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EXPANSIONS OF KAMPÉ DE FÉRIET HYPERGEOMETRIC FUNCTIONS

When studying the properties of the hypergeometric functions in two variables, expansion formulas are very important, allowing one to represent a function of two variables in the form of an infinite sum of products of several hypergeometric Gaussian functions, and this in turn facilitates the process of studying the properties of functions in two variables. Burchnall and Chaundy, in 1940–41, using the symbolic method, obtained more than 15 pairs of expansions for the second-order double hypergeometric Appell and Humbert functions. In order to find expansion formulas for functions depending on three or more variables, Hasanov and Srivastava introduced symbolic operators, with the help of which they were able to expand a whole class of hypergeometric functions of several variables. Hasanov, Turaev and Choi defined so-called H-operators that make it possible to find expansions for generalized hypergeometric functions of one variable. In addition, applications of these H-operators to the expansion of the hypergeometric functions of two and three variables of second order are known. On the other hand, thanks to the Kampé de Fériet functions, solutions of the boundary value problems for some degenerate and singular partial differential equations can be written in explicit forms. In this paper, expansion formulae are obtained for the hypergeometric Kampé de Fériet functions of the superior order. Some Kampé de Fériet functions are expanded in terms of the Appell and Humbert functions as illustrative examples.

Key words: Hypergeometric function in two variables, Kampé de Fériet function, generalized hypergeometric function, expansion formula, Burchnall-Chaundy method, symbolic H-operator, Appell and Humbert hypergeometric functions.

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Кампе-де-Ферье гипергеометриялық функцияларын жіктеу формулалары

Екі айнымалы гипергеометриялық функциясын зерттеу үшін екі айнымалы функциясын бірнеше Гаусс гипергеометриялық функциялар көбейтідісінің шексіз қосындысы ретінде көрсетуге мүмкіндік беретін жіктеу формулалары өте маңызды, бұл өз кезегінде екі айнымалы функцияларының қасиеттерін зерттеу процесін жеңілдетеді. Берчнелл мен Ченди 1940-1941 жылдары символдық әдіспен Аппель мен Гумберттің екінші ретті гипергеометриялық функциялары үшін 15-тен астам жіктеу жұбын алды. Үш немесе одан да көп айнымалыларға тәуелді функциялардың жіктеу формулаларын табу үшін Хасанов және Сривастава 2006-2007 жылдары символдық операторларды енгізді, олардың көмегімен бірнеше айнымалы гипергеометриялық функцияларының бүкіл класын жіктей алды.

Алайда, бұл символдық операторлар екінші ретті гипергеометриялық функциялармен шектелді, сондықтан 2010 жылы Хасанов, Тураев және Чой жоғары ретті бір айнымалы жалпыланған гипергеометриялық функцияларын жіктеуге мүмкіндік беретін Н-операторларын енгізді. Сонымен қатар, осы Н-операторларының екі және үш айнымалы екінші ретті гипергеометриялық функцияларын жіктеуге арналған қосымшалары белгілі. Екінші жағынан, Кампе-де-Ферье функцияларының көмегімен кейбір өзгешеленген және сингулярлық дербес туындылы дифференциалдық теңдеулер үшін шеттік есептердің шешімдері айқын түрде жазылуы мүмкін. Бұл жұмыста жоғары ретті Кампе-де-Ферьенің гипергеометриялық функциялары үшін жіктеу формулалары алынды. Көрнекі мысалдар ретінде Кампе-де-Ферьенің кейбір функциялары Аппель мен Гумберттің екінші ретті гипергеометриялық функциялары бойынша жіктеледі.
Түйін сөздер: екі айнымалы гипергеометриялық функциясы, Кампе-де-Ферье функциясы, жалпыланған гипергеометриялық функция, жіктеу формуласы, Берчнелл-Чанди әдісі, символдық H-операторы, Аппель мен Гумберттің гипергеометриялық функциялары

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Разложения гипергеометрических функций Кампе-де-Ферье

Для исследования гипергеометрической функции двух переменных очень важны формулы разложения, которые позволяют представить функцию двух переменных в виде бесконечной суммы произведений нескольких гипергеометрических функций Гаусса, а это, в свою очередь, облегчает процесс изучения свойств функций двух переменных. Берчнелл и Ченди в 1940–41 гг. символическим методом получили более 15 пар разложений для гипергеометрических функций Аппеля и Гумберта второго порядка. Чтобы найти формулы разложения функций, зависящих от трех и более переменных, Хасанов и Сривастава в 2006–07 гг. ввели символические операторы, с помощью которых они смогли разложить целый класс гипергеометрических функций нескольких переменных. Однако, эти символические операторы ограничились гипергеометрическими функциями второго порядка, поэтому в 2010 г. Хасанов, Тураев и Чой ввели в рассмотрение так называемые Н-операторы, позволяющие разложить обобщенные гипергеометрические функции одной переменной высокого порядка. Кроме того, известны приложения этих Н-операторов к разложению гипергеометрических функций двух и трех переменных второго порядка. С другой стороны, благодаря функциям Кампе-де-Ферье решения краевых задач для некоторых вырождающихся и сингулярных уравнений в частных производных могут быть записаны в явном виде. В данной работе получены формулы разложения для гипергеометрических функций Кампе-де-Ферье высшего порядка. В качестве наглядных примеров, некоторые функции Кампе-де- Ферье разложены по гипергеометрическим функциям Аппеля и Гумберта второго порядка.
Ключевые слова: гипергеометрическая функция двух переменных, функция Кампе-де-Ферье, обобщенная гипергеометрическая функция, формула разложения, метод Берчнелла-Ченди, символический H-оператор, гипергеометрические функции Аппеля и Гумберта.

1 Introduction

A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of one, two and more variables) is motivated essentially by the fact that solutions of many applied problems involving thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, and quantum mechanics and potential theory are obtainable with the help of hypergeometric (higher and special or transcendent) functions. Such kinds of functions are often referred to as special functions of mathematical physics.

A sum of the following power series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\qquad |z|<1, \tag{1}$$

is known as the Gaussian hypergeometric function, where a, b, c are independent of z. We call a, b, c the parameters of the hypergeometric function; they are arbitrary complex numbers with $c\neq 0,-1,-2,....$ Here $(\nu)_n$ is a Pochhammer symbol:

$$(\nu)_0:=1,\quad (\nu)_n:=\nu(\nu+1)...(\nu+n-1)=\frac{\Gamma(\nu+n)}{\Gamma(\nu)};$$

$\Gamma(z)$ is a well-known gamma function.

The great success of the theory of hypergeometric series in one variable has stimulated the development of a corresponding theory in two and more variables. Appell has defined, in 1880, four series, F_1 to F_4 (cf. equations (3) to (6) *infra*) which are all analogous to Gauss' $F(a,b;c;z)$. Picard has pointed out that one of these series is intimately related to a function studied by Pochhammer in 1870, and Picard and Goursat also constructed a theory of Appell's series which is analogous to Riemann's theory of Gauss' hypergeometric series. P.Humbert has studied confluent hypergeometric series in two variables (cf. equations (7) to (13) *infra*). An exposition of the results of the French school together with references to the original literature are to be found in the monograph by Appell and Kampé de Fériet [1], which is the standart work on the subject until the middle of the last century. This work also contains an extensive bibliography of all relevant papers up to 1926. In 1953, a five-volume book on special functions appeared, the first book [2] of which, dedicated to hypergeometric functions, contains brief but very clearly written conclusions of the main properties of the functions under study, from which a person who does not know the theory can study it. But it also includes numerous lists of formulas relating to the most important special functions. Currently, the monograph by Srivastava and Karlsson [3], published in 1985, is highly respected among researchers.

Gauss' hypergeometric series (1) has been generalized [4] by the introduction of p parameters of the nature of a, b, and of q parameters of the nature of c. The ensuing series

$${}_pF_q\left[\begin{array}{l}a_1,...,a_p;\\ b_1,...,b_q;\end{array} z\right]:={}_pF_q\left(a_1,...,a_p;b_1,...,b_q;z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{z^n}{n!}, \tag{2}$$

is known as the generalized hypergeometric series of the order r, where $r=\max(p,\,q+1)$.

Just as the Gaussian series $F(a,b;c;z)$ was generalized to ${}_pF_q$ by increasing the numbers of the numerator and denominator parameters, the four Appell series were unified and generalized by Kampé de Fériet [5] who defined a general hypergeometric series in two variables [1, p. 150, equation (29)]. The notation introduced by Kampé de Fériet for his double hypergeometric series of superior order was subsequently abbreviated by Burchnall and Chaundy [6, p. 112]. We recall here (see Section 2 *infra*) the definition of a more general double hypergeometric series (than the one defined by Kampé de Fériet) in a slightly modified notation (see, for example, [7]). Although the double hypergeometric series defined in [7] reduces to the Kampé de Fériet series in the special case, yet it is usually referred to in the literature as the Kampé de Fériet series.

There are many works devoted to the Kampé de Fériet hypergeometric function, but here we note only some works in which the issues of convergence [8] and reducibility [9, 10] are studied, summation formulas [11] and transformations [12], integral representations [13] are obtained. .

The hypergeometric function in one variable has been sufficiently fully studied in all respects, therefore, for the study of the hypergeometric function of two variables, expansion formulas are very important, which allow us to represent the hypergeometric function of two variables in the form of an infinite sum of products of either two hypergeometric functions in one variable, or relatively well-studied functions in two variables , and this, in turn, facilitates the process of studying the properties of the functions in two variables under consideration.

In 1940–41, Burchnall and Chaundy [6,14] systematically presented a number of expansion and decomposition formulas for double hypergeometric functions (of second order only) in series of simpler hypergeometric functions. In 2006–07, Hasanov and Srivastava [15, 16] introduced operators generalizing the Burcnall-Chaundy operators and found expansion formulas for many triple hypergeometric functions, and they proved recurrent formulas when the dimension of hypergeometric function exceeds three. Hasanov, Turaev and Choi [17] defined the so-called H-operators that make it possible to find expansions for generalized hypergeometric functions of one variable. In addition, applications of these operators to the expansion of second-order hypergeometric functions in two and three variables are known [18, 19]. When the order of the hypergeometric function exceeds two, an applications of H-operators to some third- and fourth-order Kampé de Fériet functions can be found in the works [20, 21]. Namely, thanks to the Kampé de Fériet functions, solutions to boundary value problems for some degenerate partial differential equations can be written in explicit forms [22, 23].

This work is devoted to the application of one-dimensional and two-dimensional H-operators to hypergeometric Kampé de Fériet functions of superior order. To give an examples, some Kampé de Fériet functions are expanded in terms of the Appell and Humbert hypergeometric functions.

2 Hypergeometric functions in two variables and its generalizations

Hypergeometric Appell functions are usually defined as the sums of the following series [24]:

$$F_1\left(a, b, b'; c; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| < 1, \quad |y| < 1, \tag{3}$$

$$F_2\left(a, b, b'; c, c'; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| + |y| < 1, \tag{4}$$

$$F_3\left(a,a',b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<1,\ \ |y|<1, \tag{5}$$

$$F_4\left(a,b;c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},\quad \sqrt{x}+\sqrt{y}<1, \tag{6}$$

Seven confluent forms of the four Appell functions were defined by Humbert [25]:

$$\Phi_1\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<1, \tag{7}$$

$$\Phi_2\left(b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{8}$$

$$\Phi_3\left(b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{9}$$

$$\Psi_1\left(a,b;c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<1, \tag{10}$$

$$\Psi_2\left(a;c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{11}$$

$$\Xi_1\left(a,a',b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<1, \tag{12}$$

$$\Xi_2\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<1. \tag{13}$$

In equations (3) – (13) all parameters $a,a',b,b';c,c'$ and variables x,y take complex values and, as usual, the denominator parameters $c,c'\neq 0,-1,-2,...$ are neither zero nor a negative integer.

The Kampé de Fériet hypergeometric function of order (P,Q) is defined by [7](see also [3, p.27]:

$$F^{p:\,k;l}_{q:\,r;s}\left[\begin{array}{l}(a_p):\ (b_k);\ (c_l);\\ (\alpha_q):(\beta_r);(\gamma_s);\end{array}\ x,y\right]=\sum_{m,n=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_{m+n}\prod\limits_{j=1}^{k}(b_j)_m\prod\limits_{j=1}^{l}(c_j)_n}{\prod\limits_{j=1}^{q}(\alpha_j)_{m+n}\prod\limits_{j=1}^{r}(\beta_j)_m\prod\limits_{j=1}^{s}(\gamma_j)_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{14}$$

$$= \sum_{m,n=0}^{\infty} \frac{[a,p]_{m+n}\,[b,k]_m\,[c,l]_n}{[\alpha,p]_{m+n}\,[\beta,k]_m\,[\gamma,l]_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

where $p, q,\ k, l, r, s$ are nonnegative integers with $p+k+l \neq 0$ and $q+r+s \neq 0$; $P = \max(p+k,\ p+l)$ is an order by x, and $Q = \max(q+r+1,\ q+s+1)$ is an order by y; $(a_p) := (a_1,\ ...,a_p)$ is a vector of p components; for convergence,

1. $p+k < q+r+1,\quad p+l < q+s+1,\quad |x| < \infty,\quad |y| < \infty;$

2. $p+k = q+r+1,\quad p+l = q+s+1,$

$$\begin{cases} |x|^{\frac{1}{p-q}} + |y|^{\frac{1}{p-q}} < 1, & p > q, \\ \max\{|x|,\ |y|\} < 1, & p \leq q; \end{cases}$$

3. $p+k = q+r+1,\quad p+l < q+s+1,\quad p \leq q,\quad |x| < 1,\quad |y| < \infty;$

4. $p+k < q+r+1,\quad p+l = q+s+1,\quad p \leq q,\quad |x| < \infty,\quad |y| < 1.$

Hypergeometric functions of superior order are usually divided into two types. If $p+k = q+r+1 = p+l = q+s+1 = P = Q$, then the Kampé de Fériet hypergeometric function is called *a complete hypergeometric function*, otherwise, *a confluent hypergeometric function.*

3 Operator identities for the Kampé de Fériet functions of superior order

Burchnall and Chaundy [7, 14] expanded the Appell and Humbert hypergeometric functions into series in terms of simpler hypergeometric functions. Their method is based on a mutually inverse pair of symbolic operators

$$\nabla(h) = \frac{\Gamma(h)\,\Gamma(h+\delta+\sigma)}{\Gamma(h+\delta)\,\Gamma(h+\sigma)}, \Delta(h) = \frac{\Gamma(\delta+h)\,\Gamma(\sigma+h)}{\Gamma(h)\,\Gamma(\delta+\sigma+h)},$$

where $\Gamma(z)$ is a gamma function and

$$\delta \equiv x\frac{\partial}{\partial x}, \qquad \sigma \equiv y\frac{\partial}{\partial y}, \tag{15}$$

These symbolic forms are used to obtain a large number of expansions of Appell's functions in terms of each other, of Appell's functions in terms of products of ordinary hypergeometric functions, or vice versa. By these methods Burchnall and Chaundy obtained 15 pairs of expansions involving Appell's functions $F_1 - F_4$, defined in (3) – (6), and ordinary hypergeometric functions, и обычные гипергеометрические функции, as well as a significant number of expansions containing confluent hypergeometric functions Φ, Ψ and Ξ, defined by equations (7) – (13). Burchnall-Chaundy expansions have applications in applied problems. For example, it is precisely thanks to the expansion of the Appel function F_2 [14, equation (26)] solutions to several boundary value problems were written in explicit forms [26, 27].

However, the Burchnall-Chaundy method was limited to second order functions in two variables. In order to find expansion formulas for generalized hypergeometric functions (of superior order) defined in (2), Hasanov, Turaev and Choi [17] first introduced symbolic operators:

$$H_x(A,B)=\frac{\Gamma(B)\Gamma(A+\delta)}{\Gamma(A)\Gamma(B+\delta)}, \tag{16}$$

$$H_y(A,B)=\frac{\Gamma(B)\Gamma(A+\sigma)}{\Gamma(A)\Gamma(B+\sigma)}, \tag{17}$$

where δ и σ are defined in (15). Hereinafter, A and B can take complex values, and $A\neq B$ and $A,\ B\neq 0,\ -1,-2,....$. Currently, multidimensional analogues of the operators (16) and (17) are known: in the work [18] a symbolic operator of the form

$$H_{x,y}(A,B)=\frac{\Gamma(B)\Gamma(A+\delta+\sigma)}{\Gamma(A)\Gamma(B+\delta+\sigma)}, \tag{18}$$

are applied to the expansion of second order confluent hypergeometric functions in two variables included in Horn's list [2, p. 225] . In the work [28], using the one-dimensional operator (16), infinite summation formulas for second order hypergeometric Lauricella functions in three variables are proved. H-operators are applied to the transformation of second order hypergeometric functions in three and four variables [29].

Further study of the properties of symbolic H-operators showed that they can be used in the expansion of hypergeometric functions of several variables when their order exceeds 2.

The main result of this work is

Theorem 1 *The following operator identities are valid:*

$$F^{p:k+1;l}_{q:r+1;s}\left[\begin{matrix}(a_p):(b_k),A;\ (c_l);\\ (\alpha_q):(\beta_r),B;(\gamma_s);\end{matrix}\ x,y\right]=H_x(A,\ B)\,F^{p:k;l}_{q:r;s}\left[\begin{matrix}(a_p):(b_k);\ (c_l);\\ (\alpha_q):(\beta_r);(\gamma_s);\end{matrix}\ x,y\right], \tag{19}$$

$$F^{p:k;l+1}_{q:r;s+1}\left[\begin{matrix}(a_p):(b_k);\ (c_l),C;\\ (\alpha_q):(\beta_r);(\gamma_s),D;\end{matrix}\ x,y\right]=H_y(C,\ D)\,F^{p:k;l}_{q:r;s}\left[\begin{matrix}(a_p):(b_k);\ (c_l);\\ (\alpha_q):(\beta_r);(\gamma_s);\end{matrix}\ x,y\right], \tag{20}$$

$$F^{p:k+1;l+1}_{q:r+1;s+1}\left[\begin{matrix}(a_p):\ (b_k),A;\ (c_l),\ C;\\ (\alpha_q):\ (\beta_r),B;(\gamma_s),D;\end{matrix}\ x,y\right]$$

$$=H_x(A,\ B)\,H_y(C,\ D)\,F^{p:k;l}_{q:r;s}\left[\begin{matrix}(a_p):(b_k);\ (c_l);\\ (\alpha_q):(\beta_r);(\gamma_s);\end{matrix}\ x,y\right], \tag{21}$$

$$F^{p+1:k;l}_{q+1:r;s}\left[\begin{matrix}(a_p),\ A:(b_k);\ (c_l);\\ (\alpha_q),B:(\beta_r);(\gamma_s);\end{matrix}\ x,y\right]=H_{x,y}(A,\ B)F^{p:k;l}_{q:r;s}\left[\begin{matrix}(a_p):\ (b_k);\ (c_l);\\ (\alpha_q):(\beta_r);(\gamma_s);\end{matrix}\ x,y\right]. \tag{22}$$

Proof. Using the Burchnall-Chaundy method [7], we have

$$F^{p:k+1;l}_{q:r+1;s}\left[\begin{array}{l}(a_p):(b_k),\,A;\,(c_l);\\(\alpha_q):(\beta_r),B;(\gamma_s);\end{array}x,y\right]=\frac{\Gamma(A+\delta)\Gamma(B)}{\Gamma(A)\Gamma(B+\delta)}F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\,(c_l);\\(\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right],$$

$$F^{p:k;l+1}_{q:r;s+1}\left[\begin{array}{l}(a_p):(b_k);\,(c_l),C;\\(\alpha_q):(\beta_r);(\gamma_s),D;\end{array}x,y\right]=\frac{\Gamma(C+\sigma)\Gamma(D)}{\Gamma(C)\Gamma(D+\sigma)}F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\,(c_l);\\(\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right],$$

$$F^{p:k+1;l+1}_{q:r+1;s+1}\left[\begin{array}{l}(a_p):(b_k),A;\,(c_l),\,C;\\(\alpha_q):(\beta_r),B;(\gamma_s),D;\end{array}x,y\right]$$
$$=\frac{\Gamma(A+\delta)\Gamma(B)}{\Gamma(A)\Gamma(B+\delta)}\frac{\Gamma(C+\sigma)\Gamma(D)}{\Gamma(C)\Gamma(D+\sigma)}F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\,(c_l);\\(\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right],$$

$$F^{p+1:k;l}_{q+1:r;s}\left[\begin{array}{l}(a_p),\,A:(b_k);(c_l);\\(\alpha_q),B:(\beta_r);\quad\gamma_s;\end{array}x,y\right]=\frac{\Gamma(A+\delta+\sigma)\Gamma(B)}{\Gamma(A)\Gamma(B+\delta+\sigma)}F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\,(c_l);\\(\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right].$$

Next, using the definitions of H-operators, we obtain the required identities. Q.E.D.

Corollary 1 *If $k=l$ and $r=s$, then the Kampé de Fériet function is expressed by the generalized hypergeometric function of $x+y$:*

$$F^{p:1;1}_{q:1;1}\left[\begin{array}{l}(a_p):A;\;C;\\(\alpha_q):B;\;D;\end{array}x,y\right]=H_x(A,\;B)\,H_y(C,\;D)\,{}_pF_q\left[(a_p);(\alpha_q);x+y\right],\tag{23}$$

$$F^{p:1;0}_{q:1;0}\left[\begin{array}{l}(a_p):\;A;\;-;\\(\alpha_q):B;\;-;\end{array}x,y\right]=H_x(A,\;B)\,{}_pF_q\left[(a_p);(\alpha_q);x+y\right],\tag{24}$$

$$F^{p:0;1}_{q:0;1}\left[\begin{array}{l}(a_p):-;\;C;\\(\alpha_q):-;\;D;\end{array}x,y\right]=H_y(C,\;D)\,{}_pF_q\left[(a_p);(\alpha_q);x+y\right],\tag{25}$$

where ${}_pF_q$ is a generalized hypergeometric function, defined in (2).

Corollary 2 *If $p=q=1$, then the Kampé de Fériet function is expressed through the product of two generalized hypergeometric functions:*

$$F^{1:k;l}_{1:r;s}\left[\begin{array}{l}A:(b_k);\;(c_l);\\B:(\beta_r);(\gamma_s);\end{array}x,y\right]=H_{x,y}(A,\;B)\;{}_kF_r\left[(b_k);\;(\beta_r);x\right]\;\;{}_lF_s\left[(c_l);\;(\gamma_s);y\right],\tag{26}$$

where ${}_kF_r$ and ${}_lF_s$ are generalized hypergeometric functions, defined in (2).

Symbolic forms (19) – (26) are used to obtain a large number of expansions of Kampé de Fériet functions in terms of each order of lower order, of Kampé de Fériet functions in terms of products of two generalized hypergeometric functions:

$$F^{p:k+1;l}_{q:r+1;s}\left[\begin{array}{l}(a_p):(b_k),A;\,(c_l);\\(\alpha_q):(\beta_r),B;(\gamma_s);\end{array}x,y\right]=\sum_{m=0}^{\infty}(-1)^m\frac{(B-A)_m}{m!\,(B)_m}\times$$

$$\times\frac{[a,p]_m\,[b,k]_m}{[\alpha,q]_m\,[\beta,r]_m}x^m F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p)+m:(\,b_k)+m;\,(c_l);\\ (\alpha_q)+m:(\,\beta_r)+m;(\gamma_s);\end{array}\,x,y\right],\tag{27}$$

$$F^{p:k;l+1}_{q:r;s+1}\left[\begin{array}{l}(a_p):\,(b_k);\,(c_l),\,C;\\ (\alpha_q):(\beta_r);(\gamma_s),\,D;\end{array}\,x,y\right]=\sum_{n=0}^{\infty}(-1)^n\frac{(D-C)_n}{n!\,(D)_n}\times$$

$$\times\frac{[a,p]_n\,[c,l]_n}{[\alpha,q]_n\,[\gamma,s]_n}y^n F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p)+n:\,(b_k);\,(c_l)+n;\\ (\alpha_q)+n:(\beta_r);(\gamma_s)+n;\end{array}\,x,y\right],\tag{28}$$

$$F^{p:k+1;l+1}_{q:r+1;s+1}\left[\begin{array}{l}(a_p):(b_k),A;\,(c_l),\,C;\\ (\alpha_q):(\beta_r),B;(\gamma_s),D;\end{array}\,x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times\frac{[a,p]_{m+n}\,[b,k]_m\,[c,l]_n}{[\alpha,q]_{m+n}\,[\beta,r]_m\,[\gamma,s]_n}x^m y^n F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p)+m+n:(b_k)+m;\,(c_l)+n;\\ (\alpha_q)+m+n:(\beta_r)+m;(\gamma_s)+n;\end{array}\,x,y\right],\tag{29}$$

$$F^{p+1:k;l}_{q+1:r;s}\left[\begin{array}{l}(a_p),\,A:(b_k);\,(c_l);\\ (\alpha_q),B:(\beta_r);(\gamma_s);\end{array}\,x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{[a,p]_{m+n}\,[b,k]_m\,[c,l]_n}{[\alpha,q]_{m+n}\,[\beta,r]_m\,[\gamma,s]_n}x^m y^n F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p)+m+n:(b_k)+m;\,(c_l)+n;\\ (\alpha_q)+m+n:(\beta_r)+m;(\gamma_s)+n;\end{array}\,x,y\right],\tag{30}$$

$$F^{p:1;1}_{q:1;1}\left[\begin{array}{l}(a_p):\,A;\,C;\\ (\alpha_q):B;\,D;\end{array}\,x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times\frac{[a,p]_{m+n}}{[\alpha,q]_{m+n}}x^m y^n\,{}_pF_q\left[\begin{array}{l}(a_p)+m+n;\\ (\alpha_q)+m+n;\end{array}\,x+y\right],\tag{31}$$

$$F^{p:1;0}_{q:1;0}\left[\begin{array}{l}(a_p):A;\;\;-;\\ (\alpha_q):B;\;\;-;\end{array}\,x,y\right]=\sum_{m=0}^{\infty}(-1)^m\frac{(B-A)_m}{m!\,(B)_m}\frac{[a,p]_m}{[\alpha,q]_m}x^m\,{}_pF_q\left[\begin{array}{l}(a_p)+m;\\ (\alpha_q)+m;\end{array}\,x+y\right],\tag{32}$$

$$F^{p:0;1}_{q:0;1}\left[\begin{array}{l}(a_p):-;\,C;\\ (\alpha_q):-;\,D;\end{array}\,x,y\right]=\sum_{n=0}^{\infty}(-1)^n\frac{(D-C)_n}{n!\,(D)_n}\frac{[a,p]_n}{[\alpha,q]_n}y^n\,{}_pF_q\left[\begin{array}{l}(a_p)+n;\\ (\alpha_q)+n;\end{array}\,x+y\right],\tag{33}$$

$$F^{1:k;l}_{1:r;s}\left[\begin{array}{l}A:(b_k);\,(c_l);\\ B:(\beta_r);(\gamma_s);\end{array}\,x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{[b,k]_m\,[c,l]_n}{[\beta,r]_m\,[\gamma,s]_n}x^m y^n{}_kF_r\left[(b_k)+m;\ (\beta_r)+m;x\right]\ {}_lF_s\left[(c_l)+n;\ (\gamma_s)+n;y\right]. \tag{34}$$

Hereinafter,the notation $(\nu_\lambda)+\mu$ denotes the vector $(\nu_1+\mu,...,\nu_\lambda+\mu)$.

Let us show the validity of the established expansions (27) – (34) . To give an example, by Gauss' formula [2, p. 112, equation (46)]

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, Re(c-a-b)>0, c\neq 0,-1,-2,\dots, \tag{35}$$

we have symbolically for $H_x(A,\ B)$

$$H_x\left(A,B\right)=\sum_{m=0}^{\infty}\frac{(B-A)_m(-\delta)_m}{(B)_m m!}. \tag{36}$$

Similarly, we have

$$H_y\left(C,D\right)=\sum_{n=0}^{\infty}\frac{(D-C)_n(-\delta)_n}{(D)_n n!}. \tag{37}$$

By virtue of Poole's formula [30, p. 26]

$$(-\delta)_m f(x)=(-1)^m x^m\frac{d^m f(x)}{dx^m}, \tag{38}$$

we get

$$(-\delta)_m F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\ (c_l);\\ (\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right]$$

$$=(-1)^m\frac{[a,p]_m\,[b,k]_m}{[\alpha,q]_m\,[\beta,r]_m}x^m F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p)+m:(b_k)+m;\ (c_l);\\ (\alpha_q)+m:(\beta_r)+m;(\gamma_s);\end{array}x,y\right]$$

and therefore, by virtue of (36), the symbolic form (19) leads to the expansion (27).

Further, using the summation formula for the hypergeometric Appell function F_1 in the form [3, p.34, equation (7)]

$$F_1(a,b,b';c;1,1)=\frac{\Gamma(c)\Gamma(c-a-b-b')}{\Gamma(c-a)\Gamma(c-b-b')},\ \ Re(c-a-b-b')>0,\ \ c\neq 0,-1,-2,...,$$

we have

$$H_{x,y}\left(A,B\right)=\sum_{m,n=0}^{\infty}\frac{(B-A)_{m+n}(-\delta)_m(-\sigma)_n}{(B)_{m+n}m!n!}.$$

Using the Poole's formula (38) по x и y, from the symbolic form (22) we obtain an expansion (30). The symbolic form (37) is used when proving the expansions (28), (29), (31) и (33).

Note that when proving the expansions (31) – (33) we should keep in mind the well-known identity [31, p. 52]:

$$\sum_{m,n=0}^{\infty}f(m+n)\frac{x^m}{m!}\frac{y^n}{n!}=\sum_{k=0}^{\infty}f(k)\frac{(x+y)^k}{k!}.$$

4 Examples

Some Kampé de Fériet functions can be expanded into Appell and Humbert functions (of second order!):

$$F^{1:\,2;\,2}_{1:\,1;\,1}\left[\begin{array}{l} a : A, b; C, b'; \\ c : \;\; B \;; \;\; D \;; \end{array} x, y\right] = \sum_{m,\,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m \,(D-C)_n}{m!n!\,(B)_m\,(D)_n} \times$$

$$\times \frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_{m+n}} x^m y^n F_1\left(a+m+n, b+m, b'+n; c+m+n; x, y\right), \tag{39}$$

$$F^{1:\,2;\,2}_{0:\,2;\,2}\left[\begin{array}{l} \;a : A, b;\; C, b'; \\ - : B, c; D, c'; \end{array} x, y\right] = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m \,(D-C)_n}{m!n!\,(B)_m\,(D)_n} \times$$

$$\times \frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_m\,(c')_n} x^m y^n F_2\left(a+m+n, b+m, b'+n; c+m, c'+n; x, y\right), \tag{40}$$

$$F^{0:3;3}_{1:1;1}\left[\begin{array}{l} - : A, a, b;\; C, a', b'; \\ \;c : \quad\;\; B; \quad\;\; D \;; \end{array} x, y\right] = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m \,(D-C)_n}{m!n!\,(B)_m\,(D)_n} \times$$

$$\times \frac{(a)_m\,(a')_n\,(b)_m\,(b')_n}{(c)_{m+n}} x^m y^n F_3\left(a+m, a'+n, b+m, b'+n; c+m+n; x, y\right), \tag{41}$$

$$F^{2:1;1}_{0:2;2}\left[\begin{array}{l} a, b : \quad A; \quad\; C; \\ \quad - : B, c; D, c'; \end{array} x, y\right] = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m \,(D-C)_n}{m!n!\,(B)_m\,(D)_n} \times$$

$$\times \frac{(a)_{m+n}\,(b)_{m+n}}{(b)_m\,(b')_n} x^m y^n F_4\left(a+m+n, b+m+n; c+m, c'+n; x, y\right), \tag{42}$$

$$F^{1:\,2;\,1}_{1:\,1;\,1}\left[\begin{array}{l} a : A, b; C; \\ c : \;\; B\,; D; \end{array} x, y\right] = \sum_{m,\,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m \,(D-C)_n}{m!n!\,(B)_m\,(D)_n} \times$$

$$\times \frac{(a)_{m+n}\,(b)_m}{(c)_{m+n}} x^m y^n \Phi_1\left(a+m+n, b+m; c+m+n; x, y\right), \tag{43}$$

$$F^{0:2;2}_{1:1;1}\left[\begin{array}{l} - : A, b;\; C, b'; \\ \;c : \quad B; \quad\; D; \end{array} x, y\right] = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m \,(D-C)_n}{m!n!\,(B)_m\,(D)_n} \times$$

$$\times \frac{(b)_m\,(b')_n}{(c)_{m+n}} x^m y^n \Phi_2\left(b+m, b'+n; c+m+n; x, y\right), \tag{44}$$

$$F_{1:1;1}^{0:2;1}\left[\begin{array}{l} -:A,b;\ C; \\ c:\quad B\ ;\ D; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n\,(b)_m}{m!n!\,(B)_m\,(D)_n\,(c)_{m+n}}\times$$

$$\times x^m y^n \Phi_3\left(b+m;c+m+n;x,y\right), \tag{45}$$

$$F_{0:\,2;\,2}^{1:\,2;\,1}\left[\begin{array}{l} a:A,\ b;\quad C; \\ -:B,c;D,c'; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times\frac{(a)_{m+n}\,(b)_m}{(c)_m\,(c')_n}x^m y^n \Psi_1\left(a+m+n,b+m;c+m,c'+n;x,y\right), \tag{46}$$

$$F_{0:\,2;\,2}^{1:\,1;\,1}\left[\begin{array}{l} a:\quad A;\quad C; \\ -:B,c;D,c'; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times\frac{(a)_{m+n}}{(c)_m\,(c')_n}x^m y^n \Psi_2\left(a+m+n;c+m,c'+n;x,y\right), \tag{47}$$

$$F_{1:1;1}^{0:3;2}\left[\begin{array}{l} -:A,a,b;\ C,a'; \\ c:\qquad B;\qquad D; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times\frac{(a)_m\,(a')_n\,(b)_m}{(c)_{m+n}}x^m y^n \Xi_1\left(a+m,a'+n,b+m;c+m+n;x,y\right), \tag{48}$$

$$F_{1:1;1}^{0:3;1}\left[\begin{array}{l} -:A,a,b;\ C; \\ c:\qquad B;\ D; \end{array} x,y\right]=\sum_{m,\,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times\frac{(a)_m\,(b)_m}{(c)_{m+n}}x^m y^n \Xi_2\left(a+m,b+m;c+m+n;x,y\right), \tag{49}$$

$$F_{2:0;0}^{2:1;1}\left[\begin{array}{l} a,A:\ b;\ b'; \\ c,B:-;-; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_{m+n}}x^m y^n F_1\left(a+m+n,b+m,b'+n;c+m+n;x,y\right), \tag{50}$$

$$F_{1:1;1}^{2:1;1}\left[\begin{array}{l} a,A:b;b'; \\ \quad B:c;c'; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_m\,(c')_n}x^m y^n F_2\left(a+m+n,b+m,b'+n;c+m,c'+n;x,y\right), \tag{51}$$

$$F^{1:2;2}_{2:0;0}\left[\begin{array}{r} A: a,b;a',b'; \\ B,c: \quad -; \quad -; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_m\,(a')_n\,(b)_m\,(b')_n}{(c)_{m+n}}x^m y^n F_3\left(a+m,a'+n,b+m,b'+n;c+m+n;x,y\right), \tag{52}$$

$$F^{3:0;0}_{1:1;1}\left[\begin{array}{r} a,b,A: -;-; \\ B:\ c;\ c'; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}\,(b)_{m+n}}{(c)_m\,(c')_n}x^m y^n F_4\left(a+m+n,b+m+n;c+m,c'+n;x,y\right), \tag{53}$$

$$F^{2:1;0}_{2:0;0}\left[\begin{array}{r} a,A:\ b;-; \\ c,B: -;-; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}\,(b)_m}{(c)_{m+n}}x^m y^n \Phi_1\left(a+m+n,b+m;c+m+n;x,y\right), \tag{54}$$

$$F^{1:1;1}_{2:0;0}\left[\begin{array}{r} A:\ b;\ b'; \\ c,B: -;-; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(b)_m\,(b')_n}{(c)_{m+n}}x^m y^n \Phi_2\left(b+m,b'+n;c+m+n;x,y\right), \tag{55}$$

$$F^{1:1;0}_{2:0;0}\left[\begin{array}{r} A:\ b;-; \\ c,B: -;-; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}\,(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(b)_m}{(c)_{m+n}}x^m y^n \Phi_3\left(b+m;c+m+n;x,y\right), \tag{56}$$

$$F^{2:1;0}_{1:1;1}\left[\begin{array}{r} a,A: b;-; \\ B: c;\ c'; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}\,(b)_m}{(c)_m\,(c')_n}x^m y^n \Psi_1\left(a+m+n,b+m;c+m,c'+n;x,y\right), \tag{57}$$

$$F^{2:0;0}_{1:1;1}\left[\begin{array}{r} a,A: -;-; \\ B:\ c;\ c'; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}}{(c)_m\,(c')_n}x^m y^n \Psi_2\left(a+m+n;c+m,c'+n;x,y\right), \tag{58}$$

$$F^{1:2;1}_{2:0;0}\left[\begin{array}{r} A:a,b;a';\\ c,B:\quad -;-;\end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_m\,(a')_n\,(b)_m}{(c)_{m+n}}x^m y^n \Xi_1\left(a+m,a'+n,b+m;c+m+n;x,y\right), \tag{59}$$

$$F^{1:2;0}_{2:0;0}\left[\begin{array}{r} A:a,b;-;\\ c,B:\quad -;-;\end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_m\,(b)_m}{(c)_{m+n}}x^m y^n \Xi_2\left(a+m,b+m;c+m+n;x,y\right). \tag{60}$$

Note, these expansions (27) – (34) and (39) – (60) can be proved without symbolic methods by comparing coefficients of equal powers of x and y on both sides. Indeed, for example, consider the expansion (27), the right-hand side of which we denote by

$$K:=\sum_{j=0}^{\infty}(-1)^j\frac{(B-A)_j}{j!\,(B)_j}\frac{[a,p]_j\,[b,k]_j}{[\alpha,p]_j\,[\beta,k]_j}x^j F^{p:k;l}_{q:r;s}\left[\begin{array}{c} a_p+j:b_k+j;\,c_l;\\ \alpha_q+j:\beta_r+j;\gamma_s;\end{array} x,y\right]. \tag{61}$$

Using the definition of the Kampé de Fériet function (14), the infinite sum (61) we represent in the form

$$K:=\sum_{n=0}^{\infty}\frac{[c,l]_n}{n!\,[\gamma,s]_n}y^n\sum_{j,m=0}^{\infty}\frac{(-1)^j\,(B-A)_j}{j!\,(B)_j}\frac{[a,p]_{m+n+j}\,[b,k]_{m+j}}{(m-j)!\,[\alpha,p]_{m+n+j}\,[\beta,k]_{m+j}}x^{m+j}.$$

Considering the following easily verifiable equality for power series

$$\sum_{j,m=0}^{\infty}A(j,m)x^{j+m}=\sum_{m=0}^{\infty}\sum_{j=0}^{m}A(j,\,m-j)x^m$$

and the definition of the Gaussian hypergeometric function, we obtain

$$K:=\sum_{m=0}^{\infty}\frac{[a,p]_{m+n}\,[b,k]_m}{m!\,[\alpha,p]_{m+n}\,[\beta,k]_m}F\left(-m,B-A;B;1\right)x^m\sum_{n=0}^{\infty}\frac{[c,l]_n}{n!\,[\gamma,s]_n}y^n.$$

Hence, by virtue of the Gauss' formula (35) for the value of the hypergeometric function at unity, we have

$$K:=\sum_{m,\,n=0}^{\infty}\frac{[a,p]_{m+n}\,[b,k]_m\,(A)_m\,[c,l]_n}{m!n!\,[\alpha,p]_{m+n}\,[\beta,k]_m\,(B)_m\,[\gamma,s]_n}x^m y^n.$$

Q.E.D.

5 Applications

Using the properties (expansion formulas, transformations) of the Kampé de Fériet hypergeometric functions, the authors of the works [21] and [22] managed to write an explicit solutions to Cauchy problems for hyperbolic equations of the second kind with single line

$$y^m u_{xx} - u_{yy} - \lambda^2 y^m u = 0, \ \ -1 < m < 0, \ \ \lambda \in R, \ y > 0$$

and two lines

$$y^m u_{xx} - x^n u_{yy} = 0, \ \ -1 < m, n < 0, x > 0, \ \ y > 0$$

of the degeneration, respectively.

Applications of Kampé de Fériet functions and other hypergeometric functions can be found in more recent works [10–13] and [32].

6 Conclusion

In conclusion, we note that in 1940–41, Burchnall and Chaundy found expansion formulas for a certain class of hypergeometric functions of two variables of order 2. In this paper, expansion formulas were established for hypergeometric Kampé de Fériet functions, the order of which exceeds 2. As examples, some the Kampé de Fériet functions are expanded into the well-known hypergeometric functions of Appel and Humbert. Applications of Kampé de Fériet hypergeometric functions to finding explicit solutions of boundary value problems for degenerate partial differential equations are indicated.

For the future, it would be interesting to solve the problem of decomposing a triple hypergeometric functions $F^{(3)}[x, y, z]$ of superior order (for definition, see [3, p. 44, equation (14)]) into known lower order hypergeometric functions of one, two, and three variables.

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