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# APPLICATIONS OF APPELL AND LAURICELLA HYPERGEOMETRIC FUNCTIONS TO SOLVING OF NEUMANN PROBLEM FOR DEGENERATE ELLIPTIC EQUATION

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**Hasanov Anvar**

*V.I.Romanovskiy Institute of Mathematics,  
Uzbekistan Academy of Sciences  
Tashkent, Uzbekistan  
and  
Ghent University  
Ghent, Belgium  
anvarhasanov@yahoo.com*

**Ergashev Tuhtasin**

*National Research University "TIAME"  
Tashkent, Uzbekistan  
V.I.Romanovskiy Institute of Mathematics,  
Uzbekistan Academy of Sciences  
Tashkent, Uzbekistan  
and  
Ghent University  
Ghent, Belgium  
ergashev.tukhtasin@gmail.com*

**Okboev Akmaljon**

*National Research University "TIAME"  
Tashkent, Uzbekistan  
akmaljon12012@gmail.com*

## Abstract

In this paper, Appell and Lauricella hypergeometric functions are investigated, and their known properties are applied to the solution of the Neumann problem for a three-dimensional degenerate elliptic equation. Fundamental solutions of the considered equation are expressed in terms of the Lauricella hypergeometric function of three variables, and an explicit solution of the Neumann problem in the first octant is obtained by means of the Appell hypergeometric function of the second kind. A decomposition formula for the Lauricella function is used to determine the order of singularity of the fundamental solutions. To justify the correctness of the obtained solution to the Neumann problem, an expansion formula for the Appell function is applied. The uniqueness of the solution of the posed problem is proved by the *abc*-method (a variant of the energy method).

**Keywords:** Appell and Lauricella hypergeometric functions; three-dimensional degenerate elliptic equation; PDE-systems of hypergeometric type; fundamental solution; Neumann problem.

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## 1. Introduction

The theory of the degenerate equations is one of the rapidly developing parts of the modern theory of partial differential equations, which has many applications in aerodynamics and gas dynamics [1] and irrigation problems [2]. It is known [3] that in the formulation of local and nonlocal boundary value problems and construction of their explicit solutions, the main role are played fundamental solutions of the considered equations. Fundamental solutions [4] for the degenerate partial differential equations contain hypergeometric functions and therefore further investigations depend on the properties of appropriate hypergeometric functions.

In particular, boundary value problems for the degenerate elliptic-hyperbolic equation

$$\operatorname{sign} y |y|^m u_{xx} + u_{yy} = 0, \quad m > 0 \quad (1)$$

were the subject of interest of many mathematicians, such as Tricomi, Gellerstedt, Holmgren, Frankl, Pulkin and others. In the case of negative parameters  $m < 0$ , a degenerate partial differential equations of type (1)

are usually called equations of the second kind. At present, interesting results [5], [6] have been obtained for the degenerate PDE of the second kind as well. It is known that the fundamental and other solutions of the equation (1) are closely related to the Gauss hypergeometric function and its properties.

In 1970, Copson [7] constructed fundamental solutions of the biaxially-symmetric potential equation

$$u_{\xi\xi} + u_{\eta\eta} + \frac{2\alpha}{\xi}u_{\xi} + \frac{2\beta}{\eta}u_{\eta} = 0, \xi > 0, \eta > 0, 0 < 2\alpha, 2\beta < 1 \tag{2}$$

in explicit forms via the Appell hypergeometric function  $F_2$ .

The equation (2) is related to a degenerate equation of elliptic type

$$\text{sign}y|y|^m u_{xx} + x^n u_{yy} = 0, m > 0, n > 0, x > 0 \tag{3}$$

which is a natural generalization of the Gellerstedt equation (1). The theory of boundary value problems for the equation (3) was developed in the last quarter of the twentieth century [8], [9]. In recent years, regular and generalized solutions to problems for a degenerate hyperbolic equation of the second kind with two lines of type change (in case  $-1 < m < 0, y < 0$  in equation (3)) have been constructed in explicit forms [10].

The three-dimensional analogue of the equation (2) in the form

$$u_{\xi\xi} + u_{\eta\eta} + u_{\zeta\zeta} + \frac{2\alpha}{\xi}u_{\xi} + \frac{2\beta}{\eta}u_{\eta} + \frac{2\gamma}{\zeta}u_{\zeta} = 0, 0 < 2\alpha, 2\beta, 2\gamma < 1 \tag{4}$$

has been studied in sufficient detail. Omitting a huge amount of papers on local and nonlocal problems for the singular elliptic equations, we note some works which are close to the present work. In the paper [11], fundamental solutions of the equation (4) are constructed, and in [12], [13], [14], [15], [16], [17], the unique solvability of the local, nonlocal and mixed problems for the equation (4) in a semi-infinite parallelepiped and a parts of unit ball are investigated. In the works [3], [18], the Neumann, Dirichlet and mixed (with the Dirichlet-Neumann conditions) problems for one four-dimensional degenerate elliptic equation are considered.

In this paper, we first investigate Neumann problem to a following three-dimensional degenerate equation

$$E_{\alpha,\beta,\gamma}(u) \equiv y^m z^k u_{xx} + x^n z^k u_{yy} + x^n y^m u_{zz} = 0, m > 0, n > 0, k > 0 \tag{5}$$

in the octant  $x > 0, y > 0, z > 0$ . In the recent paper [19], a solution to Dirichlet problem in the first octant for equation (5) is constructed in the explicit form.

The paper is organized as follows: First, we give some preliminary information, which will be used in what follows. Second, we find fundamental solutions for the equation (5). In the rest of the paper we formulate the problem and investigate it. Finally, we state our main result as a theorem.

## 2. Gauss, Appell and Lauricella hypergeometric functions and their properties

Gauss hypergeometric function is defined as following [20, p.56, Eq. 2.1(2)]

$$F(a, b; c; z) \equiv F \left[ \begin{matrix} a, b; \\ c; \end{matrix} z \right] = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, |z| < 1, \tag{6}$$

where  $a, b, c$  and  $z$  may be real or complex,  $(\lambda)_0 = 1, (\lambda)_n = \lambda(\lambda + 1)\dots(\lambda + n - 1), n = 1, 2, \dots$ , is a Pochhammer symbol.

For the Gauss hypergeometric function the transformation formula [20, p. 64, Eq. 2.1(22)]

$$F(a, b; c; x) = (1 - x)^{-b} F \left( c - a, b; c; \frac{x}{x - 1} \right) \tag{7}$$

and summation formula [20, p. 61, Eq. 2.1(14)]

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \text{Re}(c - a - b) > 0, c \neq 0, -1, -2, \dots \tag{8}$$

are valid.

The two functions  $F(a, b; c; z)$  and  $F(a', b'; c'; z)$  are said to be contiguous when one of the three differences  $a' - a, b' - b, c' - c$  is equal to  $\pm 1$ , the other two being zero. Three hypergeometric functions, contiguous two by two, are linked by a linear equation. There are fifteen relations of Gauss between contiguous functions [20, p.56, Eq. 2.8(31) - (45)].

Three functions  $F(a, b; c; z)$ ,  $F(a', b'; c'; z)$ ,  $F(a'', b''; c''; z)$  such that the differences  $a' - a$ ,  $b' - b$ ,  $c' - c$ ,  $a'' - a$ ,  $b'' - b$ ,  $c'' - c$  take one of the three values  $-1, 0, 1$  are still linked by a linear equation; there are  $\frac{26 \cdot 25}{1 \cdot 2}$  of these relations; here is an example [21, p. 3]

$$F(a + 1, b; c; z) - F(a, b; c; z) = \frac{b}{c} z F(a + 1, b + 1; c + 1; z).$$

Using the definition of the Gaussian function, it is easy to prove the contiguous relation

$$F(a, b + 1; c + 1; z) - F(a, b; c; z) = \frac{a(c - b)}{c(c + 1)} z F(a + 1, b + 1; c + 2; z). \tag{9}$$

The great success of the theory of hypergeometric function in one variable has stimulated the development of corresponding theory in two or more variables. Appell [22] has defined in 1880 four functions  $F_1$  to  $F_4$ , which are all analogues to Gauss'  $F(a, b; c; x)$ . For instance, a second Appell function  $F_2$  has a form

$$F_2(a, b_1, b_2; c_1, c_2; x, y) \equiv F_2 \left[ \begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} x, y \right] = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \frac{x^m y^n}{m! n!}, \quad |x| + |y| < 1.$$

For the Appell hypergeometric function, the differentiation formula [21, p. 19, Eq. (20)]

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} F_2 \left[ \begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} x, y \right] = \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} F_2 \left[ \begin{matrix} a + m + n, b_1 + m, b_2 + n; \\ c_1 + m, c_2 + n; \end{matrix} x, y \right], \tag{10}$$

the transformation formula [20, p.232, Eq. 5.11(8)]

$$F_2 \left[ \begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} x, y \right] = (1 - x - y)^{-a} F_2 \left[ \begin{matrix} a, c_1 - b_1, c_2 - b_2; \\ c_1, c_2; \end{matrix} \frac{x}{x + y - 1}, \frac{y}{x + y - 1} \right] \tag{11}$$

are valid.

The concept of a double hypergeometric series can be extended to triple sums. Such series were first studied by Lauricella [23], whose name they carry. Lauricella defined the four functions. In the region  $|x| + |y| + |z| < 1$ , one function is

$$\begin{aligned} F_A^{(3)} \left[ \begin{matrix} a, b_1, b_2, b_3; \\ c_1, c_2, c_3; \end{matrix} x, y, z \right] & \equiv F_A^{(3)}(a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) = \\ & = \sum_{m, n, p=0}^{\infty} \frac{(a)_{m+n+p} (b_1)_m (b_2)_n (b_3)_p}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m y^n z^p}{m! n! p!} = \\ & = \sum_{n, p=0}^{\infty} \frac{(a)_{n+p} (b_2)_n (b_3)_p}{(c_2)_n (c_3)_p} F(a + n + p; b_1; c_1; x) \frac{y^n z^p}{n! p!} = \end{aligned} \tag{12}$$

$$= \sum_{p=0}^{\infty} \frac{(a)_p (b_3)_p}{(c_3)_p} F_2(a + p, b_1, b_2; c_1, c_2; x, y) \frac{z^p}{p!}. \tag{13}$$

For Lauricella hypergeometric function, the differentiation formula [21, p. 19, Eq. (20)]

$$\begin{aligned} \frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p} F_A^{(3)} \left[ \begin{matrix} a, b_1, b_2, b_3; \\ c_1, c_2, c_3; \end{matrix} x, y, z \right] & = \frac{(a)_{m+n+p} (b_1)_m (b_2)_n (b_3)_p}{(c_1)_m (c_2)_n (c_3)_p} \times \\ & \times F_A^{(3)} \left[ \begin{matrix} a + m + n + p, b_1 + m, b_2 + n, b_3 + p; \\ c_1 + m, c_2 + n, c_3 + p; \end{matrix} x, y, z \right] \end{aligned} \tag{14}$$

is valid.

Appell states the system of partial differential equations which is satisfied by the function  $u \equiv F_A^{(3)}(a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)$  ([21, p. 117]):

$$\begin{cases} x(1-x)u_{xx} - xyu_{xy} - xzu_{xz} + [c_1 - (a + b_1 + 1)x]u_x - b_1yu_y - b_1zu_z - ab_1u = 0, \\ y(1-y)u_{yy} - xyu_{xy} - yzu_{yz} - b_2xu_x + [c_2 - (a + b_2 + 1)y]u_y - b_2zu_z - ab_2u = 0, \\ z(1-z)u_{zz} - xzu_{xz} - yzu_{yz} - b_3xu_x - b_3yu_y + [c_3 - (a + b_3 + 1)z]u_z - ab_3u = 0. \end{cases} \quad (15)$$

In investigation of the boundary-value problems for the degenerate elliptic equations, we need decompositions for hypergeometric functions of several variables in terms of simpler hypergeometric functions of the Gauss and Appell types [24]. The familiar operator method of Burcnall and Chaundy [25] has been used by them rather extensively for finding decomposition formulas for hypergeometric functions of two variables in terms of the classical Gauss hypergeometric function of one variable. The Appell function  $F_2$  has an expansion [25]:

$$F_2 \left[ \begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} x, y \right] = \sum_{r=0}^{\infty} \frac{(a)_r (b_1)_r (b_2)_r}{r! (c_1)_r (c_2)_r} x^r y^r F \left[ \begin{matrix} a + r, b_1 + r; \\ c_1 + r; \end{matrix} x \right] F \left[ \begin{matrix} a + r, b_2 + r; \\ c_2 + r; \end{matrix} y \right]. \quad (16)$$

Following the work [25], Hasanov and Srivastava [26] introduced operators generalizing the Burcnall-Chaundy operators and found expansion formulas for many triple hypergeometric functions, and they proved recurrent formulas when the dimension of hypergeometric function exceeds three. For instance, we have [26, Eq. (14)]

$$\begin{aligned} F_A^{(3)} \left[ \begin{matrix} a, b_1, b_2, b_3; \\ c_1, c_2, c_3; \end{matrix} x, y, z \right] &= \sum_{i,j,k=0}^{\infty} \frac{(a)_{i+j+k} (b_1)_{j+k} (b_2)_{i+k} (b_3)_{i+j}}{i! j! k! (c_1)_{j+k} (c_2)_{i+k} (c_3)_{i+j}} x^{j+k} y^{i+k} z^{i+j} \times \\ &\times F \left[ \begin{matrix} a + j + k, b_1 + j + k; \\ c_1 + j + k; \end{matrix} x \right] F \left[ \begin{matrix} a + i + j + k, b_2 + i + k; \\ c_2 + i + k; \end{matrix} y \right] F \left[ \begin{matrix} a + i + j + k, b_3 + i + j; \\ c_3 + i + j; \end{matrix} z \right]. \end{aligned} \quad (17)$$

In addition, the following relations between four contiguous Appell functions [21, p. 21]

$$\begin{aligned} \frac{b_1}{c_1} x F_2(a + 1, b_1 + 1, b_2; c_1 + 1, c_2; x, y) + \frac{b_2}{c_2} y F_2(a + 1, b_1, b_2 + 1; c_1, c_2 + 1; x, y) \\ = F_2(a + 1, b_1, b_2; c_1, c_2; x, y) - F_2(a, b_1, b_2; c_1, c_2; x, y) \end{aligned} \quad (18)$$

and between five contiguous Lauricella functions [27]

$$\begin{aligned} \frac{b_1}{c_1} x F_A^{(3)}(a + 1, b_1 + 1, b_2, b_3; c_1 + 1, c_2, c_3; x, y, z) \\ + \frac{b_2}{c_2} y F_A^{(3)}(a + 1, b_1, b_2 + 1, b_3; c_1, c_2 + 1, c_3; x, y, z) \\ + \frac{b_3}{c_3} z F_A^{(3)}(a + 1, b_1, b_2, b_3 + 1; c_1, c_2, c_3 + 1; x, y, z) \\ = F_A^{(3)}(a + 1, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) - F_A^{(3)}(a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) \end{aligned} \quad (19)$$

are valid.

Applying (9) and (12), we get new relation for the contiguous Lauricella functions

$$\begin{aligned} F_A^{(3)}(a, b_1 + 1, b_2, b_3; c_1 + 1, c_2, c_3; x, y, z) - F_A^{(3)}(a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) = \\ = \frac{a(c_1 - b_1)}{c_1(c_1 + 1)} x F_A^{(3)}(a + 1, b_1 + 1, b_2, b_3; c_1 + 2, c_2, c_3; x, y, z). \end{aligned} \quad (20)$$

We note that multidimensional analogous of the expansions (16), (17) and contiguous relations (18), (19) are found in [28] and [29], respectively (see, also [30], [31]).

The above formulas for the Appell and Lauricella hypergeometric functions are valid within the unit circle and the unit sphere, respectively. If the variables of these functions extend beyond the specified domains, then analytic continuations of the hypergeometric functions are used [32], [33]. Confluent forms the multiple Lauricella hypergeometric functions have also important applications [34].

### 3. Fundamental solutions of a degenerate three-dimensional elliptic equation

The first half-quarter of the three-dimensional Euclidean space  $\mathbb{R}_3$  is denoted by

$$D = \{(x, y, z) : x > 0, y > 0, z > 0\}.$$

Consider a degenerate three-dimensional elliptic equation (5):  $E_{\alpha,\beta,\gamma}(u) = 0$  in domain  $D$ .

Let  $(x, y, z)$  and  $(\xi, \eta, \zeta)$  be two points of the domain  $D$ . We seek the solution of the equation (5) in the form

$$u = r^{-2\delta} \omega(\rho, \sigma, \theta), \tag{21}$$

where  $\omega$  is a new unknown function,

$$\begin{aligned} \alpha &= \frac{n}{2(n+2)}, \quad \beta = \frac{m}{2(m+2)}, \quad \gamma = \frac{k}{2(k+2)}, \quad \delta = \alpha + \beta + \gamma + \frac{1}{2}; \\ q &= \frac{n+2}{2}, \quad p = \frac{m+2}{2}, \quad l = \frac{k+2}{2}; \quad \rho = -\frac{4x^q \xi^q}{q^2 r^2}, \quad \sigma = -\frac{4y^p \eta^p}{p^2 r^2}, \quad \theta = -\frac{4z^l \zeta^l}{l^2 r^2}; \\ r^2 &= \frac{1}{q^2} (x^q - \xi^q)^2 + \frac{1}{p^2} (y^p - \eta^p)^2 + \frac{1}{l^2} (z^l - \zeta^l)^2. \end{aligned}$$

It is obvious that

$$0 < 2\alpha < 1, \quad 0 < 2\beta < 1, \quad 0 < 2\gamma < 1; \quad q > 1, \quad p > 1, \quad l > 1.$$

Substituting (21) into the equation (5), we obtain a system of differential equations of hypergeometric type

$$\begin{cases} \rho(1-\rho)\omega_{\rho\rho} - \rho\sigma\omega_{\rho\sigma} - \rho\theta\omega_{\rho\theta} + [2\alpha - (\alpha + \delta + 1)\rho]\omega_{\rho} - \alpha\sigma\omega_{\sigma} - \alpha\theta\omega_{\theta} - \alpha\delta\omega = 0, \\ \sigma(1-\sigma)\omega_{\sigma\sigma} - \rho\sigma\omega_{\rho\sigma} - \sigma\theta\omega_{\sigma\theta} + [2\beta - (\beta + \delta + 1)\sigma]\omega_{\sigma} - \beta\rho\omega_{\rho} - \beta\theta\omega_{\theta} - \beta\delta\omega = 0, \\ \theta(1-\theta)\omega_{\theta\theta} - \rho\theta\omega_{\rho\theta} - \sigma\theta\omega_{\sigma\theta} + [2\gamma - (\gamma + \delta + 1)\theta]\omega_{\theta} - \gamma\rho\omega_{\rho} - \gamma\sigma\omega_{\sigma} - \gamma\delta\omega = 0. \end{cases} \tag{22}$$

Comparing the system (22) with the system (15), according to Lauricella's hypergeometric function theory [21, p.117], we obtain 8 particular solutions of the degenerate elliptic equation (5). One of these solutions looks like:

$$V(x, y, z; \xi, \eta, \zeta) = \kappa r^{-2\delta} F_A^{(3)}(\delta; \alpha, \beta, \gamma; 2\alpha, 2\beta, 2\gamma; \rho, \sigma, \theta), \tag{23}$$

where  $\kappa$  is constant, which is determined when solving boundary value problem for the equation (5).

**Lemma 3.1.** *The constructed function (23) has the following property:*

$$\left. \frac{\partial V}{\partial x} \right|_{x=0} = \left. \frac{\partial V}{\partial y} \right|_{y=0} = \left. \frac{\partial V}{\partial z} \right|_{z=0} = 0. \tag{24}$$

**Proof.** Using the differentiation formula (14) for Lauricella hypergeometric function  $F_A^{(3)}$ , we calculate the partial derivative with respect to  $x$  of the function  $V$ :

$$\begin{aligned} \frac{\partial V}{\partial x} &= -\delta \kappa r^{-2\delta-2} F_A^{(3)}(\delta; \alpha, \beta, \gamma; 2\alpha, 2\beta, 2\gamma; \rho, \sigma, \theta) \frac{\partial r^2}{\partial x} + \\ &+ \frac{\delta}{2} \kappa r^{-2\delta} F_A^{(3)}(\delta + 1; 1 + \alpha, \beta, \gamma; 1 + 2\alpha, 2\beta, 2\gamma; \rho, \sigma, \theta) \frac{\partial \rho}{\partial x} + \\ &+ \frac{\delta}{2} \kappa r^{-2\delta} F_A^{(3)}(\delta + 1; \alpha, 1 + \beta, \gamma; 2\alpha, 1 + 2\beta, 2\gamma; \rho, \sigma, \theta) \frac{\partial \sigma}{\partial x} + \\ &+ \frac{\delta}{2} \kappa r^{-2\delta} F_A^{(3)}(\delta + 1; \alpha, \beta, 1 + \gamma; 2\alpha, 2\beta, 1 + 2\gamma; \rho, \sigma, \theta) \frac{\partial \theta}{\partial x}. \end{aligned}$$

A contiguous relation (19) allows us to simplify the expression of this derivative to the form

$$\begin{aligned} \frac{\partial V}{\partial x} &= -\frac{2\delta\kappa}{q} x^{q-1} (x^q - \xi^q) r^{-2\delta-2} F_A^{(3)}(1 + \delta; \alpha, \beta, \gamma; 2\alpha, 2\beta, 2\gamma; \rho, \sigma, \theta) - \\ &- \frac{2\delta\kappa}{q} x^{q-1} \xi^q r^{-2\delta-2} F_A^{(3)}(1 + \delta; 1 + \alpha, \beta, \gamma; 1 + 2\alpha, 2\beta, 2\gamma; \rho, \sigma, \theta). \end{aligned}$$

Next, by virtue of (20), we obtain

$$\begin{aligned} \frac{\partial V}{\partial x} &= -\frac{2\delta\kappa}{q} x^{2q-1} r^{-2\delta-2} F_A^{(3)}(\delta+1; \alpha, \beta, \gamma; 2\alpha, 2\beta, 2\gamma; \rho, \sigma, \theta) - \\ &- \frac{\alpha\delta(\delta+1)\kappa}{q(1+\alpha)(1+2\alpha)} x^{q-1} \xi^q \rho r^{-2\delta-2} F_A^{(3)}(\delta+1; \alpha+1, \beta, \gamma; 2\alpha+2, 2\beta, 2\gamma; \rho, \sigma, \theta). \end{aligned}$$

Hence, since  $q > 1$ , then  $\left. \frac{\partial V}{\partial x} \right|_{x=0} = 0$ . The remaining statements in (24) are proved similarly. □

**Lemma 3.2.** *If  $0 < 2\alpha, 2\beta, 2\gamma < 1$ , then a function  $V$  has a singularity of the order  $\frac{1}{r}$  as  $r \rightarrow 0$ .*

Proof. Due to the expansion (17), the particular solution (23) can be reduced to the form

$$\begin{aligned} V &= \kappa r^{-2\delta} \sum_{i,j,k=0}^{\infty} \frac{(\delta)_{i+j+k}(\alpha)_{j+k}(\beta)_{i+k}(\gamma)_{i+j}}{i!j!k!(2\alpha)_{j+k}(2\beta)_{i+k}(2\gamma)_{i+j}} \rho^{j+k} \sigma^{i+k} \theta^{i+j} F \left[ \begin{matrix} \delta+j+k, \alpha+j+k; \\ 2\alpha+j+k; \end{matrix} \rho \right] \times \\ &\times F \left[ \begin{matrix} \delta+i+j+k, \beta+i+k; \\ 2\beta+i+k; \end{matrix} \sigma \right] F \left[ \begin{matrix} \delta+i+j+k, \gamma+i+j; \\ 2\gamma+i+j; \end{matrix} \theta \right]. \end{aligned} \tag{25}$$

Using the autotransformation formula (7) for the Gaussian hypergeometric function, from (25) we obtain

$$V(x, y, z; \xi, \eta, \zeta) = \frac{1}{r} \cdot r_1^{-2\alpha} r_2^{-2\beta} r_3^{-2\gamma} V^*(x, y, z; \xi, \eta, \zeta), \tag{26}$$

where

$$\begin{aligned} V^*(x, y, z; \xi, \eta, \zeta) &= \kappa \sum_{i,j,k=0}^{\infty} \frac{(\delta)_{i+j+k}(\alpha)_{j+k}(\beta)_{i+k}(\gamma)_{i+j}}{i!j!k!(2\alpha)_{j+k}(2\beta)_{i+k}(2\gamma)_{i+j}} \times \\ &\times \left(1 - \frac{r^2}{r_1^2}\right)^{j+k} F\left(\alpha - \beta - \gamma - \frac{1}{2}, \alpha + j + k; 2\alpha + j + k; 1 - \frac{r^2}{r_1^2}\right) \times \\ &\times \left(1 - \frac{r^2}{r_2^2}\right)^{i+k} F\left(\beta - \alpha - \gamma - \frac{1}{2} - j, \beta + i + k; 2\beta + i + k; 1 - \frac{r^2}{r_2^2}\right) \times \\ &\times \left(1 - \frac{r^2}{r_3^2}\right)^{i+j} F\left(\gamma - \alpha - \beta - \frac{1}{2} - k, \gamma + i + j; 2\gamma + i + j; 1 - \frac{r^2}{r_3^2}\right), \end{aligned} \tag{27}$$

where

$$\left. \begin{matrix} r_1^2 \\ r_2^2 \\ r_3^2 \end{matrix} \right\} = \left( \frac{1}{q} x^q + \frac{1}{q} \xi^q \right)^2 + \left( \frac{1}{p} y^p + \frac{1}{p} \eta^p \right)^2 + \left( \frac{1}{l} z^l + \frac{1}{l} \zeta^l \right)^2.$$

Let us show that the function  $V^*(x, y, z; \xi, \eta, \zeta)$  is bounded at  $r \rightarrow 0$ . Applying the summation formula (8) to each hypergeometric function included in the expansion formula (27) after passing to the limit, we obtain

$$\lim_{r \rightarrow 0} V^*(x, y, z; \xi, \eta, \zeta) = \frac{\Gamma(2\alpha)\Gamma(2\beta)\Gamma(2\gamma)\Gamma(\delta-\alpha)\Gamma(\delta-\beta)\Gamma(\delta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma^3(\delta)} I(\alpha, \beta, \gamma),$$

where

$$I(\alpha, \beta, \gamma) = \sum_{i,j,k=0}^{\infty} \frac{(\alpha)_{j+k}(\beta)_{i+k}(\gamma)_{i+j}(\delta-\beta)_j(\delta-\gamma)_k}{(\delta)_{i+j+k}(\delta)_{j+k}i!j!k!}. \tag{28}$$

Using the famous properties of the Pochhammer symbol, we transform an expression  $I(\alpha, \beta, \gamma)$  defined in (28). Applying the summation formula (8) and the definition (6) of the Gauss hypergeometric function several times, we obtain

$$I(\alpha, \beta, \gamma) = \sum_{j,k=0}^{\infty} \frac{(\alpha)_{j+k}(\beta)_k(\gamma)_j(\delta-\beta)_j(\delta-\gamma)_k}{(\delta)_{j+k}(\delta)_{j+k}j!k!} F(\beta+k, \gamma+j; \delta+j+k; 1) =$$

$$\begin{aligned}
 &= \frac{\Gamma(\delta)\Gamma(\delta - \beta - \gamma)}{\Gamma(\delta - \beta)\Gamma(\delta - \gamma)} \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\delta)_k k!} F(\alpha + k, \gamma; \delta + k; 1) = \\
 &= \frac{\Gamma^2(\delta)\Gamma(\delta - \alpha - \gamma)\Gamma(\delta - \beta - \gamma)}{\Gamma(\delta - \alpha)\Gamma(\delta - \beta)\Gamma^2(\delta - \gamma)} F(\alpha, \beta; \delta - \gamma; 1) = \frac{\Gamma^2(\delta)\Gamma(\delta - \alpha - \beta - \gamma)}{\Gamma(\delta - \alpha)\Gamma(\delta - \beta)\Gamma(\delta - \gamma)}.
 \end{aligned}$$

Hence,

$$\lim_{r \rightarrow 0} V^*(x, y, z; \xi, \eta, \zeta) = \frac{\sqrt{\pi}\Gamma(2\alpha)\Gamma(2\beta)\Gamma(2\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}. \tag{29}$$

Thus, by virtue of (29) out of (26) at  $r \rightarrow 0$ , we finally have the estimate

$$|V(x, y, z; \xi, \eta, \zeta)| \leq \frac{C}{r}, \tag{30}$$

where  $C$  is a constant.

An inequality (30) proves that the function  $V(x, y, z; \xi, \eta, \zeta)$  at  $r \rightarrow 0$  has a singularity of the order  $1/r$  and, therefore, it is a fundamental solution of the equation (5).  $\square$

#### 4. Statement of the Neumann problem and the uniqueness theorem

We introduce the following notation:

$$D = \{(x, y, z) : x > 0, y > 0, z > 0\}, \quad \bar{D} = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\},$$

$$S_1 = \{(x, y, 0) : x > 0, y > 0, z = 0\}, \quad S_2 = \{(x, 0, z) : x > 0, y = 0, z > 0\},$$

$$S_3 = \{(0, y, z) : x = 0, y > 0, z > 0\}, \quad R^2 = \frac{1}{q^2}x^{2q} + \frac{1}{p^2}y^{2p} + \frac{1}{l^2}z^{2l}.$$

**Neumann problem.** Find a solution  $u(x, y, z)$  of the equation (5) from the class  $C(\bar{D}) \cap C^1(D \cup S_1 \cup S_2 \cup S_3) C^2(D)$  that satisfies the conditions

$$\left. \frac{\partial u(x, y, z)}{\partial z} \right|_{z=0} = \nu_1(x, y), (x, y) \in S_1, \tag{31}$$

$$\left. \frac{\partial u(x, y, z)}{\partial y} \right|_{y=0} = \nu_2(x, z), (x, z) \in S_2, \tag{32}$$

$$\left. \frac{\partial u(x, y, z)}{\partial x} \right|_{x=0} = \nu_3(y, z), (y, z) \in S_3, \tag{33}$$

$$\lim_{R \rightarrow \infty} u(x, y, z) = 0, \tag{34}$$

where  $\nu_1(y, z), \nu_2(x, z), \nu_3(x, y)$  are given functions, and for sufficiently large values of  $R$  the inequalities are satisfied

$$\nu_1(x, y) = \frac{\tilde{\nu}_1(x, y)}{\left(1 + \frac{1}{q^2}x^{2q} + \frac{1}{p^2}y^{2p}\right)^{1/2-\gamma+\varepsilon_1}}, \quad \tilde{\nu}_1(x, y) \in C(0 < x, y < \infty), \tag{35}$$

$$\nu_2(x, z) = \frac{\tilde{\nu}_2(x, z)}{\left(1 + \frac{1}{q^2}x^{2q} + \frac{1}{l^2}z^{2l}\right)^{1/2-\beta+\varepsilon_2}}, \quad \tilde{\nu}_2(x, z) \in C(0 < x, z < \infty), \tag{36}$$

$$\nu_3(y, z) = \frac{\tilde{\nu}_3(y, z)}{\left(1 + \frac{1}{p^2}y^{2p} + \frac{1}{l^2}z^{2l}\right)^{1/2-\alpha+\varepsilon_3}}, \quad \tilde{\nu}_3(y, z) \in C(0 < y, z < \infty), \tag{37}$$

$\varepsilon_1, \varepsilon_2, \varepsilon_3$  are sufficiently small positive numbers.

**The uniqueness of the solution.** One can readily check the validity of the following relation:

$$uE(v) - vE(u) = y^m z^k \frac{\partial}{\partial x} \left( u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right) + x^n z^k \frac{\partial}{\partial y} \left( u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y} \right) + x^n y^m \frac{\partial}{\partial z} \left( u \frac{\partial v}{\partial z} - v \frac{\partial u}{\partial z} \right).$$

We denote by  $D_R$  the finite part of the region  $D$ , bounded by the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$  and the half-quarter part

$$S_R := \left\{ (x, y, z) : \frac{1}{q^2}x^{2q} + \frac{1}{p^2}y^{2p} + \frac{1}{l^2}z^{2l} = R^2, \quad q > 1, \quad p > 1, \quad l > 1 \right\}$$

of the higher-order ellipsoid.

Let  $D_{R,\varepsilon}$  be a sub-domain of  $D_R$  at a distance  $\varepsilon > 0$  from its boundary and  $\cos(N, x)dS = dydz$ ,  $\cos(N, y)dS = dx dz$ ,  $\cos(N, z)dS = dx dy$ ,  $N$  is the outer normal to  $\partial D_{R,\varepsilon}$ .

Integrate both sides of above given equality on the domain  $D_R$  and use the classical formula of Gauss-Ostrogradsky:

$$\begin{aligned} \iiint_{D_{R,\varepsilon}} (uE(v) - vE(u)) dx dy dz &= \iint_{\partial D_{R,\varepsilon}} \left[ y^m z^k \left( u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right) \cos(N, x) + \right. \\ &\quad \left. + x^n z^k \left( u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y} \right) \cos(N, y) + x^n y^m \left( u \frac{\partial v}{\partial z} - v \frac{\partial u}{\partial z} \right) \cos(N, z) \right] dS. \end{aligned}$$

Using the equality

$$uE_{\alpha,\beta,\gamma}(u) + y^m z^k u_x^2 + x^n z^k u_y^2 + x^n y^m u_z^2 = y^m z^k \frac{\partial}{\partial x} (uu_x) + x^n z^k \frac{\partial}{\partial y} (uu_y) + x^n y^m \frac{\partial}{\partial z} (uu_z),$$

we obtain

$$\begin{aligned} \iiint_{D_{R,\varepsilon}} uE(u) dx dy dz + \iiint_{D_{R,\varepsilon}} [y^m z^k u_x^2 + x^n z^k u_y^2 + x^n y^m u_z^2] dx dy dz &= \\ = \iiint_{D_{R,\varepsilon}} \left[ y^m z^k \frac{\partial}{\partial x} (uu_x) + x^n z^k \frac{\partial}{\partial y} (uu_y) + x^n y^m \frac{\partial}{\partial z} (uu_z) \right] dx dy dz. \end{aligned}$$

Applying again the formula of Gauss-Ostrogradskii to this equality and letting  $\varepsilon \rightarrow 0$ , we get

$$\begin{aligned} \iiint_D (y^m z^k u_x^2 + x^n z^k u_y^2 + x^n y^m u_z^2) dx dy dz &= \iint_{S_1} x^n y^m \tau_1(x, y) \nu_1(x, y) dx dy + \\ + \iint_{S_2} x^n z^k \tau_2(x, z) \nu_2(x, z) dx dz &+ \iint_{S_3} y^m z^k \tau_3(y, z) \nu_3(y, z) dy dz + \iint_{S_R} uC[u] dS_R, \end{aligned} \quad (38)$$

where  $\tau_1(x, y) = u(x, y, 0)$ ,  $(x, y) \in \bar{S}_1$ ;  $\tau_2(x, z) = u(x, 0, z)$ ,  $(x, z) \in \bar{S}_2$ ;  $\tau_3(y, z) = u(0, y, z)$ ,  $(y, z) \in \bar{S}_3$ ;  $C(u) = y^m z^k u_x \cos(N, x) + x^n z^k u_y \cos(N, y) + x^n y^m u_z \cos(N, z)$ .

**Theorem 4.1.** *The Neumann problem (31) – (34) for equation (5) can have at most one solution.*

**Proof.** To prove the uniqueness of the solution, as usual, we suppose that the problem has two  $v, w$  solutions. Denoting  $u = v - w$  we have that  $u$  satisfies homogeneous Neumann problem ( $\nu_1 = 0$ ,  $\nu_2 = 0$ ,  $\nu_3 = 0$ ,  $\lim_{R \rightarrow \infty} u = 0$ ). Further we have to prove that the homogeneous problem has only trivial solution. In this case from (38) one can easily get

$$\iiint_D (y^m z^k u_x^2 + x^n z^k u_y^2 + x^n y^m u_z^2) dx dy dz = 0.$$

Hence, it follows that  $u_x = u_y = u_z = 0$ , which implies that  $u$  is a constant function. Considering condition (34), we conclude that  $u(x, y, z) \equiv 0$  in  $\bar{D}$ .  $\square$

## 5. Existence of a solution to the Neumann problem

Consider a function

$$\begin{aligned} u(x, y, z) &= - \int_0^\infty \int_0^\infty \nu_1(t, s) t^n s^m V(x, y, z; t, s, 0) dt ds - \\ &- \int_0^\infty \int_0^\infty \nu_2(t, s) t^n s^k V(x, y, z; t, 0, s) dt ds - \int_0^\infty \int_0^\infty \nu_3(t, s) t^m s^k V(x, y, z; 0, t, s) dt ds, \end{aligned} \quad (39)$$

where  $V(x, y, z; \xi, \eta, \zeta)$  is a fundamental solution defined in (23). By virtue of (13), from (39) we get

$$u(x, y, z) = u_1(x, y, z) + u_2(x, y, z) + u_3(x, y, z), \tag{40}$$

where

$$u_1(x, y, z) = -\kappa \int_0^\infty \int_0^\infty \frac{\nu_1(t, s) t^n s^m}{r_{z0}^{2\delta}} F_2 \left[ \begin{matrix} \delta, \alpha, \beta; \\ 2\alpha, 2\beta; \end{matrix} -\frac{4x^q t^q}{q^2 r_{z0}^2}, -\frac{4y^p s^p}{p^2 r_{z0}^2} \right] dt ds, \tag{41}$$

$$u_2(x, y, z) = -\kappa \int_0^\infty \int_0^\infty \frac{\nu_2(t, s) t^n s^k}{r_{y0}^{2\delta}} F_2 \left[ \begin{matrix} \delta, \alpha, \gamma; \\ 2\alpha, 2\gamma; \end{matrix} -\frac{4x^q t^q}{q^2 r_{y0}^2}, -\frac{4z^l s^l}{l^2 r_{y0}^2} \right] dt ds, \tag{42}$$

$$u_3(x, y, z) = -\kappa \int_0^\infty \int_0^\infty \frac{\nu_3(t, s) t^m s^k}{r_{x0}^{2\delta}} F_2 \left[ \begin{matrix} \delta, \beta, \gamma; \\ 2\beta, 2\gamma; \end{matrix} -\frac{4y^p t^p}{p^2 r_{x0}^2}, -\frac{4z^l s^l}{l^2 r_{x0}^2} \right] dt ds, \tag{43}$$

$$\begin{aligned} r_{z0}^2 &= \frac{1}{q^2} (x^q - t^q)^2 + \frac{1}{p^2} (y^p - s^p)^2 + \frac{1}{l^2} z^{2l}, \quad r_{y0}^2 = \frac{1}{q^2} (x^q - t^q)^2 + \frac{1}{p^2} y^{2p} + \frac{1}{l^2} (z^l - s^l)^2, \\ r_{x0}^2 &= \frac{1}{q^2} x^{2q} + \frac{1}{p^2} (y^p - t^p)^2 + \frac{1}{l^2} (z^l - s^l)^2, \\ \kappa &= \frac{1}{2\pi} q^{-2\alpha} p^{-2\beta} l^{-2\gamma} \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(2\alpha+2\beta+2\gamma)}{\Gamma(2\alpha)\Gamma(2\beta)\Gamma(2\gamma)\Gamma(\alpha+\beta+\gamma)}. \end{aligned} \tag{44}$$

**Lemma 5.1.** *If the function  $\nu_1(x, y)$  can be represented as (35), then the function  $u_1(x, y, z)$  defined by the equality (41) is a regular solution of the equation (5) in the domain  $D$  satisfying the condition (34) and boundary conditions*

$$\left. \frac{\partial u_1(x, y, z)}{\partial z} \right|_{z=0} = \nu_1(x, y), \quad \left. \frac{\partial u_1(x, y, z)}{\partial y} \right|_{y=0} = 0, \quad \left. \frac{\partial u_1(x, y, z)}{\partial x} \right|_{x=0} = 0. \tag{45}$$

**Proof.** Using the differentiation formula (10) for Appell’s hypergeometric function  $F_2$ , we calculate the partial derivative with respect to  $z$  of the function  $u_1(x, y, z)$ . By virtue of the contiguous relation (18), we obtain

$$\frac{\partial u_1(x, y, z)}{\partial z} = \frac{2\delta k}{l} z^{2l-1} \int_0^\infty \int_0^\infty \frac{\nu_1(t, s) t^n s^m}{r_{z0}^{2\delta+2}} F_2 \left[ \begin{matrix} \delta+1, \alpha, \beta; \\ 2\alpha, 2\beta; \end{matrix} 1 - \frac{r_{z1}^2}{r_{z0}^2}, 1 - \frac{r_{z2}^2}{r_{z0}^2} \right] dt ds, \tag{46}$$

where  $r_{z1} = r_1|_{\zeta=0}$ ,  $r_{z2} = r_2|_{\zeta=0}$ .

Applying the expansion (16) and transformation (7) to the integrand  $F_2$  in (46) successively, we obtain

$$\frac{\partial u_1(x, y, z)}{\partial z} = \frac{2\delta k}{l} z^{2l-1} \int_0^\infty \int_0^\infty \frac{\nu_1(t, s) t^{2q-2} s^{2p-2}}{r_{z1}^{2\alpha} r_{z2}^{2\beta} r_{z0}^{2\gamma+3}} P(x, y, z; t, s) dt ds, \tag{47}$$

where

$$\begin{aligned} P(x, y, z; t, s) &= \sum_{r=0}^\infty \frac{(\delta+1)_r (\alpha)_r (\beta)_r}{(2\alpha)_r (2\beta)_r r!} \left(1 - \frac{r_{z0}^2}{r_{z1}^2}\right)^r \left(1 - \frac{r_{z0}^2}{r_{z2}^2}\right)^r \times \\ &\times F\left(\alpha - \beta - \gamma - \frac{3}{2}, \alpha + r; 2\alpha + r; 1 - \frac{r_{z0}^2}{r_{z1}^2}\right) F\left(\beta - \alpha - \gamma - \frac{3}{2}, \beta + r; 2\beta + r; 1 - \frac{r_{z0}^2}{r_{z2}^2}\right). \end{aligned}$$

By virtue of the summation formula (8), it is easy to calculate a value of the integrand  $P(x, y, z; t, s)$  at  $z = 0$ :

$$\lim_{z \rightarrow 0} P(x, y, z; t, s) = \frac{\Gamma(2\alpha)\Gamma(2\beta)\Gamma(\gamma+3/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta+1)}. \tag{48}$$

Introduce in the integrand in (47) instead of  $t$  and  $s$  new variables

$$\frac{1}{q}t^q = \frac{1}{q}x^q + \frac{1}{l}z^l\xi, \quad \frac{1}{p}s^p = \frac{1}{p}y^p + \frac{1}{l}z^l\eta,$$

and pass to the limit at  $z \rightarrow 0$ . Then, taking into account the expression (44) for the coefficient  $\kappa$  and the limit value (48), we have

$$\lim_{z \rightarrow 0} \frac{\partial u_1}{\partial z} = \frac{1 + 2\gamma}{2\pi} \nu_1(x, y) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\lambda d\mu}{(1 + \lambda^2 + \mu^2)^{\gamma+3/2}}.$$

Passing to polar coordinates  $\lambda = r \cos \varphi$ ,  $\mu = r \sin \varphi$ , it is easy to calculate the double improper integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\lambda d\mu}{(1 + \lambda^2 + \mu^2)^{\gamma+3/2}} = \frac{1}{2} \int_0^{2\pi} d\varphi \int_0^{\infty} \frac{d(1 + r^2)}{(1 + r^2)^{\gamma+3/2}} = \frac{2\pi}{1 + 2\gamma}.$$

So,

$$\left. \frac{\partial u_1(x, y, z)}{\partial z} \right|_{z=0} = \nu_1(x, y). \tag{49}$$

Similarly, one can also verify that

$$\left. \frac{\partial u_1(x, y, z)}{\partial y} \right|_{y=0} = 0, \quad \left. \frac{\partial u_1(x, y, z)}{\partial x} \right|_{x=0} = 0. \tag{50}$$

Now from (49) and (50) it follows that the conditions (45) are satisfied.

Next, we show that if the function  $\nu_1(x, y)$  can be represented as (35), then the function  $u_1(x, y, z)$  defined by the equality (41) tends to zero at infinity.

Using the transformation formula (11) for Appell function, we write the function in the form

$$u_1(x, y, z) = \kappa \int_0^{\infty} \int_0^{\infty} \frac{\nu_1(t, s) t^n s^m}{\rho^{2\delta}} F_2 \left[ \begin{matrix} \delta, \alpha, \beta; \\ 2\alpha, 2\beta; \end{matrix} \frac{4x^q t^q}{q^2 \rho^2}, \frac{4y^p s^p}{p^2 \rho^2} \right] dt ds, \tag{51}$$

where

$$\rho^2 = \frac{1}{q^2} (x^q + t^q)^2 + \frac{1}{p^2} (y^p + s^p)^2 + \frac{1}{l^2} z^{2l}.$$

It is easy to see that in (51) the following inequality holds

$$\frac{4x^q t^q}{q^2 \rho^2} + \frac{4y^p s^p}{p^2 \rho^2} < 1, \quad x > 0, y > 0, z > 0, t > 0, s > 0.$$

Let us prove the when the point  $(x, y, z)$  tends to infinity, i.e. when  $R \rightarrow \infty$ , the function (51) tends to zero. It is known from the theory of Appell functions, that if  $|x| + |y| < 1$ , then for any values of the numerical parameters the Appell hypergeometric function  $F_2$  is bounded:

$$|F_2(a, b_1, b_2; c_1, c_2; x, y)| \leq C_1, \quad |x| + |y| < 1.$$

Next, applying the representation (35) for given function  $\nu_1(x, y)$ , we obtain

$$|u_1| \leq C_2 \int_0^{\infty} \int_0^{\infty} \frac{t^n s^m dt ds}{\left(1 + \frac{1}{q^2} t^{2q} + \frac{1}{p^2} s^{2p}\right)^{1/2-\gamma+\varepsilon_1} \left[\frac{1}{q^2} (x^q + t^q)^2 + \frac{1}{p^2} (y^p + s^p)^2 + \frac{1}{l^2} z^{2l}\right]^{\delta}}. \tag{52}$$

Substituting

$$t = (qR\mu)^{1/q}, \quad s = (pR\nu)^{1/p}$$

for  $t$  and  $s$  in the last double improper integral (52), we get

$$|u_1| \leq C_3 \frac{q^{2\alpha} p^{2\beta}}{R^{2\varepsilon_1}} K(x, y; R), \tag{53}$$

where

$$K(x, y; R) = \int_0^\infty \int_0^\infty \frac{\mu^{2\alpha} \nu^{2\beta} d\mu d\nu}{(\mu^2 + \nu^2 + \frac{1}{R^2})^{1/2-\gamma+\varepsilon_1} \left(1 + \mu^2 + \nu^2 + \frac{2x^q}{qR} + \frac{2y^p}{pR}\right)^\delta}. \tag{54}$$

We will show that the double improper integral on the right-hand side (54) is bounded as  $R \rightarrow \infty$ . Indeed, using the formula [35]

$$\underbrace{\int_0^{+\infty} \dots \int_0^{+\infty}}_n \frac{x_1^{p_1-1} \dots x_n^{p_n-1} dx_1 \dots dx_n}{[(r_1 x_1)^{q_1} + \dots + (r_n x_n)^{q_n}]^t [1 + (r_1 x_1)^{q_1} + \dots + (r_n x_n)^{q_n}]^s} = \frac{\Gamma(p_1/q_1) \dots \Gamma(p_n/q_n) \Gamma(P-t) \Gamma(s+t-P)}{q_1 q_2 \dots q_n r_1^{p_1 q_1} \dots r_n^{p_n q_n} \Gamma(P) \Gamma(s)}, \quad P := \frac{p_1}{q_1} + \dots + \frac{p_n}{q_n},$$

where  $p_k, q_k, r_k$  and  $s$  are positive numbers ( $k = \overline{1, n}$ ),  $0 < P - t < s$ , and passing to the limit as  $R \rightarrow \infty$ , we will have the relation

$$\lim_{R \rightarrow \infty} K(x, y; R) \leq \frac{\Gamma(\frac{1}{2} + \alpha) \Gamma(\frac{1}{2} + \beta) \Gamma(\delta - \varepsilon_1) \Gamma(\varepsilon_1 - \gamma)}{4\Gamma(1 + \alpha + \beta)\Gamma(\delta)}, \quad \gamma < \varepsilon_1 < \delta. \tag{55}$$

Thus, by virtue of (53) and (55) the following estimate is valid:

$$|u_1| \leq \frac{C_3}{R^{2\varepsilon_1}}, \quad \gamma < \varepsilon_1 < \delta, \quad R \rightarrow \infty.$$

So, we conclude that the function (41) vanishes at infinity. □

**Remark 5.1.** Repeating the arguments given in Lemma 5.1, we can prove two more lemmas concerning the functions  $u_2(x, y, z)$  and  $u_3(x, y, z)$  defined by the equalities (42) and (43), respectively. Thus, if the representations (36) and (37) are valid for the given functions  $\nu_2(x, z)$  and  $\nu_3(y, z)$ , then each of the functions  $u_2(x, y, z)$  and  $u_3(x, y, z)$  is a solution to the degenerate elliptic equation (5) that vanishes at infinity and satisfies the set of conditions

$$\begin{aligned} \frac{\partial u_2(x, y, z)}{\partial z} \Big|_{z=0} &= 0, \quad \frac{\partial u_2(x, y, z)}{\partial y} \Big|_{y=0} = \nu_2(x, z), \quad \frac{\partial u_2(x, y, z)}{\partial x} \Big|_{x=0} = 0, \\ \frac{\partial u_3(x, y, z)}{\partial z} \Big|_{z=0} &= 0, \quad \frac{\partial u_3(x, y, z)}{\partial y} \Big|_{y=0} = 0, \quad \frac{\partial u_3(x, y, z)}{\partial x} \Big|_{x=0} = \nu_3(y, z), \end{aligned}$$

respectively.

**Theorem 5.1.** *If the functions  $\nu_1(x, y), \nu_2(x, z)$  and  $\nu_3(y, z)$  satisfy the conditions (35)–(37) then the function  $u(x, y, z)$  defined in (40) is a regular solution of the equation (5) in the domain  $D$  satisfying the conditions (31) – (34).*

Proof. The statement of the Theorem 5.1 follows from the Lemma 5.1 and Remark 5.1. □

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