ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2024, Vol. 45, No. 3, pp. 1072–1085. © Pleiades Publishing, Ltd., 2024.

Some Infinite Expansions of the Lauricella Functions and Their Application in the Study of Fundamental Solutions of a Singular Elliptic Equation

T. G. Ergashev[1,2*], A. Hasanov[2,3], and T. K. Yuldashev[4***]**

(Submitted by A. M. Elizarov)

[1]*National Research University "Tashkent Institute of Irrigation and Agricultural Mechanization Engineers", Tashkent, 100000 Uzbekistan*

[2]*Ghent University, Ghent, 9000 Belgium*

[3]*Romanovskii Institute of Mathematics, Uzbek Academy of Sciences, Tashkent, 100174 Uzbekistan*

[4]*Tashkent State University of Economics, Tashkent, 100066 Uzbekistan*

Received September 16, 2023; revised revised December 6, 2023; accepted accepted December 26, 2023

Abstract—In this article, a new inverse pair of symbolic operators with the multidimensional analogues is introduced. The properties of inverse pair of symbolic operators with the multidimensional analogues are studied. Formulas for the infinite expansion of multiple Lauricella functions are established. The application of some expansions in studying the properties of fundamental solutions of singular elliptic equations is shown.

DOI: 10.1134/S1995080224600742

Keywords and phrases: *Appell functions, Lauricella hypergeometric functions, the inverse pair symbolic operators, Poole formula, infinity expansion formulas, the singularity order of the fundamental solutions.*

1. INTRODUCTION AND DEFINITIONS

Hypergeometric functions in one and more variables occur naturally in a wide variety of problems in applied mathematics, statistics, operations research, theoretical physics, and engineering sciences. For instance, Srivastava and Kashyap [1] presented a number of interesting applications of hypergeometric functions in one and more variables in queuing theory and related stochastic processes. The work of Niukkanen [2] on the multiple hypergeometric functions is motivated by various physical and quantum chemical applications of such functions. Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [3].

The success of the theory of hypergeometric functions in one variable, as we note in our papers, has stimulated the development of a corresponding theory in two and more variables. Appell [4] has defined, in 1880, four functions: F_1–F_4, which are all analogous to Gauss' $F(a,b;c;z)$. A great merit in the further development of the theory of the hypergeometric series in two variables belongs to Horn [5], who gave a general definition and order classification of double hypergeometric series. He has investigated the convergence of hypergeometric series of two variables and established the systems of partial differential equations which they satisfy. Horn investigated in particular hypergeometric series

*E-mail: ergashev.tukhtasin@gmail.com
**E-mail: anvarhasanov@yahoo.com
***E-mail: t.yuldashev@tsue.uz

of order two and found that, apart from certain series which are either expressible in terms of one variable or are products of two hypergeometric series, each in one variable, there are essentially 34 (14 complete and 20 confluent) convergent series of order two.

Lauricella, further in [6], generalized the four Appell series F_1, F_2, F_3, and F_4 to series in n variables and defined his multiple hypergeometric series, denoted them by $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$. He also showed some important applications of these functions. For instance, explicit fundamental solutions of a multidimensional singular elliptic equation [7, 8] are expressed in terms of the Lauricella hypergeometric function $F_A^{(n)}$. For a given multiple hypergeometric function, it is useful to fund a decomposition formula, which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. The familiar operator method of Burchnall and Chaundy (see [9, 10]) has been used rather extensively to find decomposition formulas for hypergeometric functions of two variables in terms of the classical Gauss hypergeometric function of one variable. In the papers [11–14] interesting results were obtained on the study of the double hypergeometric functions. Recently, in the work [15], for the first time, formulas of analytic continuation for the Lauricella's hypergeometric functions in three variables are obtained. In [16], the validity of integral representations for a Kampé de Fériet function of general form are proved. In [17], exceptions of formulating the normal-regular solutions of confluent hypergeometric systems, obtained by passage to the limit from the Lauricella system have been studied. A series of theorems regarding the existence of normal-regular solutions to many variables and their connection with the confluent function, introduced by Khudozhnikov have been proved. The connection between the Khudozhnikov's new function and the Humbert's function of many variables have been found out. Some works are devoted to the study of the Lauricella hypergeometric function for partial values of the numerical parameters and their application to the solution of boundary value problems for the Laplace equation in parts of a ball [18, 19]. In [20], are studied the exact solutions for a partial system of second-order hypergeometric equations and some decomposition formulas.

In the works [9, 10, 21, 22] are introduced operators generalizing the Burchnall–Chaundy operators and are found expansion formulas for many triple hypergeometric functions. The authors of these works proved recurrent formulas, when the dimension of hypergeometric function exceeds three. However, due to the recurrence, additional difficulties may arise in the applications of those decomposition formulas. Recently, in the work [23], new non-recurrence decomposition formulas (expansions) for n-variable Lauricella functions $F_A^{(n)}$ and $F_B^{(n)}$ are obtained. In our present study we apply new expansions to the solving boundary value problems for multidimensional singular elliptic equation.

In the above mentioned works [9, 10], Burchnall and Chaundy obtained 15 pairs of expansions involving Appell's functions and ordinary hypergeometric functions, as well as a further considerable number of expansions involving hypergeometric series of higher order, and Humbert confluent hypergeometric series. However, firstly, the Burchnall–Chaundy method is limited to two variables and, secondly, this method does not allow expanding the hypergeometric function into an infinite sum through the function of the same name. In present our work, inspired by the work Wang [24], we establish the infinite summation formulas for Lauricella functions in n variables $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$.

The plan of this paper is as follows. In Section 2, we briefly give some preliminary information, which will be used later. In Section 3, we define inverse pair symbolic operators $H_x(a,b)$, $\bar{H}_x(a,b)$ and their multidimensional analogues. In Sections 4 and 5, we write the infinity summation formulas associated with one- and multi-dimensional inverse pair operators, respectively. Section 6 is devoted to the proof of some relations for the functions F_A^n and F_B^n. In Section 7, to give an example, the expansion formula and its expression at the limit values of the arguments are applied to the definition of the singularity order of the fundamental solution of the singular elliptic equation.

2. PRELIMINARIES

Throughout this work it is convenient to employ the Pochhammer symbol $(\lambda)_n$ defined by

$$(\lambda)_n = \lambda(\lambda+1)...(\lambda+n-1), \quad ,n=1,2,...; \quad (\lambda)_0 \equiv 1,$$

for which the following equalities are true

$$(1)_n = n!, \quad (\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda), \quad (\lambda)_{n+k} = (\lambda)_n(\lambda+n)_k,$$

where $\Gamma(z)$ is a famous Euler gamma-function.

It is known that the Euler gamma-function $\Gamma(a)$ has property ([25], p. 17, Eq. (2)) $\Gamma(a+m) = \Gamma(a)\,(a)_m$, where $(a)_m$ is a Pochhammer symbol, for which the equality $(a)_{m+n} = (a)_m(a+m)_n$ and its particular case $(a)_{2m} = (a)_m(a+m)_m$ are true ([25], p. 67, Eq. (5)).

A function

$$F(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k, \quad c \neq 0,-1,-2,... \tag{1}$$

is known as the Gaussian hypergeometric function.

If $\mathrm{Re}c > \mathrm{Re}b > 0$, for Gaussian hypergeometric function (1) we have Euler's formula ([25,] p. 114, Eq. (1))

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt, \tag{2}$$

where the right-hand side of (2) is a one-valued analytic function of the argument z within the domain $|arg(1-z)| < \pi$. Therefore, the function (2) gives also analytic continuation of the function $F(a,b;c;z)$.

The integral representation (2) allows to derive the Boltz formula ([25], p. 105, Eq. (3))

$$F(a,b;c;z) = (1-z)^{-b} F\left(c-a,b;c;\frac{z}{z-1}\right) \tag{3}$$

and to get the value of the Gaussian function in unity (the summation formula) ([25], p. 73, Eq. (73))

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0, \quad c \neq 0,-1,-2,... \tag{4}$$

When we consider the properties of a hypergeometric function (1) , it is very important to study the infinite sums of this function by the aid of formulas (3) and (4). For example, the following formula shows that the infinite sum of a hypergeometric function can be given by the elementary function ([26], p. 413, Eq. 6.7.1(8))

$$\sum_{k=0}^{\infty} \frac{(a)_k(c-b)_k}{k!(c)_k} z^k F(a+k,b;c+k;z) = (1-z)^{-a}. \tag{5}$$

The double Appell hypergeometric functions are defined as follows [4]

$$F_1\big(a,b,b';c;x,y\big) = \sum_{m,k=0}^{\infty} \frac{(a)_{m+k}(b)_m\,(b')_k}{(c)_{m+k}} \frac{x^m}{m!}\frac{y^k}{k!} \left(|x|+|y|<1\right), \tag{6}$$

$$F_2\big(a,b,b';c,c';x,y\big) = \sum_{m,k=0}^{\infty} \frac{(a)_{m+k}(b)_m\,(b')_k}{(c)_m\,(c')_k} \frac{x^m}{m!}\frac{y^k}{k!} \left(|x|<1,\ |y|<1\right), \tag{7}$$

$$F_3\big(a,a',b,b';c;x,y\big) = \sum_{m,k=0}^{\infty} \frac{(a)_m\,(a')_k\,(b)_m\,(b')_k}{(c)_{m+k}} \frac{x^m}{m!}\frac{y^k}{k!} \left(\sqrt{|x|}+\sqrt{|y|}<1\right), \tag{8}$$

$$F_4\big(a,b;c,c';x,y\big) = \sum_{m,k=0}^{\infty} \frac{(a)_{m+k}(b)_{m+k}}{(c)_m\,(c')_k} \frac{x^m}{m!}\frac{y^k}{k!} \left(|x|<1,\ |y|<1\right). \tag{9}$$

We introduce the following notations:

$$\mathbf{a} := \big(a_1,...,a_n\big), \quad \mathbf{b} := \big(b_1,...,b_n\big), \quad \mathbf{c} := \big(c_1,...,c_n\big); \quad \mathbf{x} := \big(x_1,...,x_n\big);$$

$$(\mathbf{a})_{\mathbf{k}} := \prod_{j=1}^{n} (a_j)_{k_j}\,, \quad (\mathbf{b})_{\mathbf{k}} := \prod_{j=1}^{n} (b_j)_{k_j}\,, \quad (\mathbf{c})_{\mathbf{k}} := \prod_{j=1}^{n} (c_j)_{k_j}\,; \quad \mathbf{x}^{\mathbf{k}} := \prod_{j=1}^{n} x_j^{k_j};$$

$$\mathbf{k} := (k_1, ..., k_n)\,, \quad |\mathbf{k}| := k_1 + ... + k_n, \quad k_1 \geq 0, ..., k_n \geq 0;\; K! := k_1!k_2!...k_n!.$$

The Lauricella hypergeometric functions in many (real or complex) variables are defined as following [6] (see, also, [27], p. 114)

$$F_A^{(n)}\,(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a)_{|\mathbf{k}|}\,(\mathbf{b})_{\mathbf{k}}}{(\mathbf{c})_{\mathbf{k}}} \frac{\mathbf{x}^{\mathbf{k}}}{K!} \quad (|x_1| + ... + |x_n| < 1)\,, \tag{10}$$

$$F_B^{(n)}\,(\mathbf{a}, \mathbf{b}; c; \mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{(\mathbf{a})_{\mathbf{k}}\,(\mathbf{b})_{\mathbf{k}}}{(c)_{|\mathbf{k}|}} \frac{\mathbf{x}^{\mathbf{k}}}{K!} \quad (|x_1| < 1,\, ...,\, |x_n| < 1)\,, \tag{11}$$

$$F_C^{(n)}\,(a, b; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a)_{|\mathbf{k}|}(b)_{|\mathbf{k}|}}{(\mathbf{c})_{\mathbf{k}}} \frac{\mathbf{x}^{\mathbf{k}}}{K!} \quad \left(\sqrt{|x_1|} + ... + \sqrt{|x_n|} < 1\right), \tag{12}$$

$$F_D^{(n)}\,(a, \mathbf{b}; c; \mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a)_{|\mathbf{k}|}\,(\mathbf{b})_{\mathbf{k}}}{(c)_{|\mathbf{k}|}} \frac{\mathbf{x}^{\mathbf{k}}}{K!} \quad (|x_1| < 1,\, ...,\, |x_n| < 1)\,. \tag{13}$$

In all definitions (10)–(13) of Lauricella hypergeometric functions, as usual, the denominator parameters $c, c_1, ..., c_n$ are neither zero nor a negative integer. Clearly, we have

$$F_A^{(2)} = F_2, \quad F_B^{(2)} = F_3, \quad F_C^{(2)} = F_4, \quad F_D^{(2)} = F_1, \quad F_A^{(1)} = F_B^{(1)} = F_C^{(1)} = F_D^{(1)} \equiv F,$$

where F_1, ..., F_4 are the Appell series defined by (6)–(9) and F is Gaussian hypergeometric function defined in (1).

Indeed, in all definitions (10)–(13) a number of variables is natural: $n \in N := \{1, 2, ...\}$. However, in our further studies, if $n = 0$, then we accept that

$$F_A^{(0)} = F_B^{(0)} = F_C^{(0)} = F_D^{(0)} \equiv 1.$$

The following summation formula ([27], p. 117)

$$F_D^{(n)}\,(a, \mathbf{b}; c; 1, ..., 1) = \frac{\Gamma(c)\Gamma\,(c - a - b_1 - ... - b_n)}{\Gamma(c - a)\Gamma\,(c - b_1 - ... - b_n)}, \quad \mathrm{Re}\,(c - a - b_1 - ... - b_n) > 0 \tag{14}$$

is valid.

It is easy to see that the formula (14) generalizes the famous Gaussian summation formula (4).

3. THE INVERSE PAIR OPERATORS $H_x(a, b)$ and $\bar{H}_x(a, b)$

Burchnall and Chaundy in [9, 10] systematically presented a number of expansions for some double hypergeometric functions in series of simpler hypergeometric functions. However, the Burchnall–Chaundy method is limited to functions of two variables.

Developing the idea of Burchnall and Chaundy, we introduce the following mutually inverse symbolic operators for n variables

$$H_{\mathbf{x}}\,(\alpha, \beta) := \frac{\Gamma\big(\alpha + \delta_1 + ... + \delta_n\big)\Gamma\,(\beta)}{\Gamma\,(\alpha)\,\Gamma\big(\beta + \delta_1 + ... + \delta_n\big)} \tag{15}$$

and

$$\bar{H}_{\mathbf{x}}\,(\alpha, \beta) := \frac{\Gamma\,(\alpha)\,\Gamma\big(\beta + \delta_1 + ... + \delta_n\big)}{\Gamma\,(\beta)\,\Gamma\big(\alpha + \delta_1 + ... + \delta_n\big)}, \tag{16}$$

where $\mathbf{x} := \left(x_1, ..., x_n\right)$; $\delta_j = x_j \frac{\partial}{\partial x_j}$, $j = 1, ..., n$.

In the one-dimensional case, these operators (15) and (16) look like

$$H_{x_j}\left(\alpha, \beta\right) := \frac{\Gamma\left(\alpha+\delta_j\right)\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta+\delta_j\right)}, \quad j = 1, ..., n \tag{17}$$

and

$$\bar{H}_{x_j}\left(\alpha, \beta\right) := \frac{\Gamma\left(\alpha\right)\Gamma\left(\beta+\delta_j\right)}{\Gamma\left(\beta\right)\Gamma\left(\alpha+\delta_j\right)}, \quad j = 1, ..., n. \tag{18}$$

Using the Gaussian formula (4) for $F(a, b; c; 1)$, we get

$$H_{x_j}\left(\alpha, \beta\right) = \sum_{r=0}^{\infty} \frac{\left(\beta-\alpha\right)_r\left(-\delta_j\right)_r}{\left(\beta\right)_r r!}, \quad , j = 1, ..., n \tag{19}$$

and

$$\bar{H}_{x_j}\left(\alpha, \beta\right) = \sum_{r=0}^{\infty} \frac{\left(\beta-\alpha\right)_r\left(-\delta_j\right)_r}{\left(1-\alpha-\delta_j\right)_r r!}, \quad j = 1, ..., n. \tag{20}$$

Similarly, using the summation formula (14) for $F_D^{(n)}\left(a, \mathbf{b}; c; 1, ..., 1\right)$, we have

$$H_{\mathbf{x}}\left(\alpha, \beta\right) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{\left(\beta-\alpha\right)_{|\mathbf{k}|}}{K!(\beta)_{|\mathbf{k}|}} \prod_{j=1}^{n} \left(-\delta_j\right)_{k_j} \tag{21}$$

and

$$\bar{H}_{\mathbf{x}}\left(\alpha, \beta\right) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{\left(\beta-\alpha\right)_{|\mathbf{k}|}}{K!(1-\alpha-\delta)_{|\mathbf{k}|}} \prod_{j=1}^{n} \left(-\delta_j\right)_{k_j}. \tag{22}$$

Note that for every analytic function $f\left(z\right)$ the following Poole formula ([28], p. 26, Eq. (33))

$$\left(-z\frac{\partial}{\partial z}\right)_r \left\{f(z)\right\} = (-1)^r z^r \frac{d^r}{dz^r}\left\{f(z)\right\} \tag{23}$$

is valid.

The companion to the Poole formula (23), the following operator identity ([28], p. 93, Eq. (7))

$$\left(a+z\frac{\partial}{\partial z}\right)_r \left\{f(z)\right\} = z^{1-a} \frac{d^r}{dz^r}\left\{z^{a+r-1} f\left(z\right)\right\} \tag{24}$$

is valid for every analytic function $f\left(z\right)$.

The operator identities (23) and (24) can be proved by the mathematical induction method.

4. THE INFINITY SUMMATION FORMULAS ASSOCIATED WITH ONE-DIMENSIONAL INVERSE PAIR OPERATORS

Consider the inverse pair operators $H_{x_j}\left(\alpha, \beta\right)$ and $\bar{H}_{x_j}\left(\alpha, \beta\right)$, defined in (17) and (18), respectively $(1 \le j \le n)$. A composition of these operators r times will be denoted by

$$H^r_{\mathbf{x}_r}\left(\mathbf{a}_r, \mathbf{b}_r\right) := H_{x_1}\left(a_1, b_1\right) H_{x_2}\left(a_2, b_2\right) ... H_{x_r}\left(a_r, b_r\right),$$

$$\bar{H}^r_{\mathbf{x}_r}\left(\mathbf{a}_r, \mathbf{b}_r\right) := \bar{H}_{x_1}\left(a_1, b_1\right) \bar{H}_{x_2}\left(a_2, b_2\right) ... \bar{H}_{x_r}\left(a_r, b_r\right),$$

where $\mathbf{a}_r := \left(a_1, ..., a_r\right)$, $\mathbf{b}_r := \left(b_1, ..., b_r\right)$ and $\mathbf{x}_r := \left(x_1, ..., x_r\right)$ are vectors with r components $(1 \le r \le n)$.

Theorem 1. *Let n be a number of the variables of Lauricella functions defined in (10)–(13). If r is a natural number and $1 \le r \le n$, then the following symbolic forms hold*

$$F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; \mathbf{x}\right) = H^r_{\mathbf{x}_r}\left(\mathbf{b}_r, \mathbf{d}_r\right) F_A^{(n)}\left(a, \mathbf{d}_r, \mathbf{b}_{r+1,n}; \mathbf{c}; \mathbf{x}\right), \tag{25}$$

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\bar{H}_{\mathbf{x}_r}^{r}\left(\mathbf{d}_r,\mathbf{b}_r\right)F_A^{(n)}\left(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};\mathbf{c};\mathbf{x}\right),\tag{26}$$

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=H_{\mathbf{x}_r}^{r}\left(\mathbf{b}_r,\mathbf{c}_r\right)\left(1-X_r\right)^{-a}$$
$$\times F_A^{(n-r)}\left(a,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r},...,\frac{x_n}{1-X_r}\right);\tag{27}$$

$$F_A^{(n-r)}\left(a,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r},...,\frac{x_n}{1-X_r}\right)$$
$$=\left(1-X_r\right)^{a}\bar{H}_{\mathbf{x}_r}^{r}\left(\mathbf{b}_r,\mathbf{c}_r\right)F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right);\tag{28}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=H_{\mathbf{x}_r}^{r}\left(\mathbf{a}_r,\mathbf{d}_r\right)F_B^{(n)}\left(\mathbf{d}_r,\mathbf{a}_{r+1,n};c;\mathbf{x}\right),\tag{29}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\bar{H}_{\mathbf{x}_r}^{r}\left(\mathbf{d}_r,\mathbf{a}_r\right)F_B^{(n)}\left(\mathbf{d}_r,\mathbf{a}_{r+1,n};c;\mathbf{x}\right);\tag{30}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=H_{\mathbf{x}_r}^{r}\left(\mathbf{d}_r,\mathbf{c}_r\right)F_C^{(n)}\left(a,b;\mathbf{d}_r,\mathbf{c}_{r+1,n};\mathbf{x}\right);\tag{31}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=H_{\mathbf{x}_r}^{r}\left(\mathbf{b}_r,\mathbf{d}_r\right)F_D^{(n)}\left(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};c;\mathbf{x}\right),\tag{32}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=\bar{H}_{\mathbf{x}_r}^{r}\left(\mathbf{d}_r,\mathbf{b}_r\right)F_D^{(n)}\left(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};c;\mathbf{x}\right),\tag{33}$$

where

$$\mathbf{b}_{r+1,n}:=\left(b_{r+1},...,b_n\right),\quad \mathbf{b}_{n+1,n}=\emptyset,\quad \mathbf{c}_{r+1,n}:=\left(c_{r+1},...,c_n\right),\quad \mathbf{c}_{n+1,n}=\emptyset,$$

$$\mathbf{d}_r:=\left(d_1,...,d_r\right),\quad X_r:=x_1+...+x_r,\quad 1\le r\le n.$$

The symbolic forms (25)–(33) are used to obtain a large number of infinite summation formulas of the multiple Lauricella hypergeometric functions. Namely, using the formulas (19), (20) and applying them many times, by virtue of the Poole formulas (23) and (24), we obtain

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{\left(d_j-b_j\right)_{k_j}}{k_j!\left(c_j\right)_{k_j}}\left(-x_j\right)^{k_j}\right]$$
$$\times F_A^{(n)}\left(a+|\mathbf{k}_r|,\mathbf{d}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};\mathbf{c}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x}\right),\tag{34}$$

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{\left(b_j-d_j\right)_{k_j}}{k_j!\left(c_j\right)_{k_j}}x_j^{k_j}\right]$$
$$\times F_A^{(n)}\left(a+|\mathbf{k}_r|,\mathbf{d}_r,\mathbf{b}_{r+1,n};\mathbf{c}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x}\right),\tag{35}$$

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\left(1-X_r\right)^{-a}\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{\left(c_j-b_j\right)_{k_j}}{k_j!\left(c_j\right)_{k_j}}\left(\frac{-x_j}{1-X_r}\right)^{k_j}\right]$$
$$\times F_A^{(n-r)}\left(a+|\mathbf{k}_r|,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r},...,\frac{x_n}{1-X_r}\right);\tag{36}$$

$$F_A^{(n-r)}\left(a,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r},...,\frac{x_n}{1-X_r}\right)=\left(1-X_r\right)^{a}$$
$$\times\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{\left(c_j-b_j\right)_{k_j}}{k_j!\left(c_j\right)_{k_j}}x_j^{k_j}\right]F_A^{(n)}\left(a+|\mathbf{k}_r|,\mathbf{b};\mathbf{c}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x}\right);\tag{37}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}\frac{1}{(c)_{|\mathbf{k}_r|}}\prod_{j=1}^{r}\left[\frac{(d_j-a_j)_{k_j}(b_j)_{k_j}}{k_j!}(-x_j)^{k_j}\right]$$
$$\times F_B^{(n)}\left(\mathbf{d}_r+\mathbf{k}_r,\mathbf{a}_{r+1,n},\mathbf{b}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};c+|\mathbf{k}_r|;\mathbf{x}\right),\tag{38}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}\frac{1}{(c)_{|\mathbf{k}_r|}}\prod_{j=1}^{r}\left[\frac{(a_j-d_j)_{k_j}(b_j)_{k_j}}{k_j!}x_j^{k_j}\right]$$
$$\times F_B^{(n)}\left(\mathbf{d}_r,\mathbf{a}_{r+1,n},\mathbf{b}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};c+|\mathbf{k}_r|;\mathbf{x}\right),\tag{39}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}(b)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(c_j-d_j)_{k_j}}{k_j!\,(c_j)_{k_j}}(-x_j)^{k_j}\right]$$
$$\times F_C^{(n)}\left(a+|\mathbf{k}_r|,b+|\mathbf{k}_r|;\mathbf{d}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x}\right);\tag{40}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}\frac{(a)_{|\mathbf{k}_r|}}{(c)_{|\mathbf{k}_r|}}\prod_{j=1}^{r}\left[\frac{(d_j-b_j)_{k_j}}{k_j!}(-x_j)^{k_j}\right]$$
$$\times F_D^{(n)}\left(a+|\mathbf{k}_r|,\mathbf{d}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};c+|\mathbf{k}_r|;\mathbf{x}\right),\tag{41}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}\frac{(a)_{|\mathbf{k}_r|}}{(c)_{|\mathbf{k}_r|}}\prod_{j=1}^{r}\left[\frac{(b_j-d_j)_{k_j}}{k_j!}x_j^{k_j}\right]$$
$$\times F_D^{(n)}\left(a+|\mathbf{k}_r|,\mathbf{d}_r,\mathbf{b}_{r+1,n};c+|\mathbf{k}_r|;\mathbf{x}\right).\tag{42}$$

In all formulas (34)–(42), n and r are natural numbers and $1\le r\le n$.

If $n=r$ in the equality (36), then we obtain a famous formula for the Lauricella function $F_A^{(n)}$ ([27], p. 116, Eq. (9))

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\left(1-X_n\right)^{-a}F_A^{(n)}\left(a,\mathbf{c}-\mathbf{b};\mathbf{c};\frac{x_1}{X_n-1},...,\frac{x_n}{X_n-1}\right).\tag{43}$$

We note that the formula (43) is a natural generalization of the Boltz formula (3).

If $n=r$ in the equality (37), then the infinity summation of the Lauricella function $F_A^{(n)}$ is written by an elementary function

$$\sum_{|\mathbf{k}|=0}^{\infty}(a)_{|\mathbf{k}|}\prod_{j=1}^{n}\left[\frac{(c_j-b_j)_{k_j}}{k_j!\,(c_j)_{k_j}}x_j^{k_j}\right]F_A^{(n)}\left(a+|\mathbf{k}|,\mathbf{b};\mathbf{c}+\mathbf{k};\mathbf{x}\right)=\left(1-X_n\right)^{-a}.\tag{44}$$

It is easy to see, that the equality (44) generalizes a famous infinity summation formula for the Gaussian hypergeometric function (5).

5. THE INFINITY SUMMATION FORMULAS ASSOCIATED WITH MULTI-DIMENSIONAL INVERSE PAIR OPERATORS

In this section, we consider the multi-dimensional inverse pair operators defined in (15) and (16).

Theorem 2. *The following symbolic forms are valid:*

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=H_{\mathbf{x}}\left(a,d\right)F_A^{(n)}\left(d,\mathbf{b};\mathbf{c};\mathbf{x}\right),\tag{45}$$

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\bar{H}_{\mathbf{x}}\left(d,a\right)F_A^{(n)}\left(d,\mathbf{b};\mathbf{c};\mathbf{x}\right);\tag{46}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=H_{\mathbf{x}}\left(d,c\right)F_B^{(n)}\left(\mathbf{a},\mathbf{b};d;\mathbf{x}\right), \tag{47}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\bar{H}_{\mathbf{x}}\left(c,d\right)F_B^{(n)}\left(\mathbf{a},\mathbf{b};d;\mathbf{x}\right); \tag{48}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=H_{\mathbf{x}}\left(a,d\right)F_C^{(n)}\left(d,b;\mathbf{c};\mathbf{x}\right), \tag{49}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=\bar{H}_{\mathbf{x}}\left(d,a\right)F_C^{(n)}\left(d,b;\mathbf{c};\mathbf{x}\right), \tag{50}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=H_{\mathbf{x}}\left(a,d_1\right)H_{\mathbf{x}}\left(b,d_2\right)F_C^{(n)}\left(d_1,d_2;\mathbf{c};\mathbf{x}\right), \tag{51}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=\bar{H}_{\mathbf{x}}\left(d_1,a\right)\bar{H}_{\mathbf{x}}\left(d_2,b\right)F_C^{(n)}\left(d_1,d_2;\mathbf{c};\mathbf{x}\right); \tag{52}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=H_{\mathbf{x}}\left(a,d\right)F_D^{(n)}\left(d,\mathbf{b};c;\mathbf{x}\right), \tag{53}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=\bar{H}_{\mathbf{x}}\left(d,a\right)F_D^{(n)}\left(d,\mathbf{b};c;\mathbf{x}\right), \tag{54}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=H_{\mathbf{x}}\left(d,c\right)F_D^{(n)}\left(a,\mathbf{b};d;\mathbf{x}\right), \tag{55}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=H_{\mathbf{x}}\left(a,c\right)\prod_{j=1}^{n}\left(1-x_j\right)^{-b_j}, \tag{56}$$

$$\prod_{j=1}^{n}\left(1-x_j\right)^{-b_j}=\bar{H}_{\mathbf{x}}(a,c)F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right). \tag{57}$$

As can be seen from the Theorem 2, each of $F_A^{(n)}$ and $F_B^{(n)}$ has two symbolic forms, and each of $F_C^{(n)}$ and $F_D^{(n)}$ has four symbolic forms. The symbolic forms (45)–(57) are used to obtain a large number of the infinity summation formulas of multiple Lauricella functions. For this purpose, in addition to the formulas (21) and (22), we will also use the following equalities

$$H_{\mathbf{x}}\left(a,d_1\right)H_{\mathbf{x}}\left(b,d_2\right)=\sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{\left(d_1-a\right)_{|\mathbf{k}|}\left(d_2-b\right)_{|\mathbf{l}|}\left(b\right)_{|\mathbf{k}|}}{K!L!\left(d_1\right)_{|\mathbf{k}|}\left(d_2\right)_{|\mathbf{k}|+|\mathbf{l}|}}\prod_{j=1}^{n}\left(-\delta_j\right)_{k_j+l_j}, \tag{58}$$

$$\bar{H}_{\mathbf{x}}\left(d_1,a\right)\bar{H}_{\mathbf{x}}\left(d_2,b\right)=\sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\left(a-d_1\right)_{|\mathbf{k}|}}{K!L!\left(b-d_2\right)_{|\mathbf{k}|}\left(1-d_1-\delta_1-...-\delta_n\right)_{|\mathbf{k}|}}$$
$$\times\frac{\left(b-d_2\right)_{|\mathbf{k}|+|\mathbf{l}|}\left(b\right)_{|\mathbf{k}|}}{\left(1-d_2-\delta_1-...-\delta_n\right)_{|\mathbf{k}|+|\mathbf{l}|}}\prod_{j=1}^{n}\left(-\delta_j\right)_{k_j+l_j}. \tag{59}$$

Applying the formulas (21) and (22) twice, one can easily obtain the equalities (58) and (59), respectively. Therefore, we have the following infinity summation formulas

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\left(d-a\right)_{|\mathbf{k}|}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}F_A^{(n)}\left(d+|\mathbf{k}|,\mathbf{b}+\mathbf{k};\mathbf{c}+\mathbf{k};\mathbf{x}\right), \tag{60}$$

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{\left(a-d\right)_{|\mathbf{k}|}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}F_A^{(n)}\left(d,\mathbf{b}+\mathbf{k};\mathbf{c}+\mathbf{k};\mathbf{x}\right); \tag{61}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\left(c-d\right)_{|\mathbf{k}|}\left(\mathbf{a}\right)_{\mathbf{k}}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!\left(d\right)_{|\mathbf{k}|}\left(c\right)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_B^{(n)}\left(\mathbf{a}+\mathbf{k},\mathbf{b}+\mathbf{k};d+|\mathbf{k}|;\mathbf{x}\right), \tag{62}$$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x})=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(d-c)_{|\mathbf{k}|}(\mathbf{a})_{\mathbf{k}}(\mathbf{b})_{\mathbf{k}}}{K!\,(d)_{|\mathbf{k}|}(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_B^{(n)}(\mathbf{a}+\mathbf{k},\mathbf{b}+\mathbf{k};d;\mathbf{x});\tag{63}$$

$$F_C^{(n)}(a,b;\mathbf{c};\mathbf{x})=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}(d-a)_{|\mathbf{k}|}(b)_{|\mathbf{k}|}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}F_C^{(n)}(d+|\mathbf{k}|,b+|\mathbf{k}|;\mathbf{c}+\mathbf{k};\mathbf{x}),\tag{64}$$

$$F_C^{(n)}(a,b;\mathbf{c};\mathbf{x})=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a-d)_{|\mathbf{k}|}(b)_{|\mathbf{k}|}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}F_C^{(n)}(d,b+|\mathbf{k}|;\mathbf{c}+\mathbf{k};\mathbf{x}),\tag{65}$$

$$F_C^{(n)}(a,b;\mathbf{c};\mathbf{x})=\sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|+|\mathbf{l}|}(d_1-a)_{|\mathbf{k}|}(d_2-b)_{|\mathbf{l}|}(b)_{|\mathbf{k}|}(d_1)_{|\mathbf{k}|+|\mathbf{l}|}}{K!L!\,(d_1)_{|\mathbf{k}|}(\mathbf{c})_{\mathbf{k}+\mathbf{l}}}\mathbf{x}^{\mathbf{k}+\mathbf{l}}$$

$$\times F_C^{(n)}(d_1+|\mathbf{k}|+|\mathbf{l}|,d_2+|\mathbf{k}|+|\mathbf{l}|;\mathbf{c}+\mathbf{k}+\mathbf{l};\mathbf{x}),\tag{66}$$

$$F_C^{(n)}(a,b;\mathbf{c};\mathbf{x})=\sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{(-1)^{|\mathbf{l}|}(a-d_1)_{|\mathbf{k}|}(b-d_2)_{|\mathbf{k}|+|\mathbf{l}|}(b)_{|\mathbf{k}|}}{K!L!\,(b-d_2)_{|\mathbf{k}|}(\mathbf{c})_{\mathbf{k}+\mathbf{l}}}\mathbf{x}^{\mathbf{k}+\mathbf{l}}$$

$$\times F_C^{(n)}(d_1+|\mathbf{l}|,d_2+|\mathbf{l}|;\mathbf{c}+\mathbf{k}+\mathbf{l};\mathbf{x});\tag{67}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x})=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}(d-a)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_D^{(n)}(d+|\mathbf{k}|,\mathbf{b}+\mathbf{k};c+|\mathbf{k}|;\mathbf{x}),\tag{68}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x})=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a-d)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_D^{(n)}(d,\mathbf{b}+\mathbf{k};c+|\mathbf{k}|;\mathbf{x}),\tag{69}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x})=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}(a)_{|\mathbf{k}|}(c-d)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(d)_{|\mathbf{k}|}(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_D^{(n)}(a+|\mathbf{k}|,\mathbf{b}+\mathbf{k};d+|\mathbf{k}|;\mathbf{x}),\tag{70}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x})=\prod_{j=1}^{n}\left[(1-x_j)^{-b_j}\right]\cdot F_D^{(n)}\left(c-a,\mathbf{b};c;\frac{x_1}{x_1-1},...,\frac{x_n}{x_n-1}\right);\tag{71}$$

$$\prod_{j=1}^{n}\left[(1-x_j)^{-b_j}\right]=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(c-a)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_D^{(n)}(a,\mathbf{b}+\mathbf{k};c+|\mathbf{k}|;\mathbf{x}).\tag{72}$$

The infinity summation formulas (34)–(42) and (60)–(72) can be proved without symbolic methods by comparing coefficients of equal powers of $x_1, x_2, ..., x_n$ on both sides.

6. EXPANSIONS ASSOCIATED WITH THE LAURICELLA FUNCTIONS $F_A^{(n)}$ AND $F_B^{(n)}$

Before proceeding to the presentation of the main result of this section, we introduce the notations

$$A(k,n)=\sum_{i=2}^{k+1}\sum_{j=i}^{n}m_{i,j},\quad B(k,n)=\sum_{i=2}^{k}m_{i,k}+\sum_{i=k+1}^{n}m_{k+1,i},$$

$$|\mathbf{m}_n| := \sum_{i=2}^{n}\sum_{j=i}^{n} m_{i,j}, \quad M_n! := \prod_{i=2}^{n}\prod_{j=i}^{n} m_{i,j}!$$

where $k, n \in N$, $k \le n$; $m_{i,j} \in N \cup \{0\}$ $(2 \le i \le j \le n)$.

It should be noted here that the sum $B(2,n) + B(3,n) + ... + B(n,n)$ has the parity property, which plays an important role in the calculation of the some values of hypergeometric functions. In fact, by virtue of equality

$$\sum_{k=2}^{n}\sum_{i=2}^{k} m_{i,k} = \sum_{k=1}^{n-1}\sum_{i=k+1}^{n} m_{k+1,i},$$

we obtain

$$\sum_{k=1}^{n} B(k,n) = 2\sum_{k=2}^{n}\sum_{i=2}^{k} m_{i,k} = 2\sum_{k=1}^{n-1}\sum_{i=k+1}^{n} m_{k+1,i}. \tag{73}$$

We present other simple properties of the functions $A(k,n)$ and $B(k,n)$:

$$A(n+1,n+1) - B(n+1,n+1) = A(n,n),$$

$$A(k+1,k+1) - B(k+1,k+1) = A(k,n) - B(k,n) + m_{2,n+1} + ... + m_{k,n+1},$$

which are easily proved if we proceed from the definitions of functions $A(k,n)$ and $B(k,n)$.

Theorem 3 [23]. *The following expansions hold at* $n \in \mathbb{N}$

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}} \times \prod_{k=1}^{n} x_k^{B(k,n)} F\left[\begin{matrix} a + A(k,n), b_k + B(k,n); \\ c_k + B(k,n); \end{matrix} x_k\right], \tag{74}$$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(-1)^{A(n,n)}}{(c)_{2A(n,n)} M_n!} \prod_{k=1}^{n} \frac{(a_k)_{B(k,n)} (b_k)_{B(k,n)}}{(c-1+A(k,n)-A(k-1,n))_{A(k,n)-A(k-1,n)}} \times \prod_{k=1}^{n} x_k^{B(k,n)} F\left[\begin{matrix} a_k + B(k,n), b_k + B(k,n); \\ c + 2A(k,n); \end{matrix} x_k\right].$$

Based on the Theorem 3, we will prove some new properties of these expansions and apply some of them to boundary value problems for the singular elliptic equations.

Theorem 4. *Let* $a, b_1, \ldots, b_n$ *are real numbers with* $a \ne 0, -1, -2, ...$ *and* $a > |\mathbf{b}|$, *where* $|\mathbf{b}| := b_1 + ... + b_n$. *Then, the following summation formulas hold at* $n \in \mathbb{N}$

$$\sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)} (a-b_k)_{A(k,n)-B(k,n)}}{(a)_{A(k,n)}} = \frac{\Gamma(a-|\mathbf{b}|)}{\Gamma(a)} \prod_{k=1}^{n} \frac{\Gamma(a)}{\Gamma(a-b_k)}, \tag{75}$$

$$\sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(-1)^{A(n,n)}}{(a)_{2A(n,n)} M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)} (a)_{2A(k,n)}}{(a-b_k)_{2A(k,n)-B(k,n)}} \times \prod_{k=1}^{n} \frac{(a-1)_{A(k,n)-A(k-1,n)}}{(a-1)_{2A(k,n)-2A(k-1,n)}} = \frac{\Gamma(a)}{\Gamma(a-|\mathbf{b}|)} \prod_{k=1}^{n} \frac{\Gamma(a-b_k)}{\Gamma(a)}.$$

Theorem 4 is proved by mathematical induction method.

Theorem 5. *Let $a, b_1, \dots, b_n$ be real numbers with $a,\, c_k,\, c_k - b_k \neq 0,\, -1,\, -2, \dots$ and $a > |\mathbf{b}|$. Then, the following limit formulas hold at $n \in \mathbb{N}$*

$$\lim_{\mathbf{x}\to 0}\left\{\mathbf{x}^{-\mathbf{b}} F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{x}}\right)\right\} = \frac{\Gamma\left(a - |\mathbf{b}|\right)}{\Gamma(a)} \prod_{k=1}^{n} \frac{\Gamma\left(c_k\right)}{\Gamma\left(c_k - b_k\right)}; \tag{76}$$

$$\lim_{\mathbf{x}\to 0}\left\{\mathbf{x}^{-\mathbf{b}} F_B^{(n)}\left(\mathbf{a}; \mathbf{b}; c; 1 - \frac{1}{\mathbf{x}}\right)\right\} = \frac{\Gamma\left(c\right)}{\Gamma\left(c - |\mathbf{b}|\right)} \prod_{k=1}^{n} \frac{\Gamma\left(a_k - b_k\right)}{\Gamma\left(a_k\right)}, \tag{77}$$

where

$$\mathbf{x}^{-\mathbf{b}} := x_1^{-b_1} \dots x_n^{-b_n}; \quad 1 - \frac{1}{\mathbf{x}} := \left(1 - \frac{1}{x_1}, \dots, 1 - \frac{1}{x_n}\right).$$

Proof. By virtue of the decomposition formula (74), we obtain

$$F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{x}}\right) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}$$
$$\times \prod_{k=1}^{n} \left(1 - \frac{1}{x_k}\right)^{B(k,n)} F\left[\begin{matrix} a + A(k,n), b_k + B(k,n); \\ c_k + B(k,n); \end{matrix} 1 - \frac{1}{x_k}\right]. \tag{78}$$

Applying now the familiar autotransformation formula (3) to each hypergeometric function included in the sum (78), we obtain

$$F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{x}}\right) = \mathbf{x}^{\mathbf{b}} \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}} \left(x_k - 1\right)^{B(k,n)}$$
$$\times \prod_{k=1}^{n} F\left[\begin{matrix} c_k - a + B(k,n) - A(k,n), b_k + B(k,n); \\ c_k + B(k,n); \end{matrix} 1 - x_k\right].$$

Using the parity property of the sum $B(2,n) + B(3,n) + \dots + B(n,n)$ (see Eq. (73)), we calculate the limit

$$\lim_{\mathbf{x}\to 0}\left\{\mathbf{x}^{-\mathbf{b}} F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{x}}\right)\right\} = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}$$
$$\times \prod_{k=1}^{n} F\left[\begin{matrix} c_k - a + B(k,n) - A(k,n), b_k + B(k,n); \\ c_k + B(k,n); \end{matrix} 1\right]$$

and applying the summation formula (4) to the Gauss hypergeometric functions in the last sum, by virtue of the equality (75), we obtain the equality (76). The equality (77) is proved similarly as proof of the equality (76). □

7. APPLICATION OF THE EXPANSIONS

The expansions of the Lauricella hypergeometric functions are very important in applications. To give an example, we consider the expansion (76).

Let $\mathbb{R}_m$ be the m -dimensional Euclidean space $(m \geq 2)$, $x := (x_1, \dots, x_m)$ and $\xi := (\xi_1, \dots, \xi_m)$ are arbitrary points in it and n is a natural number, and $n \leq m$. The 2^nth part of the Euclidean space $\mathbb{R}_m$ is defined as follows

$$\Omega \equiv \Omega_m^{n+} = \left\{x \in \mathbb{R}_m : x_i > 0,\, i = 1, \dots, n;\, -\infty < x_j < +\infty,\, j = n+1, \dots, m\right\}.$$

It is known that the fundamental solutions of the singular elliptic equation

$$\sum_{i=1}^{m}\frac{\partial^2 u}{\partial x_i^2}+\sum_{j=1}^{n}\frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j}=0 \tag{79}$$

in the hyperoctant Ω are expressed [7] by the Lauricella hypergeometric function $F_A^{(n)}$ in n variables, where $m \geq 2$ is a dimension of the Euclidean space; $n \geq 1$ is a number of the singular coefficients; $m \geq n$; α_j are real constants and $0 < 2\alpha_j < 1$ $(j = 1, ..., n)$.

For example, one of the fundamental solutions of equation (79) looks like [7]

$$q\left(x,\xi\right)=\gamma\, r^{-2\beta}F_A^{(n)}\left[\begin{matrix}\beta,\alpha_1,...,\alpha_n;\\ 2\alpha_1,...,2\alpha_n;\end{matrix}1-\frac{r_1^2}{r^2},...,1-\frac{r_n^2}{r^2}\right], \tag{80}$$

where

$$\beta:=\frac{m-2}{2}+\alpha_1+...+\alpha_n, \tag{81}$$

$$\gamma=2^{2\beta-m}\frac{\Gamma\left(\beta\right)}{\pi^{m/2}}\prod_{k=1}^{n}\frac{\Gamma\left(\alpha_k\right)}{\Gamma\left(2\alpha_k\right)}, \tag{82}$$

$$r^2=\sum_{i=1}^{m}\left(x_i-\xi_i\right)^2,\quad r_k^2=\left(x_k+\xi_k\right)^2+\sum_{i=1,\,i\neq k}^{m}\left(x_i-\xi_i\right)^2,\quad k=1,...,n.$$

When solving boundary value problems for an equation (79), it is very important to know the order of the singularity of the fundamental solution (80) at $r \to 0$, which can be determined using the expansion formula (74) and the limit relation (76). Indeed, using the expression (81) of β, the fundamental solution (80) can be easily reduced to the form

$$q(x,\xi)=\frac{1}{r^{m-2}}\tilde{q}(x,\xi),$$

where

$$\tilde{q}(x,\xi)=\gamma\prod_{k=1}^{n}r^{-2\alpha_k}\,F_A^{(n)}\left[\begin{matrix}\beta,\alpha_1,...,\alpha_n;\\ 2\alpha_1,...,2\alpha_n;\end{matrix}1-\frac{r_1^2}{r^2},...,1-\frac{r_n^2}{r^2}\right].$$

Now it is enough to prove that the expression $\tilde{q}(x,\xi)$ is bounded. Passing to the limit at $r \to 0$ and, by virtue of the limit relation (76) and (82), we obtain

$$\lim_{r\to 0}\tilde{q}(x,\xi)=\frac{1}{4\pi^{m/2}}\Gamma\left(\frac{m-2}{2}\right).$$

Thus, we have shown that the fundamental solution $q(x;\xi)$ has a singularity of the order $m-2$ at $r \to 0$, which is very important in further studies.

Other applications of the expansions of the Lauricella function F_A^n are found, also, in [29—32].

FUNDING

The research of first and second authors was supported by the Intercontinental Research Center has been initiated during the visit of authors to the intercontinental research center “Analysis and PDE” (Ghent University, Belgium), supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and partial differential equations and by the Methusalem programme of the Ghent University Special Research Fund (BOF) (Grant no. 01M01021).

The research of third author is funded by the Ministry of Higher Education, Sciences and Innovative development of the Republic of Uzbekistan (Grant F-FA-2021-424).

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. H. M. Srivastava and B. R. K. Kashyap, *Special Functions in Queuing Theory and Related Stochastic Processes* (Academic, New York, 1982).
2. A. W. Niukkanen, "Generalised hypergeometric series $^{N}F\,(x_1,...,x_N)$ arising in physical and quantum chemical applications," J. Phys. A: Math. Gen. **16**, 1813–1825 (1983).
3. F. I. Frankl, *Selected Works on the Gas Dynamics* (Nauka, Moscow, 1973) [in Russian].
4. P. Appell, "Sur les séries hypergéométriques de deux variables, et sur des équations différentielles linéaires aux dérivées partielles," C. R. Acad. Sci. Paris **90**, 296–298 (1880).
5. J. Horn, "Über die convergenz der hypergeometrischen reihen zweier und dreier veränderlichen," Math. Ann. **34**, 544–600 (1889).
6. G. Lauricella, "Sulle funzione ipergeometriche a più variabili," Rend. Circ. Mat. Palermo **7**, 111–158 (1893).
7. T. G. Ergashev, "Fundamental solutions for a class of multidimensional elliptic equations with several singular coefficients," J. Sib. Fed. Univ. Math. Phys. **13**, 48–57 (2020).
8. T. G. Ergashev, "Fundamental solutions of the generalized Helmholtz equation with several singular coefficients and confluent hypergeometric functions of many variables," Lobachevskii J. Math. **41**, 15–26 (2020).
9. J. L. Burchnall and T. W. Chaundy, "Expansions of Appell double hypergeometric functions," Q. J. Math. **11**, 249–270 (1940).
10. J. L. Burchnall and T. W. Chaundy, "Expansions of Appell double hypergeometric functions (II)," Q. J. Math. **12**, 112–128 (1941).
11. Yu. A. Brychkov and N. Saad, "Some formulas for the Appell function $F_1(a,b,b';c;w,z)$," Integr. Transforms Spec. Funct. **23**, 793–802 (2012).
12. Yu. A. Brychkov and N. Saad, "Some formulas for the Appell function $F_2(a,b,b';c,c';w,z)$," Integr. Transforms Spec. Funct. **25**, 111–123 (2014).
13. Yu. A. Brychkov and N. Saad, "Some formulas for the Appell function $F_3(a,a',b,b';c;w,z)$," Integr. Transforms Spec. Funct. **26**, 910–923 (2015).
14. Yu. A. Brychkov and N. Saad, "Some formulas for the Appell function $F_4(a,b;c,c',w,z)$," Integr. Transforms Spec. Funct. **28**, 629–644 (2017).
15. A. Hasanov and T. K. Yuldashev, "Analytic continuation formulas for the hypergeometric functions in three variables of second order," Lobachevskii J. Math. **43**, 386–393 (2022).
16. T. G. Ergashev, A. Hasanov, and T. K. Yuldashev, "Multiple Euler integral representations for the Kampé de Fériet functions," Chelyab. Phys.-Math. J. **8**, 553–567 (2023). https://doi.org/10.47475/2500-0101-2023-8-4-553-567
17. Z. N. Tasmambetov and Z. K. Ubayeva, "Exceptions of formulating the normal-regular solutions of confluent hypergeometric systems obtained from the Lauricella system," Lobachevskii J. Math. **43**, 2309–2321 (2022). https://doi.org/10.1134/S1995080222110294
18. M. O. Abbasova, T. G. Ergashev and T. K. Yuldashev, "Dirichlet problem for the Laplace equation in the hyperoctant of the multidimensional ball," Lobachevskii J. Math. **44**, 1072–1079 (2023).
19. T. G. Ergashev and M. O. Abbasova, "Holmgren's problem for the Laplace equation in the hyperoctant of the multidimensional ball," Lobachevskii J. Math. **43**, 1303–1312 (2022).
20. A. Hasanov and T. K. Yuldashev, "Exact solutions for a partial system of second-order hypergeometric equations and some decomposition formulas," Lobachevskii J. Math. **43**, 3142–3150 (2022). https://doi.org/10.1134/S1995080222140128
21. A. Hasanov and H. M. Srivastava, "Decomposition formulas associated with the Lauricella function $F_A^{(r)}$ and other multiple hypergeometric functions," Appl. Math. Lett. **19**, 113–121 (2006).
22. A. Hasanov and H. M. Srivastava, "Decomposition formulas associated with the Lauricella multivariable hypergeometric functions," Comput. Math. Appl. **53**, 1119–1128 (2007).
23. A. Hasanov and T. G. Ergashev, "New decomposition formulas associated with the Lauricella multivariable hypergeometric functions," Montes Taurus J. Pure Appl. Math. **3**, 317–326 (2021).
24. X. Wang, "Infinity summation formulas of double hypergeometric functions," Integr. Transforms Spec. Funct. **27**, 347–364 (2016).
25. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1.
26. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series,* Vol. 3: *More Special Functions* (Gordon and Breach Science, New York, 1990).

27. P. Appell and F. J. Kampé, *Fonctions Hypergéometriques et Hypersphériques: Polynômes d'Hermite* (Gauthier-Villars, Paris, 1926).
28. E. G. Poole, *Introduction to the Theory of Linear Differential Equations* (Clarendon, Oxford, 1936).
29. T. G. Ergashev, "Generalized Holmgren problem for an elliptic equation with several singular coefficients," Differ. Equat. **56**, 842–856 (2020).
30. T. G. Ergashev and Z. R. Tulakova, "The Neumann problem for a multidimensional elliptic equation with several singular coefficients in an infinite domain," Lobachevskii J. Math. **43**, 199–206 (2022).
31. T. G. Ergashev and Z. R. Tulakova, "A problem with mixed boundary conditions for a singular elliptic in an infinite domain," Russ. Math. (Iz. VUZ) **66** (7), 51–63 (2022).
32. T. G. Ergashev and Z. R. Tulakova, "The Dirichlet problem for an elliptic equation with several singular coefficients in an infinite domain," Russ. Math. **65** (7), 71–80 (2021).