sides,

$$\mathbf{C}_{m,n}F^{1:q,k}_{0:m,n}\left[\begin{matrix} a:(b_q);(c_k); \\ -:(\beta_m);(\gamma_n); \end{matrix} x,y\right]=\sum_{r=0}^{\infty}\frac{(a)_r\,(b_q)_r\,(c_k)_r}{r!\,(\beta_m)_r\,(\gamma_n)_r}\mathbf{C}_{m-r,n-r}\times$$
$$\times{}_{q+1}F_m\left[\begin{matrix} a+r,(b_q+r); \\ (\beta_m+r); \end{matrix}\ x\right]\,{}_{k+1}F_n\left[\begin{matrix} a+r,(c_k+r); \\ (\gamma_n+r); \end{matrix}\ y\right]. \tag{34}$$

But, from the definition of $F^{1:q,k}_{0:m,n}$, or equivalently by (18) ,

$$\mathbf{C}_{m,n}F^{1:q,k}_{0:m,n}\left[\begin{matrix} a:(b_q);(c_k); \\ -:(\beta_m);(\gamma_n); \end{matrix} x,y\right]=\frac{\Gamma(a)\Gamma(a+m+n)}{\Gamma(a+m)\Gamma(a+n)}\times$$
$$\times\mathbf{C}_{m,n}{}_{q+1}F_m\left[\begin{matrix} a,(b_q); \\ (\beta_m); \end{matrix}\ x\right]\,{}_{k+1}F_n\left[\begin{matrix} a,(c_k); \\ (\gamma_n); \end{matrix}\ y\right];$$

and we can replace (26) by

$$(-m)_r\mathbf{C}_{m,n}F^{1:q,k}_{0:m,n}\left[\begin{matrix} a:(b_q);(c_k); \\ -:(\beta_m);(\gamma_n); \end{matrix} x,y\right]=$$
$$=(-1)^r\frac{(a)_r(b)_r}{(c)_r}\mathbf{C}_{m-r,n}{}_{q+1}F_m\left[\begin{matrix} a+r,(b_q+r); \\ (\beta_m+r); \end{matrix}\ x\right]$$

and the similar formula in y. To complete the proof of (34) we need then only to employ (29) in m, n as written.

Thus, if in the symbolic analysis we replace δ, δ' by m, n, we can adapt the argument to give term-by-term identification of the two sides of the presumed identity.[1] This is perhaps obvious from first principles, for, in operation on x^my^n, δ, δ' become replaced by m, n. Now in the term-by-term identification we are, in fact, rearranging the summation of a multiple series. This we justify later by considerations of absolute convergence. For the present we are content with purely formal results, and, as we have just seen, these are adequately established by the symbolic analysis.

[1]We may note that with m, n positive integers the summations in the five lemmas become the finite sums from 0 to $\min(m,n)$.

Then from the six identities (18)–(23), if we use (29), (31), (32) to express ∇, Δ , $\nabla\Delta$ and we shall need to appeal to the principles of (24) and (26) or (27), we obtain six expansions including the expansion for F_2 given above. By inverting these we obtain a further six. Thus we can rewrite (18) as

$$ {}_{q+1}F_m\left[\begin{matrix} a,(b_q)\,; \\ (\beta_m)\,; \end{matrix}\; x\right] {}_{k+1}F_n\left[\begin{matrix} a,(c_k)\,; \\ (\gamma_n)\,; \end{matrix}\; y\right] = \Delta(a) F^{1:q,k}_{0:m,n}\left[\begin{matrix} a:(b_q)\,;(c_k)\,; \\ -:(\beta_m)\,;(\gamma_n)\,; \end{matrix} x,y\right], $$

and so on. We must now use (30), (33) instead of (31), (32) to give Δ, $\nabla\Delta$, and we shall need to appeal to the principle of (25) or rather (28). These six inverse expansions lack full generality in the sense that the hypergeometric function to be expanded necessarily has a pair of parameters equal.

We now state these fourteen expansions without further proof, putting each inverse series directly after the corresponding direct series and arranging the six pairs in the order of the identities (18)–(23).

5. Twelve expansions

Next twelve expansions can be proved without symbolic methods by comparing coefficients of equal powers of x and y on both sides:

$$ F^{1:q,k}_{0:m,n}\left[\begin{matrix} a:(b_q)\,;(c_k)\,; \\ -:(\beta_m)\,;(\gamma_n)\,; \end{matrix} x,y\right] = \sum_{r=0}^{\infty} \frac{(a)_r\,(b_q)_r\,(c_k)_r}{r!\,(\beta_m)_r\,(\gamma_n)_r} x^r y^r \times $$
$$ \times {}_{q+1}F_m\left[\begin{matrix} a+r,(b_q+r)\,; \\ (\beta_m+r)\,; \end{matrix}\; x\right] {}_{k+1}F_n\left[\begin{matrix} a+r,(c_k+r)\,; \\ (\gamma_n+r)\,; \end{matrix}\; y\right], $$

$$ {}_{q+1}F_m\left[\begin{matrix} a,(b_q)\,; \\ (\beta_m)\,; \end{matrix}\; x\right] {}_{k+1}F_n\left[\begin{matrix} a,(c_k)\,; \\ (\gamma_n)\,; \end{matrix}\; y\right] = \sum_{r=0}^{\infty} (-1)^r \frac{(a)_r\,(b_q)_r\,(c_k)_r}{r!\,(\beta_m)_r\,(\gamma_n)_r} \times $$
$$ \times x^r y^r F^{1:q,k}_{0:m,n}\left[\begin{matrix} a+r:(b_q+r)\,;(c_k+r)\,; \\ -:(\beta_m+r)\,;(\gamma_n+r)\,; \end{matrix} x,y\right], $$

$$ F^{0:q,k}_{1:m,n}\left[\begin{matrix} -:(b_q)\,;(c_k)\,; \\ \alpha:(\beta_m)\,;(\gamma_n)\,; \end{matrix} x,y\right] = \sum_{r=0}^{\infty} \frac{(-1)^r\,(b_q)_r\,(c_k)_r}{r!(\alpha+r-1)_r(\alpha)_{2r}\,(\beta_m)_r\,(\gamma_n)_r} x^r y^r \times $$
$$ \times {}_qF_{m+1}\left[\begin{matrix} (b_q+r)\,; \\ \alpha+2r,(\beta_m+r)\,; \end{matrix}\; x\right] {}_kF_{n+1}\left[\begin{matrix} (c_k+r)\,; \\ \alpha+2r,(\gamma_n+r)\,; \end{matrix}\; y\right], $$

$$ {}_qF_{m+1}\left[\begin{matrix} (b_q)\,; \\ \alpha,(\beta_m)\,; \end{matrix}\; x\right] {}_kF_{n+1}\left[\begin{matrix} (c_k)\,; \\ \alpha,(\gamma_n)\,; \end{matrix}\; y\right] = \sum_{r=0}^{\infty} \frac{(b_q)_r\,(c_k)_r}{r!(\alpha)_r(\alpha)_{2r}\,(\beta_m)_r\,(\gamma_n)_r} \times $$
$$ \times x^r y^r F^{0:q,k}_{1:m,n}\left[\begin{matrix} -:(b_q+r)\,;(c_k+r)\,; \\ \alpha+2r:(\beta_m+r)\,;(\gamma_n+r)\,; \end{matrix} x,y\right], $$

$$ F^{1:q,k}_{1:m,n}\left[\begin{matrix} a:(b_q)\,;(c_k)\,; \\ \alpha:(\beta_m)\,;(\gamma_n)\,; \end{matrix} x,y\right] = \sum_{r=0}^{\infty} \frac{(a)_r(\alpha-a)_r\,(b_q)_r\,(c_k)_r}{r!(\alpha+r-1)_r(\alpha)_{2r}\,(\beta_m)_r\,(\gamma_n)_r} x^r y^r \times $$
$$ \times {}_{q+1}F_{m+1}\left[\begin{matrix} a+r,(b_q+r)\,; \\ \alpha+2r,(\beta_m+r)\,; \end{matrix}\; x\right] {}_{k+1}F_{n+1}\left[\begin{matrix} a+r,(c_k+r)\,; \\ \alpha+2r,(\gamma_n+r)\,; \end{matrix}\; y\right], $$

$$ {}_{q+1}F_{m+1}\left[\begin{matrix} a,(b_q)\,; \\ \alpha,(\beta_m)\,; \end{matrix}\ x\right]\ {}_{k+1}F_{n+1}\left[\begin{matrix} a,(c_k)\,; \\ \alpha,(\gamma_n)\,; \end{matrix}\ y\right] = \sum_{r=0}^{\infty}(-1)^r\frac{(a)_r(\alpha-a)_r}{r!(\alpha)_r(\alpha)_{2r}}\times $$

$$ \times\frac{(b_q)_r\,(c_k)_r}{(\beta_m)_r\,(\gamma_n)_r}x^ry^rF^{1:q,k}_{1:\,m\,,\,n}\left[\begin{matrix} a+r:(b_q+r)\,;(c_k+r)\,; \\ \alpha+2r:(\beta_m+r)\,;(\gamma_n+r)\,; \end{matrix} x,y\right], $$

$$ F^{1:q,k}_{1:\,m\,,\,n}\left[\begin{matrix} a:(b_q)\,;(c_k)\,; \\ \alpha:(\beta_m)\,;(\gamma_n)\,; \end{matrix} x,y\right] = \sum_{r=0}^{\infty}\frac{(a)_r\,(b_q)_r\,(c_k)_r}{r!(\alpha)_{2r}\,(\beta_m)_r\,(\gamma_n)_r}\times $$

$$ \times x^ry^rF^{0:\,q+1\,,k+1}_{1:\,m\,,\,n}\left[\begin{matrix} -:a+r,(b_q+r)\,;a+r,(c_k+r)\,; \\ \alpha+2r:(\beta_m+r)\,;(\gamma_n+r)\,; \end{matrix} x,y\right], $$

$$ F^{0:\,q+1\,,k+1}_{1:\,m\,,\,n}\left[\begin{matrix} -:a,(b_q)\,;a,(c_k)\,; \\ \alpha:(\beta_m)\,;(\gamma_n)\,; \end{matrix} x,y\right] = \sum_{r=0}^{\infty}(-1)^r\frac{(a)_r\,(b_q)_r\,(c_k)_r}{r!(\alpha)_{2r}\,(\beta_m)_r\,(\gamma_n)_r}\times $$

$$ \times x^ry^rF^{1:q,k}_{1:\,m\,,\,n}\left[\begin{matrix} a+r:(b_q+r)\,;(c_k+r)\,; \\ \alpha+2r:(\beta_m+r)\,;(\gamma_n+r)\,; \end{matrix} x,y\right], $$

$$ F^{1:q,k}_{1:\,m\,,\,n}\left[\begin{matrix} a:(b_q)\,;(c_k)\,; \\ \alpha:(\beta_m)\,;(\gamma_n)\,; \end{matrix} x,y\right] = \sum_{r=0}^{\infty}\frac{(-1)^r(a)_{2r}\,(b_q)_r\,(c_k)_r}{r!(\alpha+r-1)_r(\alpha)_{2r}\,(\beta_m)_r\,(\gamma_n)_r}\times $$

$$ \times x^ry^rF^{1:q,k}_{0:m+1,n+1}\left[\begin{matrix} a+2r:(b_q+r)\,;(c_k+r)\,; \\ -:\alpha+2r,(\beta_m+r)\,;\alpha+2r,(\gamma_n+r)\,; \end{matrix} x,y\right], $$

$$ F^{1:q,k}_{0:m+1,n+1}\left[\begin{matrix} a:(b_q)\,;(c_k)\,; \\ -:\alpha,(\beta_m)\,;\alpha,(\gamma_n)\,; \end{matrix} x,y\right] = \sum_{r=0}^{\infty}\frac{(a)_{2r}\,(b_q)_r\,(c_k)_r}{r!(\alpha)_r(\alpha)_{2r}\,(\beta_m)_r\,(\gamma_n)_r}\times $$

$$ \times x^ry^rF^{1:q,k}_{1:\,m\,,\,n}\left[\begin{matrix} a+2r:(b_q+r)\,;(c_k+r)\,; \\ \alpha+2r:(\beta_m+r)\,;(\gamma_n+r)\,; \end{matrix} x,y\right], $$

$$ F^{2:\,q\,,k}_{0:m,n}\left[\begin{matrix} a_1,a_2:\ (b_q)\ ;\ (c_k)\,; \\ -:(\beta_m)\,;(\gamma_n)\,; \end{matrix} x,y\right] = \sum_{r=0}^{\infty}\frac{(a_1)_{2r}\,(a_2)_r\,(b_q)_r\,(c_k)_r}{r!\,(\beta_m)_r\,(\gamma_n)_r}\times $$

$$ \times x^ry^rF^{1:q+1,k+1}_{0:m,n}\left[\begin{matrix} a_1+2r:a_2+r,(b_q+r)\,;a_2+r,(c_k+r)\,; \\ -:(\beta_m+r)\,;(\gamma_n+r)\,; \end{matrix} x,y\right], $$

$$ F^{1:q+1,k+1}_{0:\ m\ ,\ n}\left[\begin{matrix} a_1:a_2,(b_q)\,;a_2,(c_k)\,; \\ -:\ (\beta_m)\ ;\ (\gamma_n)\ ; \end{matrix} x,y\right] = \sum_{r=0}^{\infty}(-1)^r\frac{(a_1)_{2r}\,(a_2)_r\,(b_q)_r\,(c_k)_r}{r!\,(\beta_m)_r\,(\gamma_n)_r}\times $$

$$ \times x^ry^rF^{2:\,q\,,k}_{0:m,n}\left[\begin{matrix} a_1+2r,a_2+r:\ (b_q+r)\ ;\ (c_k+r)\,; \\ -:(\beta_m+r)\,;(\gamma_n+r)\,; \end{matrix} x,y\right]. $$

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Article received 27.12.2023.
Corrections received 31.01.2025.

Челябинский физико-математический журнал. 2025. Т. 10, вып. 3. С. 513–526.

УДК 517.55 DOI: 10.47475/2500-0101-2025-10-3-513-526

РАЗЛОЖЕНИЯ ДЛЯ ГИПЕРГЕОМЕТРИЧЕСКИХ ФУНКЦИЙ КАМПЕ-ДЕ-ФЕРЬЕ

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В 1940–1941 гг. Берчнелл и Ченди, используя введённые ими взаимно обратные пары символических операторов, представили гипергеометрические функции Аппеля и Гумберта второго порядка в виде бесконечной суммы произведений более простых гипергеометрических функций. В 2006–2007 гг. Хасанов и Сривастава обобщили операторы Берчнелла — Ченди и нашли разложения для класса гипергеометрических функций многих переменных, однако применения этих операторов ограничивались гипергеометрическими функциями второго порядка. В данной работе получены разложения для гипергеометрических функций Кампе-де-Ферье произвольных порядков.

Ключевые слова: *функции Аппелля, функции Гумберта, операторы Берчнелла — Ченди, функции Кампе-де-Ферье, разложения гипергеометрических функций.*

Поступила в редакцию 27.12.2023.
После переработки 31.01.2025.

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Исследование поддержано Межконтинентальным исследовательским центром «Анализ и уравнения в частных производных» (Университет Гента, Бельгия), грант G.0H94.18N и Специальным исследовательским фондом программы Мафусаила (Университет Гента), грант 01M01021.