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COEFFICIENT INVERSE PROBLEM FOR WHITHAM TYPE TWO-DIMENSIONAL DIFFERENTIAL EQUATION WITH IMPULSE EFFECTS

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In the article the questions of unique solvability and determination of the redefinition coefficient function in the initial inverse problem for two-dimensional Whitham-type partial differential equation with impulse effects are studied. The modified method of characteristics allows partial differential equations of the first order to be represented as ordinary differential equations that describe the change of an unknown function along the line of characteristics. The unique solvability of the two-dimensional inverse problem is proved by the method of successive approximations and contraction mappings. The definition of the unknown coefficient is reduced to solving the Volterra integral equation of the first kind.

Keywords: *inverse problem, two-dimensional Whitham type equation, determination of the coefficient function, method of successive approximations, unique solvability.*

1. Introduction. Problem statement

As is known, the dynamics of evolving processes sometimes undergoes abrupt changes. Often such short-term perturbations are interpreted as impulses. That is, we actually have a dynamic system with impulse effects, the solutions of which are functions with first kind "discontinuities". Differential and integro-differential equations with impulse effects have applications in biological, chemical and physical sciences, ecology, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc. [1–5]. In particular, such kind of problems appear in biophysics at micro- and nano-scales [6–10]. A lot of publications of studying on differential equations with impulse effects, describing many natural and technical processes, are appearing [11–23].

Partial differential equations of the first order are locally solved by methods of the theory of ordinary differential equations by reducing them to a characteristic system. The application of the method of characteristics to the solution of partial differential equations of the first order makes it possible to reduce the study of wave evolution [24]. In [25, 26], methods for integrating nonlinear partial differential equations of the first order were developed. Further, many papers appeared devoted to the study of questions of the unique solvability of the Cauchy problem for different types of partial differential equations of the first order (see, for example, [27–37]). In [38,39], some initial value problems for linear fractional differential equations are considered. The issues of

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determining a coefficient in various inverse problems have been considered by many authors, in particular, in [40–45].

In this paper we consider the questions of unique solvability and determination of the redefinition coefficient function in the initial inverse problem for two-dimensional Whitham-type partial differential equation with nonlinear initial value and nonlinear impulse conditions. So, in the domain $\Omega \equiv [0;T] \times \mathbb{R}^2$ for $t \neq t_i$, $i = 1, 2, \ldots, p$ we study the following two-dimensional quasilinear equation

$$\left(\frac{\partial}{\partial t} + u(t,x,y)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\right) u(t,x,y) = \alpha(t)\,\beta(x,y) + F\left(t,x,y,u(t,x,y)\right) \quad (1)$$

with nonlinear initial value condition

$$u(t,x,y)_{|t=0} = \varphi\left(x, y, \int\limits_0^T K(\xi) u(\xi,x,y) d\xi\right), \quad x, y \in \mathbb{R}, \quad (2)$$

and nonlinear impulsive condition

$$u\left(t_i^+, x, y\right) - u\left(t_i^-, x, y\right) = G_i\left(u\left(t_i, x, y\right)\right), \quad i = 1, 2, \ldots, p, \quad (3)$$

where $u(t,x,y)$ is a desired function, $\alpha(t)$ is an unknown coefficient function, $t \neq t_i$, $i = 1, 2, \ldots, p$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T < \infty$, $\beta(x,y) \in C^{1,1}(\mathbb{R}^2)$, $\mathbb{R} \equiv (-\infty, \infty)$, $F(t,x,y,u) \in C^{0,1,1,0}(\Omega \times \mathbb{R})$, $\varphi(x,y,u) \in C^1\left(\mathbb{R}^3\right)$, $\int\limits_0^T |K(\xi)|\, d\xi < \infty$, $u\left(t_i^+, x, y\right) = \lim\limits_{\nu \to 0^+} u\left(t_i + \nu, x, y\right)$, $u\left(t_i^-, x, y\right) = \lim\limits_{\nu \to 0^-} u\left(t_i - \nu, x, y\right)$ are right-hand side and left-hand side limits of the function $u(t,x,y)$ at the point $t = t_i$, respectively.

We use some Banach spaces: the space $C\left(\Omega, \mathbb{R}\right)$ consists of continuous functions $u(t,x,y)$ with the norm

$$\|u\|_C = \sup_{(t,x,y) \in \Omega} |u(t,x,y)|\,;$$

we also use the linear space

$$PC\left(\Omega, \mathbb{R}\right) = \left\{u : \Omega \to \mathbb{R}; u(t,x,y) \in C\left(\Omega_{i,i+1}, \mathbb{R}\right), i = 1, \ldots, p\right\}$$

with the following norm

$$\|\, u \,\|_{PC} = \max\left\{\|u\|_{C(\Omega_{i,i+1})}, \; i = 1, 2, \ldots, p\right\},$$

where $\Omega_{i,i+1} = (t_i, t_{i+1}] \times \mathbb{R}^2$, $u\left(t_i^+, x, y\right)$ and $u\left(t_i^-, x, y\right)$ $(i = 0, 1, \ldots, p)$ exist and are bounded; $u\left(t_i^-, x, y\right) = u\left(t_i, x, y\right)$.

To determine the redefinition coefficient function $\alpha(t)$ in the initial value problem (1)–(3), we use the following condition

$$u(t, x_0, y_0) = \psi(t), \quad (4)$$

where $x_0, y_0 \in \mathbb{R}$, $\psi(t) \in C^1[0;T]$, $\varphi\left(x_0, y_0, \int\limits_0^T K(\xi)\psi(\xi) d\xi\right) = \psi(0^+)$.

Two-dimensional direct problem. Find an unknown function $u(t,x,y) \in PC(\Omega, \mathbb{R})$, that the function $u(t,x,y)$ for all $(t,x,y) \in \Omega$, $t \neq t_i$, $i = 1, 2, \ldots, p$ satisfies differential equation (1), nonlinear initial value condition (2) and for $(t,x,y) \in \Omega$, $t = t_i$, $i = 1, 2, \ldots, p$ satisfies nonlinear limit condition (3).

Two-dimensional inverse problem. Find a pair of unknown functions $u(t,x,y) \in PC(\Omega, \mathbb{R})$ and $\alpha(t) \in C([0,T], \mathbb{R})$, that the function $u(t,x,y)$ for all $(t,x,y) \in \Omega$, $t \neq t_i$, $i = 1, 2, \ldots, p$ satisfies the differential equation (1), initial value condition (2), for $(t,x,y) \in \Omega$, $t = t_i$, $i = 1, 2, \ldots, p$ satisfies the nonlinear limit condition (3) and additional condition (4).

2. Reducing the direct problem to a functional-integral equation

We show that direct initial value problem (1)–(3) with impulse effects is reduced to solving the following nonlinear functional-integral equation

$$u(t,x,y)=\Theta(t,x,y;u)\equiv$$
$$\equiv\varphi\left(p(t,0,x,y),q(t,0,x,y),\int\limits_0^T K(\xi)u(\xi,p(t,\xi,x,y),q(t,\xi,x,y))d\xi\right)+$$
$$+\int\limits_0^t\Big[\alpha(s)\,\beta\left(p(t,s,x,y),q(t,s,x,y)\right)+$$
$$+F\left(s,p(t,s,x,y),q(t,s,x,y),u\left(s,p(t,s,x,y),q(t,s,x,y)\right)\right)\Big]ds+$$
$$+\sum_{0<t_i<t}G_i\left(u\left(t_i,p(t,t_i,x,y),q(t,t_i,x,y)\right)\right), \tag{5}$$

where extended characteristics $p(t,s,x,y)$ and $q(t,s,x,y)$ are defined from the system of integral equations

$$\begin{cases} p(t,s,x,y)=x-\int\limits_s^t u\left(\theta,p(t,\theta,x,y),q(t,\theta,x,y)\right)d\theta, & p(t,t,x,y)=x,\\ q(t,s,x,y)=y-\int\limits_s^t u\left(\theta,p(t,\theta,x,y),q(t,\theta,x,y)\right)\,d\theta, & q(t,t,x,y)=y,\end{cases} \tag{6}$$

$x,y\in\mathbb{R}$ play the role of parameters.

Let a function $u(t,x,y)\in PC(\Omega,\mathbb{R})$ is a solution of direct problem (1)–(3). We write the domain Ω as $\Omega=\Omega_{0,1}\cup\Omega_{1,2}\cup\cdots\cup\Omega_{p,p+1}$, where $\Omega_{i,i+1}=(t_i,t_{i+1}]\times\mathbb{R}^2$. On the first domain $\Omega_{0,1}$ we rewrite equation (1) as

$$D_u[u]=\alpha(t)\,\beta(x,y)+F\left(t,x,y,u(t,x,y)\right), \tag{7}$$

where $D_u=\frac{\partial}{\partial t}+u(t,x,y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$ is two-dimensional Whitham operator.

Now we accept the notation, which is called the extended system of characteristics:

$$\begin{cases} p\left(t,s,x,y\right)=x-\int\limits_s^t u\left(\theta,p(t,\theta,x,y),q(t,\theta,x,y)\right)d\theta, & p(t,t,x,y)=x,\\ q(t,s,x,y)=y-\int\limits_s^t u\left(\theta,p(t,\theta,x,y),q(t,\theta,x,y)\right)d\theta, & q(t,t,x,y)=y.\end{cases}$$

We introduce a function of four-dimensional argument as $w(t,s,x,y)=u\left(s,p(t,s,x,y),q(t,s,x,y)\right)$, such that for $t=s$ it takes the form $w(t,t,x,y)=u\left(t,p(t,t,x,y),q(t,t,x,y)\right)=u(t,x,y)$. Differentiate the function $w(t,s,x,y)$ with respect to the new argument s and obtain

$$w_s(t,s,x,y)=u_s\left(s,p(t,s,x,y),q(t,s,x,y)\right)+u_p\left(s,p(t,s,x,y),q(t,s,x,y)\right)\cdot p_s(t,s,x,y)+$$
$$+u_q\left(s,p(t,s,x,y),q(t,s,x,y)\right)\cdot q_s(t,s,x,y)=u_s\left(s,p(t,s,x,y),q(t,s,x,y)\right)+$$
$$+u\left(s,p(t,s,x,y),q(t,s,x,y)\right)\left[u_p\left(s,p(t,s,x,y),q(t,s,x,y)\right)+\right.$$
$$\left.+u_q\left(s,p(t,s,x,y),q(t,s,x,y)\right)\right].$$

Then, taking into account the last relation, we rewrite equation (7) in the following extended form

$$\frac{\partial}{\partial s}w(t,s,x,y)=\alpha(s)\,\beta\left(p(t,s,x,y),q(t,s,x,y)\right)+$$
$$+F\left(s,p(t,s,x,y),q(t,s,x,y)w(t,s,x,y)\right). \tag{8}$$

Integrating equations (8) along the extended characteristic, we obtain

$$\int\limits_0^{t_1}\left[\alpha(s)\,\beta\left(p(t,s,x,y),q(t,s,x,y)\right)+F\left(s,p(t,s,x,y),q(t,s,x,y),w\left(t,s,x,y\right)\right)\right]ds=$$
$$=w(t,t_1^-,x,y)-w(t,0^+,x,y),\quad t\in(0,t_1], \tag{9}$$

$$\int\limits_{t_1}^{t_2}\left[\alpha(s)\,\beta\left(p(t,s,x,y),q(t,s,x,y)\right)+F\left(s,p(t,s,x,y),q(t,s,x,y),w\left(t,s,x,y\right)\right)\right]ds=$$
$$=w(t,t_2^-,x,y)-w(t,t_1^+,x,y),\quad t\in(t_1,t_2], \tag{10}$$

$$\dots$$

$$\int\limits_{t_p}^{t_{p+1}}\left[\alpha(s)\,\beta\left(p(t,s,x,y),q(t,s,x,y)\right)+F\left(s,p(t,s,x,y),q(t,s,x,y),w\left(t,s,x,y\right)\right)\right]ds=$$
$$=w(t,t_{p+1}^-,x,y)-w(t,t_p^+,x,y),\quad t\in(t_p,t_{p+1}],\ \ t_{p+1}=T. \tag{11}$$

Taking $w(t,0^+,x,y)=w(t,0,x,y)$, $w(t,t_{p+1}^-,x,y)=w(t,s,x,y)$ into account, on the interval $(0,T]$ from integral relations (9)–(11) we have

$$\int\limits_0^{s}\left[\alpha(\varsigma)\,\beta\left(p(t,\varsigma,x,y),q(t,\varsigma,x,y)\right)+F\left(\varsigma,p(t,\varsigma,x,y),q(t,\varsigma,x,y),w\left(t,\varsigma,x,y\right)\right)\right]d\varsigma=$$
$$=\left[w\left(t,t_1,x,y\right)-w\left(t,0^+,x,y\right)\right]+\left[w\left(t,t_2,x,y\right)-w\left(t,t_1^+,x,y\right)\right]+$$
$$+\cdots+\left[w(t,s,x,y)-w\left(t,t_p^+,x,y\right)\right]=$$
$$=-w(t,0,x,y)-\left[w\left(t,t_1^+,x,y\right)-w\left(t,t_1,x,y\right)\right]-\left[w\left(t,t_2^+,x,y\right)-w\left(t,t_2,x,y\right)\right]-$$
$$-\cdots-\left[w\left(t,t_p^+,x,y\right)-w\left(t,t_p,x,y\right)\right]+w(t,s,x,y). \tag{12}$$

Taking into account impulsive condition (3), last equality (12) we rewrite as

$$w(t,s,x,y)=w(t,0,x,y)+$$
$$+\int\limits_0^{s}\left[\alpha(\varsigma)\beta\left(p(t,\varsigma,x,y),q(t,\varsigma,x,y)\right)+F\left(\varsigma,p(t,\varsigma,x,y),q(t,\varsigma,x,y),w\left(t,\varsigma,x,y\right)\right)\right]d\varsigma+$$
$$+\sum_{0<t_i<s}G_i\left(w\left(t,t_i,x,y\right)\right), \tag{13}$$

where $w(t,0,x,y)$ is arbitrary constant along the characteristics, which to be determined. Initial value condition (2) for equation (13) takes the form $w(t,0,x,y)=$

$\varphi\Big(p(t,0,x,y),q(t,0,x,y),\int\limits_0^T K(\xi)w(t,\xi,x,y)d\xi\Big)$. Then, taking into account this initial value condition, from (13) we obtain that

$$w(t,s,x,y)=\varphi\Bigg(p(t,0,x,y),q(t,0,x,y),\int\limits_0^T K(\xi)w(t,\xi,x,y)d\xi\Bigg)+$$

$$+\int\limits_0^s\big[\alpha(\varsigma)\beta\left(p(t,\varsigma,x,y),q(t,\varsigma,x,y)\right)+F\left(\varsigma,p(t,\varsigma,x,y),q(t,\varsigma,x,y),w\left(t,\varsigma,x,y\right)\right)\big]d\varsigma+$$

$$+\sum_{0<t_i<s}G_i\left(w\left(t,t_i,x,y\right)\right), \tag{14}$$

For $t=s$, from (14) we arrive at nonlinear functional-integral equation (5) together with system of integral equations (6).

3. Solvability of the functional-integral equation

For fixed values of a redefinition function $\alpha(t)$, we study functional-integral equation (5).

Theorem 1. *Let the following conditions be satisfied:*

1) $0<\sup\limits_{x,y\in\mathbb{R}}|\,\varphi(x,y,0)\,|\leqslant\Delta_\varphi<\infty$;
2) $0<\sup\limits_{x,y\in\mathbb{R}}|\,\beta(x,y)\,|\leqslant\Delta_\beta<\infty$;
3) $\sup\limits_{x,y\in\mathbb{R}}|\,F(t,x,y,0)\,|\leqslant\Delta_f(t),\ 0<\Delta_f(t)\in C[0;T]$;
4) $0<|\,G_i(0)\,|\leqslant\Delta_{G_i}<\infty,\ i=1,2,\ldots,p$;
5) $|\,\varphi(x_1,y_1,u_1)-\varphi(x_2,y_2,u_2)\,|\leqslant\chi_1\left(|\,x_1-x_2\,|+|\,y_1-y_2\,|+|\,u_1-u_2\,|\right),\ 0<\chi_1=$ const;
6) $|\,\beta(x_1,y_1)-\beta(x_2,y_2)\,|\leqslant\chi_2\left(|\,x_1-x_2\,|+|\,y_1-y_2\,|\right),\ 0<\chi_2=\text{const}$;
7) $|\,G_i(u_1)-G_i(u_2)\,|\leqslant\chi_{3i}\,|\,u_1-u_2\,|,\ 0<\chi_{3i}=\text{const}$;
8) $|\,F(t,x_1,y_1,u_1)-F(t,x_2,y_2,u_2)\,|\leqslant Q(t)\left(|\,x_1-x_2\,|+|\,y_1-y_2\,|\right)+P(t)\,|\,u_1-u_2\,|$;
9) $0<Q(t),P(t)\in C[0;T],\ 0<\max\limits_{t\in[0;T]}\int\limits_0^t\left[Q(s)(t-s)+P(s)\right]ds<\infty$;
10) $\rho=\max\limits_{t\in[0;T]}\int\limits_0^t H(t,s)ds+\sum\limits_{i=1}^p\chi_{3i}<1$, *where*

$$H(t,s)=\chi_1\big(2+|\,K(s)\,|\big)+2\big(Q(s)+\chi_2\,|\alpha(s)|\,\big)(t-s)+P(s).$$

Then, for fixed values of $\alpha(t)$, *the functional-integral equation* (5) *has a unique solution in the domain* Ω. *This solution can be founded by successive approximations:*

$$u_0(t,x,y)=0,\ \ u_{k+1}(t,x,y)\equiv\Theta(t,x;u_k,p_k,q_k),\ \ k=0,1,2,\ \ldots, \tag{15}$$

where $p_k(s,t,x,y)$ *and* $q_k(s,t,x,y)$ *are defined from the following iteration system*

$$\begin{cases} p_0(t,t,x,y)=x,\ \ p_k(t,s,x,y)=x-\int\limits_s^t u_{k-1}\left(\theta,p_{k-1}(t,\theta,x,y),q_{k-1}(t,\theta,x,y)\right)d\theta, \\ q_0(t,t,x,y)=y,\ \ q_k(t,s,x,y)=y-\int\limits_s^t u_{k-1}\left(\theta,p_{k-1}(t,\theta,x,y),q_{k-1}(t,\theta,x,y)\right)d\theta. \end{cases}$$

Proof. By virtue of the conditions of the theorem, we obtain that the following estimate holds for the first difference of approximation (15):

$$|\,u_1(t,x,y)-u_0(t,x,y)\,|\leqslant \sup_{x,y\in\mathbb{R}}|\,\varphi(x,y,0)\,|+\sup_{(t,x,y)\in\Omega}\int\limits_0^t|\,\alpha(s)\,\beta(x,y)\,|\,ds+$$

$$+\sum_{0<t_i<T}|\,G_i\,(0)\,|+\max_{t\in[0;T]}\int\limits_0^t\Delta_f(s)ds\leqslant\Delta_\varphi+\sum_{i=1}^{p}\Delta_{G_i}+\Delta_1+\Delta_2<\infty, \qquad (16)$$

where

$$\Delta_1=\max_{t\in[0;T]}\int\limits_0^t\Delta_f(s)ds<\infty,\ \Delta_2=\Delta_\beta\max_{t\in[0;T]}\int\limits_0^t|\,\alpha(s)\,|\,ds<\infty.$$

Taking into account estimate (16) and the conditions of the theorem, we obtain that for an arbitrary difference of approximation (15) the following estimate holds:

$$|\,u_{k+1}(t,x,y)-u_k(t,x,y)\,|\leqslant$$

$$\leqslant\left|\varphi\left(p_{k+1}(t,0,x,y),q_{k+1}(t,0,x,y),\int\limits_0^T K(\xi)u_k\left(\xi,p_k(t,\xi,x,y),q_k(t,\xi,x,y)\right)d\xi\right)-\right.$$

$$\left.-\varphi\left(p_k(t,0,x,y),q_k(t,0,x,y),\int\limits_0^T K(\xi)u_{k-1}\left(\xi,p_{k-1}(t,\xi,x,y),q_{k-1}(t,\xi,x,y)\right)d\xi\right)\right|+$$

$$+\int\limits_0^t|\,\alpha(s)\,|\cdot|\,\beta\left(p_{k+1}(t,s,x,y),q_{k+1}(t,s,x,y)\right)-\beta\left(p_k(t,s,x,y),q_k(t,s,x,y)\right)|\,ds+$$

$$+\int\limits_0^t\Big|F\left(s,p_{k+1}(t,s,x,y),q_{k+1}(t,s,x,y),u_k\left(s,p_k(t,s,x,y),q_k(t,s,x,y)\right)\right)-$$

$$-F\left(s,p_k(t,s,x,y),q_k(t,s,x,y),u_{k-1}\left(s,p_{k-1}(t,s,x,y),q_{k-1}(t,s,x,y)\right)\right)\Big|ds+$$

$$+\sum_{0<t_i<t}|\,G_i\left(u_k\left(t_i,p_k,q_k\right)\right)-G_i\left(u_{k-1}\left(t_i,p_{k-1},q_{k-1}\right)\right)|\leqslant$$

$$\leqslant\chi_1\left[2\int\limits_0^t|\,u_k(s,x,y)-u_{k-1}(s,x,y)\,|\,ds+\int\limits_0^T|\,K(s)\,|\cdot|\,u_k(s,x,y)-u_{k-1}(s,x,y)\,|\,ds\right]+$$

$$+\int\limits_0^t\left[2\left(Q(s)+\chi_2\,|\alpha(s)|\right)\int\limits_s^t|\,u_k(\theta,x,y)-u_{k-1}(\theta,x,y)\,|\,d\theta+\right.$$

$$\left.+P(s)\,|\,u_k(s,x,y)-u_{k-1}(s,x,y)\,|\,\right]ds+\sum_{0<t_i<t}\chi_{3i}\,|\,u_k\left(t_i,x,y\right)-u_{k-1}\left(t_i,x,y\right)|\leqslant$$

$$\leqslant\max_{t\in[0;T]}\int\limits_0^t H(t,s)\,|\,u_k(s,x,y)-u_{k-1}(s,x,y)\,|\,ds+$$

$$+\sum_{i=1}^{p}\chi_{3i}\,|\,u_k\,(t,x,y))-u_{k-1}\,(t,x,y)\,|\,, \tag{17}$$

where $H(t,s)=\chi_1\big(2+|\,K(s)\,|\,\big)+2\big(Q(s)+\chi_2\,|\alpha(s)|\,\big)(t-s)+P(s)$.

In estimate (17), we pass to the norm in the space $PC\,(\Omega,\mathbb{R})$ and arrive at the estimate

$$\|\,u_{k+1}(t,x,y)-u_k(t,x,y)\,\|_{PC}\leqslant\rho\cdot\|\,u_k(t,x,y)-u_{k-1}(t,x,y)\,\|_{PC}\,, \tag{18}$$

where

$$\rho=\max_{t\in[0;T]}\int_0^t H(t,s)ds+\sum_{i=1}^{p}\chi_{3i}.$$

Since $\rho<1$, it follows from estimate (18) that the sequence of functions $\{u_k(t,x,y)\,\}_{k=1}^{\infty}$, defined by formula (15), converges absolutely and uniformly in the domain Ω. In addition, it follows from the existence of a unique fixed point of the operator $\Theta(t,x,y;u)$ on the right side of (5) that functional-integral equation (5) has a unique solution in the domain Ω. The theorem has been proven.$\square$

Corollary 1. *Let all the conditions of Theorem 1 be satisfied. Then, for fixed values of the function $\alpha(t)$, two-dimensional direct initial value problem* (1)–(3) *with impulse effects has a unique solution in the domain Ω.*

4. Determination of the redefinition coefficient function

Using additional condition (4), from functional-integral equation (5) we obtain the linear Volterra integral equation of the first kind

$$\int_0^t\beta(t,s)\,\alpha(s)ds=g(t), \tag{19}$$

where $\beta(t,s)=\beta\left(p(t,s,x_0,y_0),q(t,s,x_0,y_0)\right)$,

$$g(t)=\psi(t)-\varphi\Bigg(p(t,0,x_0y_0),q(t,0,x_0,y_0),\int_0^T K(\xi)\psi(\xi)d\xi\Bigg)-$$

$$-\int_0^t F\left(s,p(t,s,x_0,y_0),q(t,s,x_0,y_0),\psi(s)\right)ds-\sum_{0<t_i<t}G_i\left(\psi\left(t_i\right)\right).$$

Due to the formulation of the inverse problem and the equality $\varphi\Big(x_0,y_0,\int_0^T K(\xi)\psi(\xi)d\xi\Big)=\psi(0^+)$, Volterra integral equation of the first kind (19) has a unique solution on the interval $[0;T]$. This equation can be reduced to the Volterra integral equation of the second kind by differentiation, and, further, the method of successive approximations can be applied. We substitute the solution of Volterra integral equation (19) into functional-integral equation (5) and obtain the desired solution $u(t,x,y)$ by the method of successive approximations.

Here the following holds true.

Theorem 3. *Let all the conditions of Theorem 1 be satisfied. Then two-dimensional inverse problem* (1)–(4) *with impulse effects has a unique pair of solutions $\big\{u(t,x,y),\,\alpha(t)\big\}$ in the domain Ω.*

5. Conclusion

In this paper the questions of unique solvability and determination of the redefinition coefficient function $\alpha(t)$ in initial inverse problem (1)–(4) for two-dimensional Whitham type partial differential equation with impulse effects are studied. The modified method of characteristics allows partial differential equations of the first order to be represented as ordinary differential equations that describe the change of an unknown function along the line of characteristics. Nonlinear functional-integral equation (5) is obtained. The unique solvability of two-dimensional inverse problem (1)–(4) is proved by the method of successive approximations and contraction mappings. The determination of the unknown coefficient function $\alpha(t)$ is reduced to solving Volterra integral equation of the first kind (19).

We will use these results in our future work to investigate other type partial differential equations of the first order with impulse effects.

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КОЭФФИЦИЕНТНАЯ ОБРАТНАЯ ЗАДАЧА ДЛЯ ДВУМЕРНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ТИПА УИЗЕМА С ИМПУЛЬСНЫМИ ВОЗДЕЙСТВИЯМИ

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Изучены вопросы однозначной разрешимости и определения коэффициентной функции в начальной обратной задаче для двумерного дифференциального уравнения в частных производных типа Уизема. Модифицированный метод характеристик позволяет дифференциальное уравнение в частных производных первого порядка представить как систему обыкновенных дифференциальных уравнений, которые описывают изменение неизвестной функции вдоль линии характеристик. Доказана однозначная разрешимость двумерной обратной задачи методами последовательных приближений и сжимающих отображений. Определение неизвестного коэффициента сведено к решению интегрального уравнения Вольтерра первого рода.

Ключевые слова: *обратная задача, двумерное уравнение типа Уизема, определение коэффициентной функции, метод последовательных приближений, однозначная разрешимость.*

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