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Lauricella hypergeometric function $F_A^{(n)}$ with applications to the solving Dirichlet problem for three-dimensional degenerate elliptic equation

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Abstract. In this paper, hypergeometric function of Lauricella $F_A^{(n)}$ has been investigated. The new properties of which are established and applied to the solution of the Dirichlet problem for the three-dimensional degenerate elliptic equation. Fundamental solutions of the named equation are expressed through the Lauricella hypergeometric function in three variables and an explicit solution of the Dirichlet problem in the first octant is written out through the Appell hypergeometric function F_2. A limit theorem for calculating the value of a function of many variables is proved, and formulas for their transformation are established. These results are used to determine the order of singularity of fundamental solutions and to prove the truth of the solution to the Dirichlet problem. The uniqueness of the solution to the Dirichlet problem is proved by the extremum principle for elliptic equations.

Keywords: Appell and Lauricella hypergeometric functions, three-dimensional degenerate elliptic equation, PDE-systems of hypergeometric type, fundamental solution, Dirichlet problem

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1. Introduction

It is known, that a special functions are used for solving many problems of mathematical physics (see [4, 18]). These include the Gauss hypergeometric series, Bessel and Hermite functions, Lauricella hypergeometric functions, etc. The Hermite functions are actively applied in algorithms and information systems that are used in medical diagnostics [16]. The Bessel functions are used in solving a number of problems of hydrodynamics, radiophysics, and thermal conductivity [14, Part 2]. Some functions that are used in astronomy can be arranged in hypergeometric series [20, Chapter 3]. Multidimensional hypergeometric functions are used in the superstrings theory [5].

The study of boundary value problems for degenerate equations is one of the important directions of the modern theory of partial differential equations. It is known that in the formulation and construction of local and nonlocal boundary value problems solutions, the main role is played by fundamental solutions. Fundamental solutions of the two-dimensional degenerate elliptic equations are expressed by the Appell function F_2, and when the dimension of the equation exceeds two – by the Lauricella hypergeometric function $F_A^{(n)}$ with three and more variables.

In this work, the established properties of the Lauricella function are applied to solving the Dirichlet problem for the three-dimensional degenerate elliptic equation

$$y^m z^k u_{xx} + x^n z^k u_{yy} + x^n y^m u_{zz} = 0, m > 0, n > 0, k > 0 \tag{1.1}$$

in the domain $\Omega = \{(x, y, z) : x > 0,\ y > 0,\ z > 0\}$.

A degenerate elliptic equation (1.1) is related to an elliptic equation with the singular coefficients

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x} u_x + \frac{2\beta}{y} u_y + \frac{2\gamma}{z} u_z = 0,\ 0 < 2\alpha, 2\beta, 2\gamma < 1. \tag{1.2}$$

Namely, if in the region of ellipticity the equation (1.1) is reduced to a canonical form, then we obtain equation (1.2). Using the fundamental solutions constructed in [9], the main boundary value problems for the equation (1.2) in the finite (first octant of the ball) were solved in explicit forms [10, 11, 22], and local and nonlocal boundary value problems for the equation (1.2) by the Fourier method in special infinite domains were investigated [12, 13].

Few works are devoted to the study of boundary value problems for the two-dimensional analogue of the equation (1.1). In works [1, 19], for the two-dimensional degenerate elliptic equation

$$y^m u_{xx} + x^n u_{yy} = 0, m > 0, n > 0$$

solutions of the Dirichlet and Neumann problems in the bounded and unbounded domains were found in explicit forms.

2. Multiple hypergeometric functions and their some new properties

The Gauss hypergeometric function can be represented by the following series [6, p.56, Eq. 2.1(2)]

$$F\left(a,b;c;x\right)=\sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m}{(c)_m}\frac{x^m}{m!},\ \ |x|<1, \tag{2.1}$$

where $(z)_n$ is a Pochhammer symbol: $(z)_n=z(z+1)...(z+n-1),\ \ n=1,2,...;\ (z)_0=1.$

The great success of the theory of hypergeometric function in one variable has stimulated the development of corresponding theory in two or more variables. Appell [2] has defined four functions F_1 to F_4, which are all analogues to Gauss' $F(a,b;c;x)$. For instance, the Appell function F_2 has a form

$$F_2\left[\begin{array}{c} a,b_1,b_2; \\ c_1,c_2; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\,(b_1)_m\,(b_2)_n}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!},\ \ |x|+|y|<1, \tag{2.2}$$

which satisfies the following system of partial differential equations [6, p. 234, Eq. 5.9(10)]:

$$\begin{cases} x(1-x)u_{xx}-xyu_{xy}+\left[c_1-(a+b_1+1)\,x\right]u_x-b_1yu_y-ab_1u=0, \\ y(1-y)u_{yy}-xyu_{xy}-b_2xu_x+\left[c_2-(a+b_2+1)\,y\right]u_y-ab_2u=0. \end{cases} \tag{2.3}$$

Lauricella hypergeometric function[15] (see also [21, p. 33])

$$F_A^{(n)}\left[\begin{array}{c} a,\mathbf{b}; \\ \mathbf{c}; \end{array} \mathbf{x}\right]=\sum_{|\mathbf{k}|=0}^{\infty}(a)_{|\mathbf{k}|}\prod_{i=1}^{n}\frac{(b_i)_{k_i}}{(c_i)_{k_i}}\frac{x_i^{k_i}}{k_i!},\ |x_1|+...+|x_n|<1$$

is a natural generalization of the classical Gauss hypergeometric function (2.1) and the Appell function (2.2) to the case of many complex variables and their corresponding complex parameters. Hereinafter

$$\mathbf{b}:=\left(b_1,...,b_n\right),\ \mathbf{c}:=\left(c_1,...,c_n\right),\ \mathbf{x}:=\left(x_1,...,x_n\right),$$

$$\mathbf{k}:=\left(k_1,...,k_n\right),\ \ |\mathbf{k}|:=k_1+...+k_n,\ \ k_1\geq 0,...,k_n\geq 0.$$

Let us list some properties of the Lauricella hypergeometric function $F_A^{(n)}$:

1) tranformation formula [3, p. 116, Eq. (9)]:

$$F_A^{(n)}\left[\begin{array}{c} a,\mathbf{b}; \\ \mathbf{c}; \end{array} \mathbf{x}\right]=(1-X)^{-a}\,F_A^{(n)}\left[\begin{array}{c} a,\mathbf{c}-\mathbf{b}; \\ \mathbf{c}; \end{array} \frac{\mathbf{x}}{X-1}\right],\ X:=\sum_{j=1}^{n}x_j; \tag{2.4}$$

2) differentation formula:

$$\frac{\partial}{\partial x_k}F_A^{(n)}\left[\begin{array}{c} a,\mathbf{b}; \\ \mathbf{c}; \end{array} \mathbf{x}\right]=\frac{ab_k}{c_k}F_A^{(n)}\left[\begin{array}{c} a+1,\mathbf{b}_k+1; \\ \mathbf{c}_k+1; \end{array} \mathbf{x}\right], \tag{2.5}$$

where the vectors $\mathbf{b}_k+1$ and $\mathbf{c}_k+1$ appear, the k-th component of which is one greater than the corresponding components of the vectors $\mathbf{b}$ and $\mathbf{c}$, respectively:

$$\mathbf{b}_k+1:=\left(b_1,...,b_{k-1},b_k+1,b_{k+1},...,b_n\right),\ \ \mathbf{c}_k+1:=\left(c_1,...,c_{k-1},c_k+1,c_{k+1},...,c_n\right),\ \ k=\overline{1,n}.$$

The Lauricella hypergeometric function of n variables satisfies the system with n equations and this system has 2^n linearly independent solutions (for details, see [3, pp. 117, 118]). In our further studies, we use the following system corresponding to a function of three variables $u=F_A^{(3)}\left[\begin{array}{c} a,b_1,b_2,b_3; \\ c_1,c_2,c_3; \end{array} x,y,z\right]$:

$$\begin{cases} x(1-x)u_{xx}-xyu_{xy}-xzu_{xz}+[c_1-(a+b_1+1)x]u_x-b_1yu_y-b_1zu_z-ab_1u=0,\\ y(1-y)u_{yy}-xyu_{xy}-yzu_{yz}-b_2xu_x+[c_2-(a+b_2+1)y]u_y-b_2zu_z-ab_2u=0,\\ z(1-z)u_{zz}-xzu_{xz}-yzu_{yz}-b_3xu_x-b_3yu_y+[c_3-(a+b_3+1)z]u_z-ab_3u=0.\end{cases} \tag{2.6}$$

The PDE-system (2.6) has 8 linearly independent solutions [3, pp. 117, 118] :

$$1\left\{F_A^{(3)}\left[\begin{matrix} a,b_1,b_2,b_3;\\ c_1,c_2,c_3;\end{matrix}\ x,y,z\right],\right. \tag{2.7}$$

$$3\begin{cases} x^{1-c_1}F_A^{(3)}\left[\begin{matrix} a+1-c_1,b_1+1-c_1,b_2,b_3;\\ 2-c_1,c_2,c_3;\end{matrix}\ x,y,z\right],\\ y^{1-c_2}F_A^{(3)}\left[\begin{matrix} a+1-c_2,b_1,b_2+1-c_2,b_3;\\ c_1,2-c_2,c_3;\end{matrix}\ x,y,z\right],\\ z^{1-c_3}F_A^{(3)}\left[\begin{matrix} a+1-c_3,b_1,b_2,b_3+1-c_3;\\ c_1,c_2,2-c_3;\end{matrix}\ x,y,z\right],\end{cases} \tag{2.8}$$

$$3\begin{cases} x^{1-c_1}y^{1-c_2}F_A^{(3)}\left[\begin{matrix} a+2-c_1-c_2,b_1+1-c_1,b_2+1-c_2,b_3;\\ 2-c_1,2-c_2,c_3;\end{matrix}\ x,y,z\right],\\ y^{1-c_2}z^{1-c_3}F_A^{(3)}\left[\begin{matrix} a+2-c_2-c_3,b_1,b_2+1-c_2,b_3+1-c_3;\\ c_1,2-c_2,2-c_3;\end{matrix}\ x,y,z\right],\\ x^{1-c_1}z^{1-c_3}F_A^{(3)}\left[\begin{matrix} a+2-c_1-c_3,b_1+1-c_1,b_2,b_3+1-c_3;\\ 2-c_1,c_2,2-c_3;\end{matrix}\ x,y,z\right],\end{cases} \tag{2.9}$$

$$1\left\{x^{1-c_1}y^{1-c_2}z^{1-c_3}F_A^{(3)}\left[\begin{matrix} a+3-c_1-c_2-c_3,b_1+1-c_1,b_2+1-c_2,b_3+1-c_3;\\ 2-c_1,2-c_2,2-c_3;\end{matrix}\ x,y,z\right].\right. \tag{2.10}$$

It can also be shown by direct calculations that the functions (2.7) – (2.10) satisfy the system (2.6).

3. Fundamental solutions of a degenerate three-dimensional elliptic equation

Let (x,y,z) and (ξ,η,ζ) be two points of the domain Ω. We are looking for a solution of the equation (1.1) in the form

$$u=r^{-2\alpha-2\beta-2\gamma-1}\omega\left(\rho,\sigma,\theta\right), \tag{3.1}$$

where ω is a new unknown function,

$$\alpha=\frac{n}{2\left(n+2\right)},\ \ \beta=\frac{m}{2\left(m+2\right)},\ \ \gamma=\frac{k}{2\left(k+2\right)};\ \ q=\frac{n+2}{2},\ \ p=\frac{m+2}{2},\ \ l=\frac{k+2}{2};$$

$$\rho=-\frac{4x^q\xi^q}{q^2r^2},\ \sigma=-\frac{4y^p\eta^p}{p^2r^2},\ \theta=-\frac{4z^l\zeta^l}{l^2r^2},\ \ r^2=\frac{1}{q^2}\left(x^q-\xi^q\right)^2+\frac{1}{p^2}\left(y^p-\eta^p\right)^2+\frac{1}{l^2}\left(z^l-\zeta^l\right)^2.$$

It is obvious that

$$0<2\alpha<1,\ \ 0<2\beta<1,\ \ 0<2\gamma<1;\quad q>1,\ \ p>1,\ \ l>1.$$

Substituting (3.1) into equation (1.1), we obtain a system of differential equations of hypergeometric type

$$\begin{cases} \rho\left(1-\rho\right)\omega_{\rho\rho}-\rho\sigma\omega_{\rho\sigma}-\rho\theta\omega_{\rho\theta}+\\ \qquad+\left[2\alpha-\left(2\alpha+\beta+\gamma+\frac{3}{2}\right)\rho\right]\omega_\rho-\alpha\sigma\omega_\sigma-\alpha\theta\omega_\theta-\alpha\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\omega=0,\\ \sigma\left(1-\sigma\right)\omega_{\sigma\sigma}-\rho\sigma\omega_{\rho\sigma}-\sigma\theta\omega_{\sigma\theta}+\\ \qquad+\left[2\beta-\left(\alpha+2\beta+\gamma+\frac{3}{2}\right)\sigma\right]\omega_\sigma-\beta\rho\omega_\rho-\beta\theta\omega_\theta-\beta\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\omega=0,\\ \theta\left(1-\theta\right)\omega_{\theta\theta}-\rho\theta\omega_{\rho\theta}-\sigma\theta\omega_{\sigma\theta}+\\ \qquad+\left[2\gamma-\left(\alpha+\beta+2\gamma+\frac{3}{2}\right)\theta\right]\omega_\theta-\gamma\rho\omega_\rho-\gamma\sigma\omega_\sigma-\gamma\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\omega=0.\end{cases} \tag{3.2}$$

Comparing the system (3.2) with the system (2.6) which has 8 particular solutions, we obtain [9]

$$q_0\left(x,y,z;\xi,\eta,\zeta\right)=k_0r^{-2\alpha-2\beta-2\gamma-1}F_A^{(3)}\left[\begin{matrix} 1/2+\alpha+\beta+\gamma,\alpha,\beta,\gamma;\\ 2\alpha,2\beta,2\gamma;\end{matrix}\ \rho,\sigma,\theta\right], \tag{3.3}$$

$$q_1(x,y,z;\xi,\eta,\zeta) = k_1 x\xi r^{2\alpha-2\beta-2\gamma-3} F_A^{(3)} \left[\begin{matrix} 3/2-\alpha+\beta+\gamma, 1-\alpha, \beta, \gamma; \\ 2-2\alpha, 2\beta, 2\gamma; \end{matrix} \rho, \sigma, \theta \right], \quad (3.4)$$

$$q_{11}(x,y,z;\xi,\eta,\zeta) = k_{11} y\eta r^{-2\alpha+2\beta-2\gamma-3} F_A^{(3)} \left[\begin{matrix} 3/2+\alpha-\beta+\gamma, \alpha, 1-\beta, \gamma; \\ 2\alpha, 2-2\beta, 2\gamma; \end{matrix} \rho, \sigma, \theta \right], \quad (3.5)$$

$$q_{12}(x,y,z;\xi,\eta,\zeta) = k_{12} z\zeta r^{-2\alpha-2\beta+2\gamma-3} F_A^{(3)} \left[\begin{matrix} 3/2+\alpha+\beta-\gamma, \alpha, \beta, 1-\gamma; \\ 2\alpha, 2\beta, 2-2\gamma; \end{matrix} \rho, \sigma, \theta \right], \quad (3.6)$$

$$q_2(x,y,z;\xi,\eta,\zeta) = k_2 xy\xi\eta r^{2\alpha+2\beta-2\gamma-5} F_A^{(3)} \left[\begin{matrix} 5/2-\alpha-\beta+\gamma, 1-\alpha, 1-\beta, \gamma; \\ 2-2\alpha, 2-2\beta, 2\gamma; \end{matrix} \rho, \sigma, \theta \right], \quad (3.7)$$

$$q_{21}(x,y,z;\xi,\eta,\zeta) = k_{21} xz\xi\zeta r^{2\alpha-2\beta+2\gamma-5} F_A^{(3)} \left[\begin{matrix} 5/2-\alpha+\beta-\gamma, 1-\alpha, \beta, 1-\gamma; \\ 2-2\alpha, 2\beta, 2-2\gamma; \end{matrix} \rho, \sigma, \theta \right], \quad (3.8)$$

$$q_{22}(x,y,z;\xi,\eta,\zeta) = k_{22} yz\eta\zeta r^{-2\alpha+2\beta+2\gamma-5} F_A^{(3)} \left[\begin{matrix} 5/2+\alpha-\beta-\gamma, \alpha, 1-\beta, 1-\gamma; \\ 2\alpha, 2-2\beta, 2-2\gamma; \end{matrix} \rho, \sigma, \theta \right], \quad (3.9)$$

$$q_3(x,y,z;\xi,\eta,\zeta) = k_3 xyz\xi\eta\zeta r^{2\alpha+2\beta+2\gamma-7} F_A^{(3)} \left[\begin{matrix} 7/2-\alpha-\beta-\gamma, 1-\alpha, 1-\beta, 1-\gamma; \\ 2-2\alpha, 2-2\beta, 2-2\gamma; \end{matrix} \rho, \sigma, \theta \right], \quad (3.10)$$

where k_0,..., k_3 are constants, which are determined when solving boundary value problems for the equation (1.1).

It is easy to see that the each of three particular solutions q_1, q_{11} and q_{12} are symmetrical to each other with respect to the numerical parameters of the Lauricella function. Hence, in further studies there is no need to consider the functions q_{11} and q_{12}, i.e. we omit them and study only the function q_1. Similar propositions can be made about the second trio of particular solutions q_2, q_{21} and q_{22}: we study only q_2, and omit the functions q_{21} and q_{22}.

It is easy to see that the constructed functions q_0, q_1, q_2 and q_3 have the following properties:

$$\left.\frac{\partial}{\partial x} q_0\right|_{x=0} = 0,\ \left.\frac{\partial}{\partial y} q_0\right|_{y=0} = 0,\ \left.\frac{\partial}{\partial z} q_0\right|_{z=0} = 0;\quad q_1|_{x=0} = 0,\ \left.\frac{\partial}{\partial y} q_1\right|_{y=0} = 0,\ \left.\frac{\partial}{\partial z} q_1\right|_{z=0} = 0,$$

$$q_2|_{x=0} = 0,\ q_2|_{y=0} = 0,\ \left.\frac{\partial}{\partial z} q_2\right|_{z=0} = 0;\quad q_3|_{x=0} = 0,\ q_3|_{y=0} = 0,\ q_3|_{z=0} = 0.$$

Note, these properties will be used in solving four (Neumann, two Dirichlet-Neumann and Dirichlet) boundary value problems for the equation (1.1).

Lemma 3.1. *If* $0 < 2\alpha, 2\beta, 2\gamma < 1$, *then every function* q_k $(k = \overline{0,3})$ *has a singularity of order* $\frac{1}{r}$ *as* $r \to 0$.

Proof. To give an example, we consider function q_0. The order of singularity of the remaining functions is determined similarly.

In the case of three variables, the transformation formula (2.4) takes the form

$$F_A^{(3)} \left[\begin{matrix} a, b_1, b_2, b_3; \\ c_1, c_2, c_3; \end{matrix} x, y, z \right] = (1-x-y-z)^{-a} \times$$
$$\times F_A^{(3)} \left[\begin{matrix} a, c_1-b_1, c_2-b_2, c_3-b_3; \\ c_1, c_2, c_3; \end{matrix} \frac{x}{x+y+z-1}, \frac{y}{x+y+z-1}, \frac{z}{x+y+z-1} \right]. \quad (3.11)$$

Using the transformation formula (3.11), the function q_0 defined in (3.3) can be reduced to the form

$$q_0(x,y,z;\xi,\eta,\zeta) = \frac{1}{r} \cdot q_0^*(x,y,z;\xi,\eta,\zeta), \quad (3.12)$$

where

$$q_0^*(x,y,z;\xi,\eta,\zeta) = k_0 \varrho^{-2\alpha-2\beta-2\gamma} F_A^{(3)} \left[\begin{matrix} \alpha+\beta+\gamma+1/2, \alpha, \beta, \gamma; \\ 2\alpha,\ 2\beta,\ 2\gamma; \end{matrix} \frac{4x^q\xi^q}{q^2\varrho^2}, \frac{4y^p\eta^p}{p^2\varrho^2}, \frac{4z^l\zeta^l}{l^2\varrho^2} \right], \quad (3.13)$$

$$\varrho^2 = \frac{1}{q^2}\left(x^q+\xi^q\right)^2 + \frac{1}{p^2}\left(y^p+\eta^p\right)^2 + \frac{1}{l^2}\left(z^l+\zeta^l\right)^2.$$

We must show that the value of $q_0^*\left(x,y,z;\xi,\eta,\zeta\right)$ as $r\to 0$, i.e. $x\to\xi$, $y\to\eta$, $z\to\zeta$, is bounded.

According to the theory of Lauricella hypergeometric functions [3, Chap. VII], if the sum of the absolute values of the variables is less than one, then the function $F_A^{(n)}$ is bounded for any values of the numerical parameters. In the case of three variables, this statement looks like

$$\left|F_A^{(3)}\left[\begin{matrix} a,b_1,b_2,b_3; \\ c_1,c_2,c_3; \end{matrix} x,y,z\right]\right| < \infty, |x|+|y|+|z|<1. \tag{3.14}$$

By virtue of (3.14), it is obvious that in (3.13):

$$\frac{4x^q\xi^q}{q^2\varrho^2} + \frac{4y^p\eta^p}{p^2\varrho^2} + \frac{4z^l\zeta^l}{l^2\varrho^2} < 1,$$

therefore the following inequality is true

$$\left|q_0^*\left(x,y,z;\xi,\eta,\zeta\right)\right| \le \frac{C}{R^{2\alpha+2\beta+2\gamma}},\quad r\to 0, \tag{3.15}$$

where $C = const > 0$ and

$$R^2 = \frac{1}{q^2}x^{2q} + \frac{1}{p^2}y^{2p} + \frac{1}{l^2}z^{2l}. \tag{3.16}$$

Now from (3.12) and (3.15) follows that the function q_0 has a singularity of order $\frac{1}{r}$ as $r\to 0$. The Lemma 3.1 is proved. □

Based on the Lemma 3.1, we conclude that the particular solutions defined in (3.3) – (3.10) are fundamental solutions of the equation (1.1).

4. Statement of the Dirichlet problem and the uniqueness theorem

Dirichlet problem. Find a solution $u\left(x,y,z\right)$ of the equation (1.1) with the regularity $C\left(\overline{\Omega}\right)\bigcap C^2\left(\Omega\right)$ that satisfies the conditions

$$u\left(x,y,z\right)\big|_{z=0} = \tau_1\left(x,y\right),\;\; 0\le x,y<\infty, \tag{4.1}$$

$$u\left(x,y,z\right)\big|_{y=0} = \tau_2\left(x,z\right),\;\; 0\le x,z<\infty, \tag{4.2}$$

$$u\left(x,y,z\right)\big|_{x=0} = \tau_3\left(y,z\right),\;\; 0\le y,z<\infty, \tag{4.3}$$

$$\lim_{R\to\infty} u\left(x,y,z\right) = 0, \tag{4.4}$$

where $\overline{\Omega} = \left\{(x,y,z):\, x\ge 0, y\ge 0,\, z\ge 0\right\}$; R is defined in (3.16); $\tau_1\left(y,z\right), \tau_2\left(x,z\right), \tau_3\left(x,y\right)$ are given continuous functions in a closed domain and have representations

$$\tau_1\left(x,y\right) = \frac{\tilde{\tau}_1\left(x,y\right)}{\left(1+\frac{1}{q^2}x^{2q}+\frac{1}{p^2}y^{2p}\right)^{\varepsilon_1}},\quad \tilde{\tau}_1(x,y)\in C(0\le x,y<\infty), \tag{4.5}$$

$$\tau_2\left(x,z\right) = \frac{\tilde{\tau}_2\left(x,z\right)}{\left(1+\frac{1}{q^2}x^{2q}+\frac{1}{l^2}z^{2l}\right)^{\varepsilon_2}},\quad \tilde{\tau}_2(x,z)\in C(0\le x,z<\infty), \tag{4.6}$$

$$\tau_3\left(y,z\right) = \frac{\tilde{\tau}_3\left(y,z\right)}{\left(1+\frac{1}{p^2}y^{2p}+\frac{1}{l^2}z^{2l}\right)^{\varepsilon_3}},\quad \tilde{\tau}_3(y,z)\in C(0\le y,z<\infty), \tag{4.7}$$

where $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$ are a real numbers with $\alpha+\beta+\gamma<\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3<2$.

In addition, the functions $\tau_1(x,y)$, $\tau_2(x,z)$ and $\tau_3(y,z)$ satisfy the matching conditions at the origin: $\tau_1(0,0)=\tau_2(0,0)=\tau_3(0,0)$ and at the lateral edges of the domain Ω:

$$\tau_1(x,0)=\tau_2(x,0),\;\tau_1(0,y)=\tau_3(y,0),\;\;\tau_2(0,z)=\tau_3(0,z),\;x,y,z\in\overline{\Omega}.$$

Theorem 4.1. *The Dirichlet problem can have at most one solution.*

Proof. To prove Theorem 4.1, it suffices to show that the corresponding homogeneous Dirichlet problem has a trivial solution. For this purpose, the finite part of the domain Ω, bounded by the planes $x=0, y=0,\ z=0$ and the sphere σ_0 :

$$\frac{1}{q^2}x^{2q}+\frac{1}{p^2}y^{2p}+\frac{1}{l^2}z^{2l}=R^2,\ x>0,\ y>0,\ z>0,$$

we denote by Ω_R. Let

$$\tau_1(y,z)=\tau_2(x,z)=\tau_3(x,y)=0. \tag{4.8}$$

Then the validity of Theorem 4.1 follows from the extremum principle for elliptic equations [17, p. 12]. Indeed, the function $u(x,y,z)$ in the domain $\bar{\Omega}_R$, by virtue of (4.8), can reach its positive maximum and negative minimum only at σ_0.

Let (x,y,z) be an arbitrary point in D_R. We take an arbitrary small number $\varepsilon>0$ and, considering (4.8), we choose R large enough that $|u(x,y,z)|<\varepsilon$ on σ_0. For R large enough, this point falls in D_R and therefore $|u(x,y,z)|<\varepsilon$. Since ε is arbitrary, we have $u(x,y,z)=0$. Then $u(x,y,z)\equiv 0$ in D. The Theorem 4.1 is proved. □

5. Existence of a solution to the Dirichlet problem

Consider a function

$$u(x,y,z)=\int\limits_0^\infty\int\limits_0^\infty t^n s^m\tau_1(t,s)\left.\frac{\partial}{\partial\zeta}q_3(x,y,z;t,s,\zeta)\right|_{\zeta=0}dtds+$$

$$+\int\limits_0^\infty\int\limits_0^\infty t^n s^k\tau_2(t,s)\left.\frac{\partial q_3}{\partial\eta}\right|_{\eta=0}dtds+\int\limits_0^\infty\int\limits_0^\infty t^m s^k\tau_3(t,s)\left.\frac{\partial q_3}{\partial\xi}\right|_{\xi=0}dtds, \tag{5.1}$$

where $q_3(x,y,z;\xi,\eta,\zeta)$ is a fundamental solution defined in (3.10). Applying a differential formula (2.5), from (5.1) we get the following function:

$$u(x,y,z)=u_1(x,y,z)+u_2(x,y,z)+u_3(x,y,z), \tag{5.2}$$

where

$$u_1(x,y,z)=k_3xyz\int\limits_0^\infty\int\limits_0^\infty\frac{\tau_1(t,s)\,t^{n+1}s^{m+1}}{r_1^{2\delta}}F_2\left[\begin{matrix}\delta,\,1-\alpha,1-\beta; & -\dfrac{4x^qt^q}{q^2r_1^2},-\dfrac{4y^ps^p}{p^2r_1^2}\\ 2-2\alpha,\,2-2\beta; & \end{matrix}\right]dtds, \tag{5.3}$$

$$u_2(x,y,z)=k_3xyz\int\limits_0^\infty\int\limits_0^\infty\frac{\tau_2(t,s;)\,t^{n+1}s^{k+1}}{r_2^{2\delta}}F_2\left[\begin{matrix}\delta,\,1-\alpha,1-\gamma; & -\dfrac{4x^qt^q}{q^2r_2^2},-\dfrac{4z^ls^l}{l^2r_2^2}\\ 2-2\alpha,\,2-2\gamma; & \end{matrix}\right]dtds, \tag{5.4}$$

$$u_3(x,y,z)=k_3xyz\int\limits_0^\infty\int\limits_0^\infty\frac{\tau_3(t,s;)\,t^{m+1}s^{k+1}}{r_3^{2\delta}}F_2\left[\begin{matrix}\delta,\,1-\beta,1-\gamma; & -\dfrac{4y^pt^p}{p^2r_3^2},-\dfrac{4z^ls^l}{l^2r_3^2}\\ 2-2\beta,\,2-2\gamma; & \end{matrix}\right]dtds, \tag{5.5}$$

$$k_3=\frac{1}{2\pi}q^{-2+2\alpha}p^{-2+2\beta}l^{-2+2\gamma}\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1-\gamma)\Gamma(6-2\alpha-2\beta-2\gamma)}{\Gamma(2-2\alpha)\Gamma(2-2\beta)\Gamma(2-2\gamma)\Gamma(3-\alpha-\beta-\gamma)}, \tag{5.6}$$

$$\delta=\frac{7}{2}-\alpha-\beta-\gamma;\qquad r_1^2=\frac{1}{q^2}(x^q-t^q)^2+\frac{1}{p^2}(y^p-s^p)^2+\frac{1}{l^2}z^{2l},$$

$$r_2^2=\frac{1}{q^2}(x^q-t^q)^2+\frac{1}{p^2}y^{2p}+\frac{1}{l^2}\left(z^l-s^l\right)^2,\ \ r_3^2=\frac{1}{q^2}x^{2q}+\frac{1}{p^2}(y^p-t^p)^2+\frac{1}{l^2}\left(z^l-s^l\right)^2.$$

Here F_2 is Appell hypergeometric function defined in (2.2).

Lemma 5.1. *If the function $\tau_1(x,y)$ can be represented as* (4.5)*, then the function $u_1(x,y,z)$ defined by equality* (5.3) *is a regular solution of equation* (1.1) *in the domain Ω satisfying the conditions* (4.4) *and*

$$u_1(x,y,0)=\tau_1(x,y),\ \ u_1(x,0,z)=0,\ \ u_1(0,y,z)=0. \tag{5.7}$$

Proof. First let us prove that the function (5.3) satisfies the degenerate elliptic equation (1.1). For this purpose, we consider the auxiliary function

$$W(x,y,z;t,s)=xyzr_1^{-2\delta}\omega(\vartheta,\varsigma), \tag{5.8}$$

where

$$\omega(\vartheta,\varsigma):=F_2\left[\begin{matrix}\delta,1-\alpha,1-\beta;\\ 2-2\alpha,\ 2-2\beta;\end{matrix}\ \vartheta,\varsigma\right],\ \ \vartheta=-\frac{4xt}{r_1^2},\ \varsigma=-\frac{4ys}{r_1^2}.$$

We calculate the necessary derivatives of the auxiliary function W with respect to the variables x, y, z and substitute them into the degenerate elliptic equation (1.1). As a result, we obtain the relation

$$\begin{gathered}y^m z^k W_{xx}+x^n z^k W_{yy}+x^n y^m W_{zz}=\\ =\vartheta yzr_1^{-2\mu}\left\{\vartheta(1-\vartheta)\omega_{\vartheta\vartheta}-\vartheta\varsigma\omega_{\vartheta\varsigma}+\left[2(1-\alpha)-(2-\alpha+\delta)\,\vartheta\right]\omega_\vartheta-(1-\alpha)\delta\omega\right\}\\ +x\varsigma zr_1^{-2\mu}\left\{\varsigma(1-\varsigma)\omega_{\varsigma\varsigma}-\vartheta\varsigma\omega_{\vartheta\varsigma}+\left[2(1-\beta)-(2-\beta+\delta)\varsigma\right]\omega_\varsigma-(1-\beta)\delta\omega\right\}=0,\end{gathered}$$

which is equivalent to the following system of hypergeometric equations

$$\begin{cases}\vartheta(1-\vartheta)\omega_{\vartheta\vartheta}-\vartheta\varsigma\omega_{\vartheta\varsigma}+\left[2(1-\alpha)-(2-\alpha+\delta)\,\vartheta\right]\omega_\vartheta-(1-\alpha)\delta\omega=0,\\ \varsigma(1-\varsigma)\omega_{\varsigma\varsigma}-\vartheta\varsigma\omega_{\vartheta\varsigma}+\left[2(1-\beta)-(2-\beta+\delta)\varsigma\right]\omega_\varsigma-(1-\beta)\delta\omega=0.\end{cases}$$

Comparing the last system of equations with the system of equations (2.3) for the Appell function F_2, we can conclude that the function (5.8) is a solution of the corresponding degenerate elliptic equation. Consequently, the function $u_1(x,\,y,\,z)$ defined by (5.3) satisfies the degenerate elliptic equation (1.1).

Now we prove that the function $u_1(x,y,z)$ satisfies the boundary conditions (5.7). Indeed, introducing in the integrand in (5.3) instead of t and s new variables

$$\mu=\frac{l\left(t^q-x^q\right)}{qz^l},\nu=\frac{l\left(s^p-y^p\right)}{pz^l},$$

we obtain

$$\begin{gathered}u_1(x,y,z)=l^{2\delta-2}k_3xyz^{2l(\alpha+\beta-2)}\int\limits_{-\frac{lx^q}{qz^l}}^{\infty}\int\limits_{-\frac{ly^p}{pz^l}}^{\infty}\frac{\left(x^q+\mu qz^l/l\right)\left(y^p+\nu qz^l/l\right)}{\left(1+\mu^2+\nu^2\right)^\delta}\times\\ \times F_2\left[\begin{matrix}\delta,\,1-\alpha,1-\beta;\\ 2-2\alpha,\ 2-2\beta;\end{matrix}\ -\frac{4l^2x^q\left(x^q+\mu qz^l/l\right)}{q^2z^{2l}\left(1+\mu^2+\nu^2\right)},-\frac{4l^2y^p\left(y^p+\nu pz^l/l\right)}{p^2z^{2l}\left(1+\mu^2+\nu^2\right)}\right]\times\\ \times\tau_1\left[\left(x^q+\mu qz^l/l\right)^{1/q},\left(y^p+\nu pz^l/l\right)^{1/p}\right]d\mu d\nu.\end{gathered}$$

Taking the expression (5.6) into account for the coefficient k_3, considering the well-known formula for calculating the double improper integral [8, p. 633, Eq. 4.623]

$$\int\limits_0^\infty\int\limits_0^\infty\varphi\left(a^2x^2+b^2y^2\right)dxdy=\frac{\pi}{4ab}\int\limits_0^\infty\varphi(x)dx$$

and Legendre's duplication formula [6, p. 5, Eq. 1.2(15)],

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

we obtain

$$\lim_{z\to 0} u_1(x,y,z) = \tau_1(x,y). \tag{5.9}$$

Using the similar transformations, we have

$$\lim_{x\to 0} u_1(x,y,z) = 0, \ \ \lim_{y\to 0} u_1(x,y,z) = 0. \tag{5.10}$$

Therefore, based on equalities (5.9) and (5.10) we conclude that the function $u_1(x,y,z)$, defined by (5.3), satisfies conditions (5.7).

Let us show that if given function τ_1 has representation (4.5), then the function $u_1(x,y,z)$ defined in (5.3) tends to zero at infinity.

Using the transformation formula for Appell function F_2 [6, p. 240, Eq. 5.11(8)]

$$F_2\left[\begin{array}{l} a, b_1, b_2; \\ \quad c_1, c_2; \end{array} x, y\right] = (1-x-y)^{-a} F_2\left[\begin{array}{l} a, c_1-b_1, c_2-b_2; \\ \qquad\quad c_1, c_2; \end{array} \frac{x}{x+y-1}, \frac{y}{x+y-1}\right],$$

we write the function (5.3) in the form

$$u_1(x,y,z) = k_3 xyz \int\limits_0^\infty \int\limits_0^\infty \frac{\tau_1(t,s)\, t^{n+1} s^{m+1}}{\rho^{2\delta}} F_2\left[\begin{array}{l} \delta, 1-\alpha, 1-\beta; \\ 2-2\alpha, \, 2-2\beta; \end{array} \frac{4x^q t^q}{q^2\rho^2}, \frac{4y^p s^p}{p^2\rho^2}\right] dtds, \tag{5.11}$$

where

$$\rho^2 = \frac{1}{q^2}\left(x^q + t^q\right)^2 + \frac{1}{p^2}\left(y^p + s^p\right)^2 + \frac{1}{l^2} z^{2l}.$$

It is easy to see that in (5.11) the following inequality holds

$$\frac{4x^q t^q}{q^2\rho^2} + \frac{4y^p s^p}{p^2\rho^2} < 1, \ \ x>0, y>0, z>0, t>0, s>0.$$

Let us prove that when the point (x,y,z) tends to infinity, i.e. when $R\to\infty$, the function (5.11) tends to zero. It known from the theory of Appell functions [2], that, if $|x|+|y|<1$, then for any values of the numerical parameters the Appell hypergeometric function F_2 is bounded:

$$\mid F_2(a, b_1, b_2; c_1, c_2; x, y) \mid \le C_1, \ \ |x|+|y|<1.$$

Next, applying the representation (4.5) for given function $\tau_1(x,y)$, we obtain

$$|u_1| \le C_2 xyz \int\limits_0^\infty \int\limits_0^\infty \frac{t^{n+1} s^{m+1} dtds}{\left(1 + \frac{1}{q^2} t^{2q} + \frac{1}{p^2} s^{2p}\right)^{\varepsilon_1} \left[\frac{1}{q^2}\left(x^q+t^q\right)^2 + \frac{1}{p^2}\left(y^p+s^p\right)^2 + \frac{1}{l^2} z^{2l}\right]^{7/2-\alpha-\beta-\gamma}}. \tag{5.12}$$

Substituting t and s for

$$\mu = \frac{1}{qR} t^q, \ \ \nu = \frac{1}{pR} s^p$$

in the last double improper integral (5.12), we get

$$|u_1| \le \frac{qpC_3}{R^{2\varepsilon_1 - 2\alpha - 2\beta - 2\gamma}} \cdot \frac{x}{R} \cdot \frac{y}{R} \cdot \frac{z}{R} \cdot K(x,y;R), \tag{5.13}$$

where $\varepsilon_1 > \alpha+\beta+\gamma$ (see condition in (4.5)) and

$$K(x,y;R) = \int\limits_0^\infty \int\limits_0^\infty \frac{\mu\nu d\mu d\nu}{\left(\mu^2+\nu^2+\frac{1}{R^2}\right)^{\varepsilon_1} \left(1+\mu^2+\nu^2+\frac{2x^q}{qR}+\frac{2y^p}{pR}\right)^{7/2-\alpha-\beta-\gamma}}. \tag{5.14}$$

It is easy to show that the double improper integral on the right-hand side (5.14) is bounded as $R\to\infty$. Indeed, using the formula [7]

$$\underbrace{\int\limits_0^{+\infty}...\int\limits_0^{+\infty}}_{n}\frac{x_1^{p_1-1}...x_n^{p_n-1}dx_1...dx_n}{\left[(r_1x_1)^{q_1}+...+(r_nx_n)^{q_n}\right]^t\left[1+(r_1x_1)^{q_1}+...+(r_nx_n)^{q_n}\right]^s}=$$

$$=\frac{\Gamma\left(p_1/q_1\right)...\Gamma\left(p_n/q_n\right)\Gamma\left(P-t\right)\Gamma\left(s+t-P\right)}{q_1q_2...q_nr_1^{p_1q_1}...r_n^{p_nq_n}\Gamma\left(P\right)\Gamma\left(s\right)},\ P:=\frac{p_1}{q_1}+...+\frac{p_n}{q_n},$$

where p_k, q_k, r_k and s are positive numbers $(k=\overline{1,n})$, $0<P-t<s$, and passing in (5.14) to the limit as $R\to\infty$, we obtain

$$\lim_{R\to\infty}K\left(x,y;R\right)=\frac{\Gamma\left(2-\varepsilon_1\right)\Gamma\left(3/2-\alpha-\beta-\gamma+\varepsilon_1\right)}{4\Gamma\left(7/2-\alpha-\beta-\gamma\right)},\ \ \alpha+\beta+\gamma<\varepsilon_1<2. \tag{5.15}$$

Thus, by virtue of (5.13) and (5.15) the following estimate is valid:

$$|u_1|\le\frac{C_4}{R^{2(\varepsilon_1-\alpha-\beta-\gamma)}},\ \alpha+\beta+\gamma<\varepsilon_1<2,\ \ R\to\infty. \tag{5.16}$$

Considering (5.16), we conclude that the function (5.3) vanishes at infinity. Lemma 5.1 is proved. □

Remark 5.2. Repeating the arguments given in Lemma 5.1, one can prove two lemmas concerning the functions $u_2(x,y,z)$ and $u_3(x,y,z)$ defined by equalities (5.4) and (5.5), respectively. Thus, if the representations (4.6) and (4.7) are valid for the given functions $\tau_2(x,z)$ and $\tau_3(y,z)$, then each of the functions $u_2(x,y,z)$ and $u_3(x,y,z)$ is a solution to the degenerate elliptic equation (1.1) that vanishes at infinity and satisfies the set of conditions

$$u_2(x,y,0)=0,\ \ u_2(x,0,z)=\tau_2(x,z),\ \ u_2(0,y,z)=0,$$

$$u_3(x,y,0)=0,\ \ u_3(x,0,z)=0,\ \ u_3(0,y,z)=\tau_3(y,z),$$

respectively.

Theorem 5.3. *If given functions $\tau_1(x,y)$, $\tau_2(x,z)$ and $\tau_3(y,z)$ have the representations* (4.5), (4.6) *and* (4.7), *respectively, then the function $u(x,y,z)$ defined in* (5.2) *is a regular solution of the equation* (1.1) *in the domain Ω satisfying the conditions* (4.1) – (4.4).

Proof of Theorem 5.3 follows from Lemma 5.1 and Remark 5.2.

References

[1] Amanov D. Some boundary value problems for a degenerate elliptic equation in an unbounded domain. Izv.AN UzSSR, Ser.Fiz.-Mat.Nauki, –1984. –1.– P. 8 – 13.

[2] Appell P. Sur les séries hypergéométriques de deux variables, et sur des équations différentielles linéaires aux dérivées partielles C.R. Acad. Sci., Paris, – 1880.–90. – P. 296 – 298.

[3] Appell P. and Kampe de Feriet J. Fonctions Hypergeometriques et Hyperspheriques; Polynomes d'Hermite, Gauthier - Villars. Paris. – 1926.

[4] Bers L. Mathematical Aspects of Subsonic and Transonic Gas Dynamics Wiley New York. – 1958.

[5] Candelas P., de la Ossa X., Greene P., Parkes L., A pair of Calabi-Yau manifolds as an exactly soluble super conformal theory. Nucl. Phys., – 1991. – B539. – 21 – 74.

[6] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. Higher Transcendental Functions 1, McGraw-Hill, New York, Toronto, London. – 1953.

[7] Ergashev T. G., Tulakova Z. R. The Neumann problem for a multidimensional elliptic equation with several singular coefficients in an infinite domain. Lobachevskii Journal of Mathematics,– 2022. – 43(1). – P. 199 – 206.

[8] Gradshteyn I. S., Ryzhik I. M Table of integrals, series, and products Academic Press Amsterdam. – 2007.

[9] Hasanov A., Karimov E. T. Fundamental solutions for a class of three-dimensional elliptic equations with singular coefficients. Applied Mathematics Letters,– 2009. – 22. – P. 1828 – 1832.

[10] Karimov E.T. A boundary value problem for 3D elliptic equation with singular coefficients. Progress in analysis and its applications, – 2010. P. 619 – 625.

[11] Karimov E. T. On the Dirichlet problem for a three-dimensional elliptic equation with singular coefficients. Dokl.AN Uz, – 2010. – 2. – P. 9 – 11.

[12] Karimov K. T. Nonlocal problem for an elliptic equation with singular coefficients in a semi-infinite parallelepiped. Lobachevskii Journal of Mathematics, 2020. – 41(1). – P. 46 – 57.

[13] Karimov K. T. Boundary value problems in a semi-infinite parallelepiped for an elliptic equation with three singular coefficients. Lobachevskii Journal of Mathematics, – 2021.– 42(3).– P. 560 – 571.

[14] Korenev B.G. Introduction to the theory of Bessel functions. Nauka, Moscow, 1971 (in Russian).

[15] Lauricella G. Sulle funzioni ipergeometriche a piu variabili Rend. Circ. Mat. Palermo, – 1893. – 7. – P. 111 – 158.

[16] Mamayev N.V., Lukin A.S., Yurin D.V., Glazkova M.A., Sinitsin V.E. Algorithm of nonlocal mean based on decompositions via Hermite functions in problems of computer tomography. Proceedings of the 23rd Inter. Conf. on Comp. Graphics and Vision GraphiCon2013, Vladivostok, Russia. (2013) Sept 1620, P. 254–258 (in Russian).

[17] Miranda C. Partial Differential Equations of Elliptic Type Berlin Springer. – 1970.

[18] Niukkanen A. W. Generalised hypergeometric series arising in physical and quantum chemical applications. J. Phys. A: Math. Gen., – 1983. – 16.– P. 1813 – 1825.

[19] Salakhitdinov M. C., Hasanov A. Tricomi problem for a mixed type equation with a non-smooth degeneracy line. Diff. Uravn., – 1983. – 19(1).– P. 110–119.

[20] Smart U.M., Celestial mechanics. Longmans, Green and Co, London - New York - Toronto, 1953

[21] Srivastava H. M. Karlsson P. W. Multiple Gaussian hypergeometric series New York, Chichester, Brisbane and Toronto Halsted Press (Ellis Horwood Limited, Chichester), Wiley. – 1985.

[22] Tulakova Z.R. Spatial mixed problems and Neumann problem for the three-dimensional elliptic equation with the two singular coefficients. Uzbek Math. Journal, – 2024. – 68(3). P. 150–157.

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