DOI 10.1007/s10958-024-06985-6
Journal of Mathematical Sciences, Vol. 279, No. 1, February, 2024

Euler-type integral representations for the Kampé de Fériet functions

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(Presented by V. Ryazanov)

Abstract. In this paper, the Kampé de Fériet functions of arbitrary orders and their Euler-type integral representations are studied. The general form of the integral representations for a Kampé de Fériet function are proved. Conditions, under which these representations are expressed in terms of products of two generalized hypergeometric functions, are found. Examples are identified in which the integral representation of the Kampé de Fériet function contains an elementary function or a known second-order hypergeometric function of two variables.

Keywords. Kampé de Fériet functions, Euler-type integral representations, generalized hypergeometric functions, second-order hypergeometric function of two variables.

1. Introduction

A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of one, two, and more variables) is motivated essentially by the fact that solutions of many applied problems involving thermal conductivity and dynamics, electromagnetic oscillations and aerodynamics, quantum mechanics and potential theory are obtainable with the help of hypergeometric (higher and special or transcendent) functions [5, 28, 29]. Such kinds of functions are often called special functions of mathematical physics.

It is known that the hypergeometric series $F(a,b;c;z)$ [see equation (2.3)] were studied by Leonhard Euler. However, the first full systematic treatment was given by Carl Friedrich Gauss in 1813. In the nineteenth century, Ernst Kummer (1836) and Bernhard Riemann (1857) proposed a fundamental characterization of the hypergeometric functions by means of differential equations. The great success of the theory of hypergeometric series in one variable stimulated the development of the corresponding theory for the case of two or more variables. In 1880, Appell defined four series, F_1 to F_4 [see equations (2.4)–(2.7)], all of which are analogous to the Gaussian hypergeometric functions $F(a,b;c;z)$. Picard pointed out that one of those series is intimately related to a function studied by Pochhammer in 1870. Picard and Goursat also constructed a theory of Appell's series, which is analogous to Riemann's theory of Gauss hypergeometric series. P. Humbert has studied confluent hypergeometric series in two variables [see equations (2.8)–(2.14)]. An extension of the results of the French school together with references to the original literature can be found in the monograph by Appell and Kampé de Fériet [3], which is a classic work on the subject. This work also contains an extensive bibliography of all relevant papers up to 1926.

A great merit in the further development of the theory of the hypergeometric series in two variables belongs to Horn, who gave a general definition and an order classification of double hypergeometric series. He has investigated the convergence of hypergeometric series of two variables and established which systems of partial differential equations they satisfy. Horn studied particular hypergeometric

Translated from Ukrains'kiĭ Matematychnyĭ Visnyk, Vol. 20, No. 4, pp. 485–504, October–Desember, 2023.
Original article submitted May 08, 2023

series of order two. He found that apart from certain series that are either expressible in terms of one variable or are products of two hypergeometric series of one variable, there are essentially 34 distinct convergent series of the second order.

Four Appell series were unified and generalized by Kampé de Fériet in [24]. He defined a general hypergeometric series in two variables. However, the notation introduced by Kampé de Fériet for his double hypergeometric series of higher orders was subsequently abbreviated by Burchnall and Chaundy in [9]. Srivastava and Panda [38] gave a definition to a more general double hypergeometric series (than that defined by Kampé de Fériet) in a slightly modified notation and announced some groups of conditions on the parameters under which the Kampé de Fériet series converges in a non-empty set. Interesting results in this direction have been obtained by many authors (see, for example, works [10, 11, 13, 25–27, 30, 36]).

Many special functions appear as solutions of differential equations or integrals of elementary functions (for instance, see [1, 20–22, 33]). Therefore, tables of integrals usually include descriptions of special functions, whereas tables of special functions include most important integrals, at least, the integral representations of special functions. Symmetries of differential equations are essential to both physical and mathematical sciences. Therefore, the theory of integral representations is closely related to the theory of special functions for certain topics in mathematical physics. For example, in works [6, 15], some Kampé de Fériet functions were studied, and owing to their properties, the authors managed to obtain a solution to a boundary value problem for a differential equation in the explicit form.

Integral representations are very important in the study of applied problems. Integral representations can be useful when solving integral equations with functions in kernels and when calculating the values of explicit solutions to some applied problems. For evaluations and extensions of the results on Euler-type integrals, we refer to paper [39]. Also, in this regard, it is noticed that the general sextic equation can be solved in terms of the Kampé de Fériet function (see [12, 34]). Therefore, the well-known reference books [17, 31, 32], where the second-order hypergeometric functions (mainly in one and two variables) are considered, are highly respected among applied scientists. Hasanov and Ruzhansky, in 2019, constructed Euler-type integral representations for 205 second-order hypergeometric series in three variables [19]. However, there are very few works on integral representations of hypergeometric functions when their order exceeds two. We note only work [18], in which 18 integral representations were constructed for some Kampé de Fériet functions of the fourth order.

In this paper, we obtain the Euler-type integral representations for the Kampé de Fériet functions of arbitrary orders.

2. Preliminaries

With a view to introducing formally the Gauss hypergeometric series and its generalizations, we recall here some definitions and identities involving the Beta function $B(x, y)$, the Gamma function $\Gamma(z)$, and the Pochhammer symbol $(\lambda)_n$.

The Gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \begin{cases} \int\limits_0^\infty t^{z-1} e^{-t} dt, \ \ \mathrm{Re}(z) > 0, \\ \\ \dfrac{\Gamma(z+1)}{z}, \ \ \mathrm{Re}(z) < 0; \ \ z \neq -1, -2, -3, ... \end{cases} \tag{2.1}$$

Definition (2.1) was used by Euler, but there are also other definitions of Gamma function (see, for instance, [14]).

The Beta function is defined by the following integral representation

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}\alpha > 0,\ \mathrm{Re}\beta > 0. \tag{2.2}$$

The function

$$F(a,b;c;z) \equiv F\left[\begin{array}{c} a,b; \\ c\ ; \end{array} z\right] := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},\ c \neq 0,-1,-2,..., \tag{2.3}$$

is known as the Gauss hypergeometric function.

In the Gauss hypergeometric series $F(a,b;c;z)$, there are two numerator parameters a and b, and one denominator parameter c. A natural generalization of this series is accomplished by introducing any arbitrary numbers of numerator and denominator parameters. The resulting series

$${}_pF_q\left[\begin{array}{c} (a_p)\,; \\ (b_q)\,; \end{array} z\right] \equiv {}_pF_q\left[(a_p)\,;(b_q)\,;z\right] := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{j=1}^{q}(b_j)_k}\frac{z^k}{k!}$$

is known as the generalized Gauss series [14, p. 182] or, simply, the generalized hypergeometric series. Here p and q are positive integers or zero (interpreting the empty product as 1), and we assume that the variable z, the numerator parameters a_1,..., a_p, and the denominator parameters b_1, ..., b_q can acquire complex values provided that $b_j \neq 0,\ -1,\ -2,\ ...\ (j=1,\ ...,\ q)$.

Gauss series (2.3) in the present notation looks like

$${}_2F_1(a,b;c;z) \equiv F\left[\begin{array}{c} a,b; \\ c\ ; \end{array} z\right] := F(a,b;c;z).$$

The double Appell hypergeometric functions are defined as following [2]:

$$F_1\left(a,b,b';c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_{m+n}m!n!}x^my^n, \tag{2.4}$$

$$F_2\left(a,b,b';c,c';x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_m\left(c'\right)_n m!n!}x^my^n, \tag{2.5}$$

$$F_3\left(a,a',b,b';c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m\left(a'\right)_n(b)_m\left(b'\right)_n}{(c)_{m+n}m!n!}x^my^n, \tag{2.6}$$

$$F_4\left(a,b;c,c';x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m\left(c'\right)_n m!n!}x^my^n, \tag{2.7}$$

where, as usual, the denominator parameters c and c' are neither zero nor negative integers.

A standard work on the theory of Appell series is the monograph wrote by Appell and Kampé de Fériet [3]. This monograph contains an extensive bibliography of all relevant papers up to 1926 (for example, by L. Pochhammer, J. Horn, É. Picard, and É. Gursat). See [14, pp. 222–245] for a review of subsequent works on the subject; see also Bailey [4], Exton [16], Slater [35], Srivastava and Karlsson [37].

Seven confluent forms of the four Appell series were defined by Humbert [23], and he denoted these confluent hypergeometric series in two variables by the following formulas:

$$\Phi_1(\alpha,\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}m!n!}x^m y^n,\quad |x|<1, \tag{2.8}$$

$$\Phi_2\left(\beta,\beta';\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n, \tag{2.9}$$

$$\Phi_3(\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n, \tag{2.10}$$

$$\Psi_1\left(\alpha,\beta;\gamma,\gamma';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n m!n!}x^m y^n,\quad |x|<1, \tag{2.11}$$

$$\Psi_2\left(\alpha;\gamma,\gamma';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n\,m!n!}x^m y^n, \tag{2.12}$$

$$\Xi_1\left(\alpha,\alpha',\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n,\quad |x|<1, \tag{2.13}$$

$$\Xi_2(\alpha,\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n,\quad |x|<1, \tag{2.14}$$

where the denominator parameters γ and γ' are neither zero nor negative integers. The hypergeometric functions defined in (2.8)–(2.14) are called *Humbert functions.*

Just as the Gauss series $F(a,b;c;z)$ was generalized to ${}_pF_q$ by increasing the numbers of the numerator and denominator parameters, the four Appell series were unified and generalized by Kampé de Fériet [24]. So he obtained a general hypergeometric series in two variables (see [3, p. 150, eq. (29)]). The notation introduced by Kampé de Fériet for his double hypergeometric series of superior order was subsequently abbreviated by Burchnall and Chaundy [9, p. 112]. Srivastava and Panda [38] (see also [37, Section 3.1]) gave a definition to the more general double hypergeometric series (than that defined by Kampé de Fériet) in a slightly modified notation

$$F_{l:m,n}^{p:q,k}\left[\begin{matrix}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{2.15}$$

where, for convergence,
(i) $p+q<l+m+1,\ p+k<l+n+1,\ |x|<\infty,\ |y|<\infty,$
or
(ii) $p+q=l+m+1,\ p+k=l+n+1$, and

$$\begin{cases}|x|^{1/(p-l)}+|y|^{1/(p-l)}<1 & \text{if } p>l,\\ \max\{|x|,\ |y|\}<1 & \text{if } p\leq l.\end{cases}$$

Although the double hypergeometric series defined in (2.15) can be reduced to the Kampé de Fériet series only in the *special* case $q = k$ and $m = n$, yet it is usually referred to in the literature as the Kampé de Fériet series.

3. Integral representations

Theorem 3.1. *Let p, q, k, l, m, and n be non-negative integers. If* $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\beta) > 0$, *then the following integral representation formulas are valid:*

$$F_{l+1:\ m\ ;\ n}^{\ p\ :q+1;k+1}\left[\begin{matrix}(a_p) & :(b_q),\alpha;(c_k),\beta; \\ (\alpha_l),\alpha+\beta: & (\beta_m)\ ;\ (\gamma_n)\ ;\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\times\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}F_{l:m;n}^{p:q;k}\left[\begin{matrix}(a_p):(b_q)\ ;(c_k); \\ (\alpha_l)\ :(\beta_m);(\gamma_n);\end{matrix}xt,y(1-t)\right]dt, \quad (3.1)$$

$$F_{l+1:m;n}^{\ p\ :\ q;k+1}\left[\begin{matrix}(a_p) & :\ (b_q)\ ;(c_k),\beta; \\ (\alpha_l),\alpha+\beta: & (\beta_m)\ ;\ (\gamma_n)\ ;\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\times\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}F_{l:m+1;n}^{p:\ q\ ;\ k}\left[\begin{matrix}(a_p): & (b_q)\ ;(c_k); \\ (\alpha_l)\ : & (\beta_m),\alpha;(\gamma_n);\end{matrix}xt,y(1-t)\right]dt, \quad (3.2)$$

$$F_{l+1:\ m\ ;\ n}^{\ p\ :q+1;k}\left[\begin{matrix}(a_p) & :(b_q),\alpha;(c_k); \\ (\alpha_l),\alpha+\beta: & (\beta_m)\ ;(\gamma_n)\ ;\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\times\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}F_{l:m;n+1}^{p:q;k}\left[\begin{matrix}(a_p):(b_q)\ ; & (c_k)\ ; \\ (\alpha_l)\ :(\beta_m); & (\gamma_n),\beta;\end{matrix}xt,y(1-t)\right]dt, \quad (3.3)$$

$$F_{l+1:m;n}^{\ p\ :\ q;\ k}\left[\begin{matrix}(a_p) & :(b_q)\ ;(c_k); \\ (\alpha_l),\alpha+\beta & :(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}\times$$
$$\times(1-t)^{\beta-1}F_{l:m+1;n+1}^{p:\ q\ ;\ k}\left[\begin{matrix}(a_p): & (b_q)\ ; & (c_k)\ ; \\ (\alpha_l)\ : & (\beta_m),\alpha; & (\gamma_n),\beta;\end{matrix}xt,y(1-t)\right]dt. \quad (3.4)$$

Proof. Equalities (3.1)–(3.4) follow easily from the definition of the Kampé de Fériet series (2.15) if we use formula (2.2) for calculating the Beta function. □

Next, we give some examples where the Kampé de Fériet functions can be represented as integrals of either elementary functions or known hypergeometric functions.

Example 3.1. The following Appell and Humbert functions are expressed by the elementary and Humbert functions, respectively, as follows:

$$F_1(a,\alpha,\beta;\alpha+\beta;x,y)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-xt-y(1-t)\right]^{-a}dt,$$

$$\Phi_1(a,\alpha;\alpha+\beta;x,y)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Psi_1\left[a,b;b,\beta;xt,y(1-t)\right]dt.$$

Example 3.2. The following Kampé de Fériet functions are expressed by the Appell and Humbert functions as follows:

$$F^{1:2;2}_{2:0;0}\left[\begin{matrix} a & :b,\alpha;b',\beta; \\ c,\alpha+\beta: & -\ ;\ -\ ; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}F_1\left[a,b,b';c;xt,y(1-t)\right]dt,$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} a & :b,\alpha;b',\beta; \\ \alpha+\beta: & c\ ;\ c'\ ; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}F_2\left[a,b,b';c,c';xt,y(1-t)\right]dt,$$

$$F^{0:3;3}_{2:0;0}\left[\begin{matrix} - & :a,b,\alpha;a',b',\beta; \\ c,\alpha+\beta: & -\ ;\ -\ ; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}F_3\left[a,a',b,b';c;xt,y(1-t)\right]dt,$$

$$F^{2:1;1}_{1:1;1}\left[\begin{matrix} a,\ b & :\alpha;\beta; \\ \alpha+\beta & :c;c'; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}F_4\left[a,b;c,c';xt,y(1-t)\right]dt,$$

$$F^{1:2;1}_{2:0;0}\left[\begin{matrix} a & :b,\alpha;\beta; \\ c,\alpha+\beta: & -\ ;-; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Phi_1\left[a,b;c;xt,y(1-t)\right]dt,$$

$$F^{0:2;2}_{2:0;0}\left[\begin{matrix} - & :b,\alpha;b',\beta; \\ c,\alpha+\beta: & -\ ;\ -\ ; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Phi_2\left[b,b';c;xt,y(1-t)\right]dt,$$

$$F^{0:2;1}_{2:0;0}\left[\begin{matrix} - & :b,\alpha;\beta; \\ c,\alpha+\beta: & -\ ;-; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Phi_3\left[b;c;xt,y(1-t)\right]dt,$$

$$F^{1:2;1}_{1:1;1}\left[\begin{matrix} a & :b,\alpha;\beta; \\ \alpha+\beta: & c\ ;c'; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Psi_1\left[a,b;c,c';xt,y(1-t)\right]dt,$$

$$F^{1:1;1}_{1:1;1}\left[\begin{matrix} a & :\alpha;\beta; \\ \alpha+\beta & :c;c'; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Psi_2\left[a;c,c';xt,y(1-t)\right]dt,$$

$$F^{0:3;2}_{2:0;0}\left[\begin{matrix} - & :a,b,\alpha;a',\beta; \\ c,\alpha+\beta: & -\ ;\ -\ ; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Xi_1\left[a,a',b;c;xt,y(1-t)\right]dt,$$

$$F^{0:3;1}_{2:0;0}\left[\begin{matrix} - & :a,b,\alpha;\beta; \\ c,\alpha+\beta: & -\ ;-; \end{matrix} x,y\right]=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Xi_2\left[a,b;c;xt,y(1-t)\right]dt.$$

Corollary 3.1. *Let* $\operatorname{Re}(\alpha) > 0$ *and* $\operatorname{Re}(\beta) > 0$. *If* $p = 0$ *and* $l = 0$, *then the Kampé de Fériet function defined in* (2.15) *can be represented as an integral of the product of two generalized hypergeometric functions, viz.*

$$F^{0:\,q+1;k+1}_{1:\ m\ ;\ n}\left[\begin{matrix} - & :(b_q),\alpha;(c_k),\beta; \\ \alpha+\beta: & (\beta_m)\ ;\ (\gamma_n)\ ; \end{matrix} x,y\right] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\times \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,{}_qF_m\left[\begin{matrix}(b_q);\\(\beta_m);\end{matrix}xt\right]\,{}_kF_n\left[\begin{matrix}(c_k);\\(\gamma_n);\end{matrix}y(1-t)\right]dt,$$

$$F^{0:\,q;k+1}_{1:m;\ n}\left[\begin{matrix} - & :(b_q);(c_k),\beta; \\ \alpha+\beta & :(\beta_m);\ (\gamma_n)\ ; \end{matrix} x,y\right] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\times \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,{}_qF_{m+1}\left[\begin{matrix}(b_q)\ ;\\(\beta_m),\alpha;\end{matrix}xt\right]\,{}_kF_n\left[\begin{matrix}(c_k);\\(\gamma_n);\end{matrix}y(1-t)\right]dt,$$

$$F^{0:\,q+1;\,k}_{1:\ m\ ;n}\left[\begin{matrix} (a_p) & :(b_q),\alpha;(c_k); \\ (\alpha_l),\alpha+\beta: & (\beta_m)\ ;(\gamma_n); \end{matrix} x,y\right] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\times \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,{}_qF_m\left[\begin{matrix}(b_q)\ ;\\(\beta_m);\end{matrix}xt\right]\,{}_kF_{n+1}\left[\begin{matrix}(c_k)\ ;\\(\gamma_n),\beta;\end{matrix}y(1-t)\right]dt,$$

$$F^{0:\ q\ ;\ k}_{1:m;n}\left[\begin{matrix} (a_p) & :(b_q)\ ;(c_k); \\ (\alpha_l),\alpha+\beta & :(\beta_m);(\gamma_n); \end{matrix} x,y\right] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\times \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,{}_qF_{m+1}\left[\begin{matrix}(b_q)\ ;\\(\beta_m),\alpha;\end{matrix}xt\right]\,{}_kF_{n+1}\left[\begin{matrix}(c_k)\ ;\\(\gamma_n),\beta;\end{matrix}y(1-t)\right]dt.$$

Next, we give some examples where the Kampé de Fériet functions can be represented as an integral of the product of the two (elementary and/or known hypergeometric) functions.

Example 3.3. The following integral representations of the Appell and Humbert functions are known [7,8]:

$$F_3(a,b,\alpha,\beta;\alpha+\beta;x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-xt)^{-a}\left[1-y+yt\right]^{-b}dt,$$

$$\Phi_2(\alpha,\beta;\alpha+\beta;x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{xt}e^{y(1-t)}dt,$$

$$\Phi_3(\alpha;\alpha+\beta;x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{xt}{}_0F_1\left[-;\beta;y(1-t)\right]dt,$$

$$\Xi_1(a,\alpha,\beta;\alpha+\beta;x,y)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-xt)^{-a}e^{y(1-t)}dt,$$

$$\Xi_2(a,\alpha;\alpha+\beta;x,y)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\times\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-xt)^{-a}{}_0F_1\left[-;\beta;y(1-t)\right]dt.$$

Theorem 3.2. *Let p, q, k, l, m, and n be non-negative integers. If*

$$\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\mathrm{Re}(\lambda)>0,\ \mathrm{Re}(\mu)>0, \tag{3.5}$$

then the following double-integral representation formulas hold true:

$$F^{\,p\,:\,q+2\,;k+2}_{l+2:\ m\ ;\ n}\left[\begin{matrix}(a_p):(b_q),\alpha,\lambda;(c_k),\beta,\mu;\\ [\alpha_l]:\ (\beta_m)\ ;\ (\gamma_n)\ ;\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}$$
$$\times\int_0^1\int_0^1 f(s,t)F^{p:\,q\,;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k);\\ (\alpha_l):(\beta_m);(\gamma_n);\end{matrix}X,Y\right]dsdt, \tag{3.6}$$

$$F^{\,p\,:\,q+1;k+1}_{l+2:\ m\ ;\ n}\left[\begin{matrix}(a_p):(b_q),\alpha;(c_k),\beta;\\ [\alpha_l]:\ (\beta_m)\ ;\ (\gamma_n)\ ;\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\times$$
$$\times\int_0^1\int_0^1 f(s,t)F^{p:\ q\ ;\ k}_{l:m+1;n+1}\left[\begin{matrix}(a_p):\ (b_q)\ ;\ (c_k)\ ;\\ (\alpha_l):(\beta_m),\lambda;(\gamma_n),\mu;\end{matrix}X,Y\right]dsdt, \tag{3.7}$$

$$F^{\,p\,:\,q\,;k}_{l+2:\,m;\,n}\left[\begin{matrix}(a_p):(b_q);(c_k);\\ [\alpha_l]:(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\times$$
$$\times\int_0^1\int_0^1 f(s,t)F^{p:\ q\ ;\ k}_{l:m+2;n+2}\left[\begin{matrix}(a_p):\ (b_q)\ ;\ (c_k)\ ;\\ (\alpha_l):(\beta_m),\alpha,\lambda;(\gamma_n),\beta,\mu;\end{matrix}X,Y\right]dsdt, \tag{3.8}$$

$$F^{\,p\,:\,q+2\,;k}_{l+2:\ m\ ;n}\left[\begin{matrix}(a_p):(b_q),\alpha,\lambda;(c_k);\\ [\alpha_l]:\ (\beta_m)\ ;(\gamma_n);\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\times$$
$$\times\int_0^1\int_0^1 f(s,t)F^{p:\,q\,;k}_{l:m;n+2}\left[\begin{matrix}(a_p):(b_q);\ (c_k)\ ;\\ (\alpha_l):(\beta_m);(\gamma_n),\beta,\mu;\end{matrix}X,Y\right]dsdt, \tag{3.9}$$

$$F^{\,p\,:\,q\,;k+2}_{l+2:m;\,n}\left[\begin{matrix}(a_p):(b_q);(c_k),\beta,\mu;\\ [\alpha_l]:(\beta_m);\ (\gamma_n)\ ;\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\times$$
$$\times\int_0^1\int_0^1 f(s,t)F^{p:\ q\ ;k}_{l:m+2;n}\left[\begin{matrix}(a_p):\ (b_q)\ ;(c_k);\\ (\alpha_l):(\beta_m),\alpha,\lambda;(\gamma_n);\end{matrix}X,Y\right]dsdt, \tag{3.10}$$

$$F_{l+2:\;m\;;\;n}^{\;p\;:q+1\;;k+2}\left[\begin{matrix}(a_p):(b_q),\alpha;(c_k),\beta,\mu;\\ [\alpha_l]:\;(\beta_m)\;\;;\;\;(\gamma_n)\;\;;\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\times$$

$$\times\int_0^1\int_0^1 f(s,t)F_{l:m+1;n}^{p:\;q\;;k}\left[\begin{matrix}(a_p):\;(b_q)\;;(c_k);\\(\alpha_l):(\beta_m),\lambda;(\gamma_n);\end{matrix}X,Y\right]dsdt, \quad (3.11)$$

$$F_{l+2:\;m\;;\;n}^{\;p\;:q+2\;;k+1}\left[\begin{matrix}(a_p):(b_q),\alpha,\lambda;\;(c_k),\beta;\\ [\alpha_l]:\;(\beta_m)\;\;;\;\;(\gamma_n)\;\;;\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\times$$

$$\times\int_0^1\int_0^1 f(s,t)F_{l:m;n+1}^{p:\;q;k}\left[\begin{matrix}(a_p):(b_q);\;(c_k);\\(\alpha_l):(\beta_m);(\gamma_n),\mu;\end{matrix}X,Y\right]dsdt, \quad (3.12)$$

$$F_{l+2:m;\;n}^{\;p\;:q\;;k+1}\left[\begin{matrix}(a_p):(b_q);\;(c_k),\beta;\\ [\alpha_l]:(\beta_m);\;(\gamma_n)\;\;;\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\times$$

$$\times\int_0^1\int_0^1 f(s,t)F_{l:m+2;n+1}^{p:\;q\;;k}\left[\begin{matrix}(a_p):\;\;(b_q)\;\;;\;\;(c_k);\\(\alpha_l):(\beta_m),\alpha,\lambda;(\gamma_n),\mu;\end{matrix}X,Y\right]dsdt, \quad (3.13)$$

$$F_{l+2:\;m\;;n}^{\;p\;:q+1\;;\;k}\left[\begin{matrix}(a_p):(b_q),\alpha;\;(c_k)\;\;;\\ [\alpha_l]:(\beta_m)\;;\;(\gamma_n)\;\;;\end{matrix}x,y\right]=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\times$$

$$\times\int_0^1\int_0^1 f(s,t)F_{l:m+1;n+2}^{p:\;q\;;k}\left[\begin{matrix}(a_p):\;\;(b_q)\;;\;\;(c_k)\;\;;\\(\alpha_l):(\beta_m),\lambda;(\gamma_n),\beta,\mu;\end{matrix}X,Y\right]dsdt. \quad (3.14)$$

In all equations (3.6)–(3.14)*, for convenience and brevity of writing, we used the notations* $[\alpha_l] := [(\alpha_l),\alpha+\beta,\lambda+\mu]$*,* $X := xst$*,* $Y := y(1-s)(1-t)$*, and* $f(s,t) = s^{\alpha-1}(1-s)^{\beta-1}t^{\lambda-1}(1-t)^{\mu-1}$*.*

Proof. The theorem statements follow easily from the definition of the Kampé de Fériet series (2.15) if we use twice formula (2.2) for calculating the Beta function (2.2). □

Now we consider examples with concrete positive integer values for p, q, k, l, m, and n.

Example 3.4. The following Kampé de Fériet functions are expressed by the double integral of the Appell and Humbert functions:

$$F_{3:0;0}^{1:3;3}\left[\begin{matrix}a\;\;:b,\alpha,\lambda;b',\beta,\mu;\\c,\alpha+\beta,\lambda+\mu:\;\;-\;\;;\;\;-\;\;;\end{matrix}x,y\right]=$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\int_0^1\int_0^1 f(s,t)F_1\left(a,b,b';c;X,Y\right)dsdt.$$

$$F_{2:1;1}^{1:3;3}\left[\begin{matrix}a\;\;:b,\alpha,\lambda;b',\beta,\mu;\\\alpha+\beta,\lambda+\mu:\;\;c\;\;;\;\;c'\;\;;\end{matrix}x,y\right]=$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\int_0^1\int_0^1 f(s,t)F_2\left(a,b,b';c,c';X,Y\right)dsdt.$$

$$F^{0:4;4}_{3:0;0}\left[\begin{matrix} - & :a,a',\alpha,\lambda;b,b',\beta,\mu; \\ c,\alpha+\beta,\lambda+\mu: & -\quad;\quad -\quad; \end{matrix} x,y\right]=$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\int\limits_0^1\int\limits_0^1 f(s,t)F_3\left(a,a',b,b';c;X,Y\right)dsdt.$$

$$F^{2:2;2}_{2:1;1}\left[\begin{matrix} a,\quad b & :\alpha,\ \lambda;\ \beta,\mu; \\ \alpha+\beta,\lambda+\mu: & c\quad;\quad c';\end{matrix} x,y\right]=$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\int\limits_0^1\int\limits_0^1 f(s,t)F_4\left(a,b;c,c';X,Y\right)dsdt.$$

$$F^{1:3;2}_{3:0;0}\left[\begin{matrix} a & :b,\alpha,\lambda;\beta,\mu; \\ c,\alpha+\beta,\lambda+\mu: & -\quad;\ -\quad; \end{matrix} x,y\right]=$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\int\limits_0^1\int\limits_0^1 f(s,t)\Phi_1(a,b;c;X,Y)dsdt.$$

$$F^{0:3;3}_{3:0;0}\left[\begin{matrix} - & :b,\alpha,\lambda;b',\beta,\mu; \\ c,\alpha+\beta,\lambda+\mu: & -\quad;\quad -\quad; \end{matrix} x,y\right]=$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\int\limits_0^1\int\limits_0^1 f(s,t)\Phi_2\left(b,b';c;X,Y\right)dsdt.$$

$$F^{0:3;2}_{3:0;0}\left[\begin{matrix} - & :b,\alpha,\lambda;\beta,\mu; \\ c,\alpha+\beta,\lambda+\mu: & -\quad;\ -\ ; \end{matrix} x,y\right]=$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\int\limits_0^1\int\limits_0^1 f(s,t)\Phi_3(b;c;X,Y)dsdt.$$

$$F^{1:3;2}_{2:1;1}\left[\begin{matrix} a & :b,\alpha,\lambda;\beta,\mu; \\ \alpha+\beta,\lambda+\mu: & c\quad;\ \ c'\ ; \end{matrix} x,y\right]=$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\int\limits_0^1\int\limits_0^1 f(s,t)\Psi_1\left(a,b;c,c';X,Y\right)dsdt.$$

$$F^{1:2;2}_{2:1;1}\left[\begin{matrix} a & :\alpha,\lambda;\beta,\mu; \\ \alpha+\beta,\lambda+\mu: & c\ ;\ \ c'\ ; \end{matrix} x,y\right]=$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\int\limits_0^1\int\limits_0^1 f(s,t)\Psi_2\left(a;c,c';X,Y\right)dsdt.$$

$$F^{0:4;3}_{3:0;0}\left[\begin{matrix} - & :a,b,\alpha,\lambda;a',\beta,\mu; \\ c,\alpha+\beta,\lambda+\mu: & - \quad ; \quad - \quad ; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)} \int_0^1\int_0^1 f(s,t)\Xi_1\left(a,a',b;c;X,Y\right)dsdt.$$

$$F^{0:4;2}_{3:0;0}\left[\begin{matrix} - & :a,b,\alpha,\lambda;\beta,\mu; \\ c,\alpha+\beta,\lambda+\mu: & - \quad ; - ; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)} \int_0^1\int_0^1 f(s,t)\Xi_2(a,b;c;X,Y)dsdt.$$

$$F^{\,p\,:2;2}_{l+2:0;0}\left[\begin{matrix} (a_p) & :\alpha,\lambda;\ \beta,\mu; \\ (\alpha_l),\alpha+\beta,\lambda+\mu: & - \ ; \ - \ ; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\alpha+\beta)\Gamma(\lambda+\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)} \int_0^1\int_0^1 f(s,t)\,{}_pF_l\left[\begin{matrix}(a_p); \\ (\alpha_l);\end{matrix} X+Y\right]dsdt.$$

Here, for convenience and brevity of writing, we used the notations $f(s,t) = s^{\alpha-1}(1-s)^{\beta-1}t^{\lambda-1}(1-t)^{\mu-1}$, $X := xst$, and $Y := y(1-s)(1-t)$.

Before presenting the following Corollary, we adopt the notations $\alpha+\beta=\xi_1$ and $\lambda+\mu=\xi_2$.

Corollary 3.2. *Let conditions* (3.5) *be satisfied. If* $p=0$ *and* $l=0$, *then the Kampé de Fériet function defined in* (2.15) *can be represented as an integral of the product of two generalized hypergeometric functions:*

$$F^{0:\,q+2;k+2}_{2:\ m\ ;\ n}\left[\begin{matrix} - & :(b_q),\alpha,\lambda;(c_k),\beta,\mu; \\ (\xi_1,\xi_2): & (\beta_m) \quad ; \quad (\gamma_n) \quad ; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\xi_1)\Gamma(\xi_2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)} \int_0^1\int_0^1 f(s,t)\,{}_qF_m\left[\begin{matrix}(b_q); \\ (\beta_m);\end{matrix} X\right]{}_kF_n\left[\begin{matrix}(c_k); \\ (\gamma_n);\end{matrix} Y\right]dsdt,$$

$$F^{0:\,q+1\,;k+1}_{2:\ m\ ;\ n}\left[\begin{matrix} - & :(b_q),\alpha;(c_k),\beta; \\ (\xi_1,\xi_2): & (\beta_m) \ ; \ (\gamma_n) \ ; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\xi_1)\Gamma(\xi_2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)} \int_0^1\int_0^1 f(s,t)\,{}_qF_{m+1}\left[\begin{matrix}(b_q) & ; \\ (\beta_m),\lambda; & \end{matrix} X\right]{}_kF_{n+1}\left[\begin{matrix}(c_k) & ; \\ (\gamma_n),\mu; & \end{matrix} Y\right]dsdt,$$

$$F^{0:\,q;\,k}_{2:\,m;\,n}\left[\begin{matrix} - & :(b_q);(c_k); \\ (\xi_1,\xi_2) & :(\beta_m);(\gamma_n); \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\xi_1)\Gamma(\xi_2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)} \int_0^1\int_0^1 f(s,t)\,{}_qF_{m+2}\left[\begin{matrix}(b_q) & ; \\ (\beta_m),\alpha,\lambda; & \end{matrix} X\right]{}_kF_{n+2}\left[\begin{matrix}(c_k) & ; \\ (\gamma_n),\beta,\mu; & \end{matrix} Y\right]dsdt,$$

$$F^{0:\,q+2\,;\,k}_{2:\ \ m\ \ ;\,n}\left[\begin{matrix} - & :(b_q)\,,\alpha,\lambda; & (c_k)\,; \\ (\xi_1,\xi_2): & (\beta_m)\ \ ; & (\gamma_n)\,; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\xi_1)\,\Gamma(\xi_2)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\lambda)\,\Gamma(\mu)} \int\limits_0^1 \int\limits_0^1 f(s,t)\,{}_qF_m\left[\begin{matrix}(b_q)\,; \\ (\beta_m)\,;\end{matrix} X\right] {}_kF_{n+2}\left[\begin{matrix}(c_k)\ \ ; \\ (\gamma_n)\,,\beta,\mu;\end{matrix} Y\right] dsdt,$$

$$F^{0:\ q\,;k+2}_{2:\,m;\ \ n}\left[\begin{matrix} - & :\ (b_q)\,;(c_k)\,,\beta,\mu; \\ (\xi_1,\xi_2) & :(\beta_m)\,;\quad (\gamma_n)\quad ; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\xi_1)\,\Gamma(\xi_2)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\lambda)\,\Gamma(\mu)} \int\limits_0^1 \int\limits_0^1 f(s,t)\,{}_qF_{m+2}\left[\begin{matrix}(b_q)\quad ; \\ (\beta_m)\,,\alpha,\lambda;\end{matrix} X\right] {}_kF_n\left[\begin{matrix}(c_k)\,; \\ (\gamma_n)\,;\end{matrix} Y\right] dsdt,$$

$$F^{0:q+1;\,k+2}_{2:\ \ m\ ;\ \ n}\left[\begin{matrix} - & :(b_q)\,,\alpha;(c_k)\,,\beta,\mu; \\ (\xi_1,\xi_2): & (\beta_m)\ \ ;\quad (\gamma_n)\quad ; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\xi_1)\,\Gamma(\xi_2)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\lambda)\,\Gamma(\mu)} \int\limits_0^1 \int\limits_0^1 f(s,t)\,{}_qF_{m+1}\left[\begin{matrix}(b_q)\quad ; \\ (\beta_m)\,,\lambda;\end{matrix} X\right] {}_kF_n\left[\begin{matrix}(c_k)\,; \\ (\gamma_n)\,;\end{matrix} Y\right] dsdt,$$

$$F^{0:\,q+2\,;k+1}_{2:\ \ m\ \ ;\ \ n}\left[\begin{matrix} - & :(b_q)\,,\alpha,\lambda;(c_k)\,,\beta; \\ (\xi_1,\xi_2): & (\beta_m)\quad ;\ \ (\gamma_n)\ \ ; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\xi_1)\,\Gamma(\xi_2)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\lambda)\,\Gamma(\mu)} \int\limits_0^1 \int\limits_0^1 f(s,t)\,{}_qF_m\left[\begin{matrix}(b_q)\quad ; \\ (\beta_m)\,;\end{matrix} X\right] {}_kF_{n+1}\left[\begin{matrix}(c_k)\ \ ; \\ (\gamma_n)\,,\mu;\end{matrix} Y\right] dsdt,$$

$$F^{0:\,q\,;k+1}_{2:m;\ \ n}\left[\begin{matrix} - & :\ (b_q)\,;\ (c_k)\,,\beta; \\ (\xi_1,\xi_2) & :(\beta_m)\,;\ \ (\gamma_n)\quad ; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\xi_1)\,\Gamma(\xi_2)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\lambda)\,\Gamma(\mu)} \int\limits_0^1 \int\limits_0^1 f(s,t)\,{}_qF_{m+2}\left[\begin{matrix}(b_q)\quad ; \\ (\beta_m)\,,\alpha,\lambda;\end{matrix} X\right] {}_kF_{n+1}\left[\begin{matrix}(c_k)\ \ ; \\ (\gamma_n)\,,\mu;\end{matrix} Y\right] dsdt,$$

$$F^{0:\,q+1\,;\,k}_{2:\ \ m\ \ ;\,n}\left[\begin{matrix} - & :(b_q)\,,\alpha;\ (c_k)\,; \\ (\xi_1,\xi_2): & (\beta_m)\ \ ;(\gamma_n)\,; \end{matrix} x,y\right] =$$

$$= \frac{\Gamma(\xi_1)\,\Gamma(\xi_2)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\lambda)\,\Gamma(\mu)} \int\limits_0^1 \int\limits_0^1 f(s,t)\,{}_qF_{m+1}\left[\begin{matrix}(b_q)\quad ; \\ (\beta_m)\,,\lambda;\end{matrix} X\right] {}_kF_{n+2}\left[\begin{matrix}(c_k)\quad ; \\ (\gamma_n)\,,\beta,\mu;\end{matrix} Y\right] dsdt,$$

where, for convenience and brevity of writing, we used the notations $f(s,t) = s^{\alpha-1}(1-s)^{\beta-1}t^{\lambda-1}(1-t)^{\mu-1}$, $X := xst$, *and* $Y := y(1-s)(1-t)$.

Declarations

Declaration of competing interest

The authors declare no potential conflict of interest with respect to the research, authorship, and publication of this article.

Data availability

All necessary data are included into the paper.

Acknowledgement

The research was supported by the Intercontinental Research Center "Analysis and PDE" (Ghent University, Belgium, grant G.0H94.18N) and the Special Research Fund of Methusalem programme (Ghent University, Grant number 01M01021).

CRediT (Contributor Roles Taxonomy) authorship contribution statement

Author 1: Visualization, Investigation, and Writing - Original draft preparation. Author 2: Conceptualization and Methodology. Author 3: Software, Validation, Reviewing, and Editing.

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