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A.R. Ryskan[1,2*], **Z.O. Arzikulov**[3,4], **T.G. Ergashev**[4,5]
[1]Abai Kazakh National Pedagogical University, Almaty, Kazakhstan
[2]Narxoz University, Almaty, Kazakhstan
[3]Fergana Polytechnic Institute, Fergana, Uzbekistan
[4]TIIAME National Research University, Tashkent, Uzbekistan
[5]Ghent University, Ghent, Belgium
*e-mail: ryskan.a727@gmail.com

PARTICULAR SOLUTIONS OF MULTIDIMENSIONAL GENERALIZED EULER-POISSON-DARBOUX EQUATIONS OF ELLIPTIC OR HYPERBOLIC TYPE

The primary outcome of this study is the construction of partial solutions for a class of multidimensional partial differential equations with multiple singular coefficients of the second order. We consider the generalized multidimensional second-order Euler-Poisson-Darboux equation. Employing a well-known method, we reduce the generalized Euler-Poisson-Darboux equation to a second-order partial differential equation of the hypergeometric type. The solutions to this second order hypergeometric equation comprise 2^n functions that contain the first Lauricella hypergeometric function. The Lauricella function, also known as an n-dimensional series, incorporates three distinct parameters - the Pohhammer polynomials. To study the properties of these particular solutions, we require a decomposition formula expressing the first Lauricell function as the product of simpler hypergeometric functions with fewer variables. Through this study of particular solutions and the determination of singularity order at the origin, we establish the uniqueness of these solutions. Thus, having proved the peculiarity of particular solutions at the origin, it can be argued that the constructed particular solutions are fundamental solutions of the generalized multidimensional second-order Euler-Poisson-Darboux equation.

Key words: multidimensional generalized Euler-Poisson-Darboux equation, particular solutions, Lauricella's hypergeometric function, expansion formula, order of the singularity.

А.Р. Рысқан[1,2*], З.О. Арзикулов[3,4], Т.Г. Эргашев[4,5]
[1]Абай атындағы Қазақ ұлттық педагогикалық университеті, Алматы қ., Қазақстан
[2]Нархоз университеті, Алматы қ., Қазақстан
[3]Ферғана политехникалық институты, Ферғана қ., Өзбекстан
[4]ТИИАШМИ Ұлттық зерттеу университеті, Ташкент қ., Өзбекстан
[5]Гент университеті, Гент қ., Бельгия
*e-mail: ryskan.a727@gmail.com

Эллиптикалық немесе гиперболалық типтегі эйлер-пуассон-дарбу көпөлшемді жалпыланған теңдеулердің дербес шешімдері

Осы жұмыстың негізгі нәтижесі екінші ретті бірнеше сингулярлық коэффициенттері бар көп айнымалы дербес туындылы дифференциалдық теңдеулер класы үшін дербес шешімдерді құру болып табылады. Эйлер-Пуассон-Дарбудың екінші ретті жалпыланған көп өлшемді теңдеуі қарастырылады. Белгілі әдіс көмегімен Эйлер-Пуассон-Дарбу жалпыланған теңдеуі гипергеометриялық типтегі екінші ретті дербес туындылы дифференциалдық теңдеуге әкеледі. Екінші ретті гипергеометриялық теңдеудің шешімдері құрамында Лауричелланың алғашқы гипергеометриялық функциясы бар 2^n функциялар болып табылады.

Лауричелланың функциясы деп аталатын функция бұл үш түрлі параметрден тұратын, яғни Похгаммер көпмүшелерінен құрылған n-өлшемді қатар. Дербес шешімдердің қасиеттерін зерттеу үшін Лауричелланың бірінші функциясын аз айнымалылары бар бірнеше қарапайым гипергеометриялық функциялардың көбейтіндісі түрінде келтіретін ыдырау формуласы қажет. Дербес шешімдердің қасиеттері зерттеледі, осылайша координаттардың басындағы ерекшеліктің реті анықталады. Координаттардың басында дербес шешімдердің ерекшелігін дәлелдей отырып, құрылған дербес шешімдер Эйлер-Пуассон-Дарбудың жалпыланған екінші ретті көп өлшемді теңдеуінің іргелі шешімдері болып табылады деп айтуға болады.
Түйін сөздер: Эйлер-Пуассон-Дарбудың көпөлшемді жалпыланған теңдеуі, ерекше шешімдер, Лауричелла гипергеометриялық функциясы, жіктеу формуласы, ерекшелік реті.

А.Р. Рысқан[1,2*], З.О. Арзикулов[3,4], Т.Г. Эргашев[4,5]
[1]Казахский национальный педагогический университет имени Абая, г. Алматы, Казахстан
[2]Университет Нархоз, г. Алматы, Казахстан
[3]Ферганский политехнический институт, г. Фергана, Узбекистан
[4]Национальный исследовательский университет ТИИИМСХ, г. Ташкент, Узбекистан
[5]Гентский университет, г. Гент, Бельгия
[*]e-mail: ryskan.a727@gmail.com

Частные решения многомерных обобщенных уравнений эйлера-пуассона-дарбу эллиптического или гиперболического типа

Основным результатом настоящей работы является построение частных решений для класса многомерных уравнений в частных производных с несколькими сингулярными коэффициентами второго порядка. Рассматривается обобщенное многомерное уравнение второго порядка Эйлера-Пуассона-Дарбу. С помощью известного метода обобщенное уравнение Эйлера-Пуассона-Дарбу приводится к дифференциальному уравнению в частных производных второго порядка гипергеометрического типа. Решениями гипергеометрического уравнения второго порядка являются 2^n функций, которые содержат в себе первую гипергеометрическую функцию Лауричелла. Так называемая функция Лауричелла представляет собой n-мерный ряд, содержащий три разных параметра - многочлены Похгаммера. Для исследования свойств частных решений необходима формула разложения, которая выражала бы первую функцию Лауричелла в терминах произведения нескольких более простых гипергеометрических функций, содержащих меньшее количество переменных. Изучаются свойства частных решений, таким образом определяется порядок особенности в начале координат. Доказав особенность частных решений в начале координат, можно утверждать о том, что построенные частные решения являются фундаментальными решениями обобщенного многомерного уравнения второго порядка Эйлера-Пуассона-Дарбу.
Ключевые слова: многомерное обобщенное уравнение Эйлера-Пуассона-Дарбу, частные решения, гипергеометрическая функция Лауричелла, формула разложения, порядок особенности.

1 Introduction

It is known that particular solutions have an essential role in studying partial differential equations. In case of the singular elliptic equations, the role of particular solutions is played by fundamental solutions. Formulation and solving of many local and non-local boundary value problems are based on these solutions. The explicit form of particular solutions gives a possibility to study the considered equation in detail. For example, in the works of Barros-Neto and Gelfand [1–3] fundamental solutions for Tricomi operator in the plane were explicitly calculated.

Particular solutions of generalized Euler-Poisson-Darboux equation

$$u_{xx}+u_{yy}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y=u_{tt}+\frac{2\gamma}{t}u_t,\ x>0,\ y>0,t>0$$

were found in [4], where α, β and γ are constants $(0<2\alpha,\ 2\beta,\ 2\gamma<1)$.

It is well known [5] that all linearly-independent fundamental solutions of the singular elliptic equation

$$\sum_{j=1}^{m}\frac{\partial^2 u}{\partial x_j^2}+\sum_{j=1}^{n}\frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j}=0,\ m\geq 2,\ n\leq m \tag{1}$$

in the first hyperoctant $x_1>0,...,x_n>0$ are expressed explicitly by a hypergeometric function $F_A^{(n)}$ in n variables introduced by Lauricella [6]. Fundamental solutions of the equation (1) in its various special cases were constructed by many authors [7–10] and applied to the solution of boundary value problems for the equation (1) up to dimension $m\leq 4$ [11–13]. Further applications of fundamental solutions of the equation (1) can be found in the works [14–16].

In this paper, we construct a particular solutions of multidimensional generalized Euler-Poisson-Darboux equation

$$\sum_{j=1}^{k}\frac{\partial^2 u}{\partial x_j^2}+\sum_{j=1}^{k}\frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j}=\sum_{j=k+1}^{n}\frac{\partial^2 u}{\partial x_j^2}+\sum_{j=k+1}^{n}\frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j},k=\overline{1,n-1} \tag{2}$$

in the n-dimensional cone

$$\Omega=\left\{(x_1,...,x_n):x_1^2+...+x_k^2>x_{k+1}^2+...+x_n^2,\ k=\overline{1,n-1}\right\},$$

where α_j are constants $(0<2\alpha_j<1\ ,\ j=\overline{1,n})$. It turns out that particular solutions of the equation (2) are also expressed in terms of the Lauricella function $F_A^{(n)}$ with variables, however, which differ from the variables of the functions involved in the fundamental solutions of the singular elliptic equation (1).

In investigation of the particular solutions for the singular partial differential equations, we need expansions for hypergeometric functions of several variables in terms of simpler hypergeometric functions of (for example) the Gauss and Appell types.

The familiar operator method of Burcnall and Chaundy [17, 18] has been used by them rather extensively for finding decomposition formulas for hypergeometric functions of two variables in terms of the classical Gauss hypergeometric function of one variable.

Following the works [17, 18], Hasanov and Srivastava [19, 20] introduced operators generalizing the Burcnall-Chaundy operators and found expansion formulas for many triple hypergeometric functions, and they proved recurrent formulas when the dimension of hypergeometric function exceeds three. However, due to the recurrence, additional difficulties may arise in the applications of those decomposition formulas. For two Lauricella hypergeometric functions in n variables are proved new expansion formulas which are free from the recurrence [21]. The most recent properties of Lauricella's hypergeometric function F_A^n can be found in [22].

The plan of this paper is as follows. In Section 2 we briefly give some preliminary information, which will be used later: definitions of Pochhammer symbol, Gaussian and Lauricella hypergeometric functions; a system of PDE satisfied by Lauricella hypergeometric function $F_A^{(n)}$ and its linearly-independent solutions. In Section 3 the expansion formula for the Lauricella function and consequences from this formula are given.

In Section 4 we describe the method of finding particular solutions for the considered equation and in Section 5 we show what order of singularity the found solutions will have.

2 Preliminaries

Below we give definition of Pochhammer symbol and some formulas for Gauss hypergeometric function, definition of Lauricella hypergeometric function $F_A^{(n)}$ and system of partial differential equations which can be satisfied by the Lauricella function $F_A^{(n)}$.

First we define a Pochhammer symbol and Gaussian hypergeometric function.

A symbol $(\kappa)_\nu$ denotes the general Pochhammer symbol or the shified factorial, since $(1)_l = l!$ $(l \in N \cup \{0\};\ N := \{1, 2, 3, ...\})$, which is defined (for $\kappa, \nu \in C$), in terms of the familiar Gamma function, by

$$(\kappa)_\nu := \frac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)} = \begin{cases} 1 & (\nu = 0; \kappa \in C\backslash\{0\}) \\ \kappa(\kappa+1)...(\kappa+l-1) & (\nu = l \in N; \kappa \in C), \end{cases}$$

it being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the $\Gamma-$ quotient exists.

A function

$$F(a, b; c; x) \equiv F\left[\begin{matrix} a, b; \\ c; \end{matrix}\ x\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},\ |x| < 1 \tag{3}$$

is known as the Gauss hypergeometric function and an equality

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, c \neq 0, -1, -2, ..., \mathrm{Re}(c-a-b) > 0 \tag{4}$$

holds [23, p.73,(14)]. Moreover, the following Boltz formula [23, p.76,(22)]

$$F(a, b; c; x) = (1-x)^{-b} F\left(c-a, b; c; \frac{x}{x-1}\right) \tag{5}$$

is valid.

Lauricella hypergeometric function $F_A^{(n)}$ in $n \in N$ (real or complex) variables is defined as following [6]

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) \equiv F\left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix}\ \mathbf{x}\right] = \sum_{|\mathbf{k}|=0}^{\infty} (a)_{|\mathbf{k}|} \prod_{j=1}^{n} \frac{(b_j)_{k_j}}{(c_j)_{k_j}} \frac{x_j^{k_j}}{k_j!},\ \sum_{j=1}^{n} |x_j| < 1, \tag{6}$$

where

$$\mathbf{a} := (a_1, ..., a_n),\ \mathbf{b} := (b_1, ..., b_n),\ \mathbf{c} := (c_1, ..., c_n);$$

$$\mathbf{x} := (x_1, ..., x_n)\,;\ |\mathbf{k}| := k_1 + ... + k_n,\ \ k_1 \geq 0, ..., k_n \geq 0.$$

In definition (6), as usual, the denominator parameters c_1,...,c_n are neither zero nor a negative integer.

Lauricella function $\omega(\mathbf{x}) = F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x})$ satisfies the system of equations [24, p. 117]

$$x_j (1 - x_j) \frac{\partial^2 \omega}{\partial x_j^2} - x_j \sum_{\substack{i=1 \\ i \neq j}}^{n} x_i \frac{\partial^2 \omega}{\partial x_i \partial x_j} - b_j \sum_{\substack{i=1 \\ i \neq j}}^{n} x_i \frac{\partial u}{\partial x_i}$$

$$+ [c_j - (a + b_j + 1)\, x_j] \frac{\partial u}{\partial x_j} - a b_j \omega = 0,\ \ j = \overline{1, n} \tag{7}$$

and, in turn, this system has 2^n linearly independent solutions [24, p.118]

$$1 : \left\{ F_A^{(n)} \left[\begin{array}{l} a, b_1, ..., b_n; \\ c_1, ..., c_n; \end{array} \mathbf{x} \right], \right.$$

$$n : \left\{ \begin{array}{l} x_1^{1-c_1} F_A^{(n)} \left[\begin{array}{l} a + 1 - c_1, b_1 + 1 - c_1, b_2, ..., b_n; \\ 2 - c_1, c_2, ..., c_n; \end{array} \mathbf{x} \right], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n^{1-c_n} F_A^{(n)} \left[\begin{array}{l} a + 1 - c_n, b_1, ..., b_{n-1}, b_n + 1 - c_n; \\ c_1, ..., c_{n-1}, 2 - c_n; \end{array} \mathbf{x} \right], \end{array} \right.$$

$$\frac{n(n-1)}{2} : \left\{ \begin{array}{l} x_1^{1-c_1} x_2^{1-c_2} F_A^{(n)} \left[\begin{array}{l} a + 2 - c_1 - c_2, b_1 + 1 - c_1, b_2 + 1 - c_2, b_3, ..., b_n; \\ 2 - c_1, 2 - c_2, c_3, ..., c_n; \end{array} \mathbf{x} \right], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_1^{1-c_1} x_n^{1-c_n} F_A^{(n)} \left[\begin{array}{l} a + 2 - c_1 - c_n, b_1 + 1 - c_1, b_2, ..., b_{n-1}, b_n + 1 - c_n; \\ 2 - c_1, c_2, ..., c_{n-1}, 2 - c_n; \end{array} \mathbf{x} \right], \\ \\ x_2^{1-c_2} x_3^{1-c_3} F_A^{(n)} \left[\begin{array}{l} a + 2 - c_2 - c_3, b_1, b_2 + 1 - c_2, b_3 + 1 - c_3, b_4, ..., b_n; \\ c_1, 2 - c_2, 2 - c_3, c_4, ..., c_n; \end{array} \mathbf{x} \right], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n-1}^{1-c_{n-1}} x_n^{1-c_n} \times \\ \quad \times F_A^{(n)} \left[\begin{array}{l} a + 2 - c_{n-1} - c_n, b_1, ..., b_{n-2}, b_{n-1} + 1 - c_{n-1}, b_n + 1 - c_n; \\ c_1, ..., c_{n-2}, 2 - c_{n-1}, 2 - c_n; \end{array} \mathbf{x} \right], \end{array} \right.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$1 : \left\{ x_1^{1-c_1} ... x_n^{1-c_n} F_A^{(n)} \left[\begin{array}{l} a + n - c_1 - ... - c_n, b_1 + 1 - c_1, ..., b_n + 1 - c_n; \\ 2 - c_1, ..., 2 - c_n; \end{array} \mathbf{x} \right]. \right.$$

3 Methods and materials. Expansions of Lauricella function $F_A^{(n)}$

For a given multivariable function, it is useful to find a decomposition formula which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables.

Burchnall and Chaundy [17, 18] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. For example, the Appell function

$$F_2\left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m}{m!}\frac{y^n}{n!},\ |x|+|y|<1$$

has the expansion [17]

$$F_2\left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} x, y\right] = \sum_{i=0}^{\infty} \frac{(a)_r(b_1)_r(b_2)_r}{r!(c_1)_r(c_2)_r} x^r y^r F\left[\begin{matrix} a+r, b_1+r; \\ c_1+r; \end{matrix} x\right] F\left[\begin{matrix} a+r, b_2+r; \\ c_2+r; \end{matrix} y\right].$$

Following the works [17,18] Hasanov and Srivastava found a decomposition formulas for all four Lauricella functions of three variables [19] and proved a recurrence formulas at arbitrary $n \in N\backslash\{1\}$ [20]. However, due to the recurrence of Hasanov-Srivastava's decomposition formulas, additional difficulties may arise in the applications of this expansions. Further study of the properties of the Lauricella function $F_A^{(n)}$ showed that known decomposition formula can be reduced to a more convenient (nonrecurrence) form.

With any natural numbers $n \in N \setminus \{1\}$ the following expansion formula holds true [22]:

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_{i,j}!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}} x_k^{B(k,n)} F\left[\begin{matrix} a+A(k,n), b_k+B(k,n); \\ c_k+B(k,n); \end{matrix} x_k\right], \quad (8)$$

where

$$|\mathbf{m}_n| = \sum_{i=2}^{n}\sum_{j=i}^{n} m_{i,j},\ m_{i,j} \geq 0,\ ,2 \leq i \leq j \leq n;$$

$$M_{i,j}! = m_{2,2}!m_{2,3}! \cdots m_{i,j}! \cdots m_{n,n}!,\ 2 \leq i \leq j \leq n;$$

$$A(k,n) = \sum_{i=2}^{k+1}\sum_{j=i}^{n} m_{i,j},\ B(k,n) = \sum_{i=2}^{k} m_{i,k} + \sum_{i=k+1}^{n} m_{k+1,i}.$$

In case $n = 1$, the formula (8) is greatly simplified and coincides with the definition of the single hypergeometric function (3).

Using expansion (8) and Boltz formula (5), it is easy to derive an analogue of the Boltz formula for the Lauricella hypergeometric function in the form

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_{i,j}!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}} \left(\frac{x_k}{1-x_k}\right)^{B(k,n)} \times$$

$$\times \prod_{k=1}^{n} (1-x_k)^{-b_k} F\left[\begin{matrix} c_k - a + B(k,n) - A(k,n), b_k + B(k,n); \\ c_k + B(k,n); \end{matrix} \frac{x_k}{x_k - 1}\right]. \quad (9)$$

Let $a, b_1, \ldots, b_n$ are real numbers with $a \neq 0, -1, -2, ...$ and $a > |\mathbf{b}|$, where $|\mathbf{b}| := b_1 + ... + b_n$. Then $n = 1, 2, ...,$ the following summation formula holds true [21]

$$\sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)} (a-b_k)_{A(k,n)-B(k,n)}}{(a)_{A(k,n)}} = \frac{\Gamma(a-|\mathbf{b}|)}{\Gamma(a)} \prod_{k=1}^{n} \frac{\Gamma(a)}{\Gamma(a-b_k)}. \tag{10}$$

It is easy to see that formula (10) is a natural generalization of the well-known summation formula (4).

Let a, b_k, c_k be real numbers, where $c_k \neq 0, -1, -2, ...$ and $a > |\mathbf{b}| > 0$ and $c_k > b_k$. Then for $n = 1, 2, ...,$ the following limit correlation is true

$$\lim_{\varepsilon \to 0} \left\{ \varepsilon^{-b_1 - \ldots - b_n} F_A^{(n)} \left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{z_1(\varepsilon)}{\varepsilon}, ..., 1 - \frac{z_n(\varepsilon)}{\varepsilon} \right) \right\} = \frac{\Gamma(a-|\mathbf{b}|)}{\Gamma(a)} \prod_{k=1}^{n} \frac{|z_k(0)|^{-b_k} \Gamma(c_k)}{\Gamma(c_k - b_k)}. \tag{11}$$

where $z_k(\varepsilon)$ are arbitrary functions, and $z_k(0) \neq 0$.

Limit correlation (11) directly follows from decomposition (9) and summation formula (10).

4 Particular solutions

Consider equation (2) in Ω. Let $x := (x_1, ..., x_n)$ be any point and $x_0 := (x_{01}, ..., x_{0n})$ be any fixed point of Ω. We search for a solution of (2) as follows:

$$u_k(x, x_0) = P(r_k)\, \omega(\xi_k), \ \ k = \overline{1, n}, \tag{12}$$

where ω is unknown function and

$$P(r_k) = r_k^{-2\beta}, \ \ \beta = \frac{n-2}{2} + \sum_{j=1}^{n} \alpha_j; \tag{13}$$

$$r_k^2 = \sum_{i=1}^{n} \mathrm{sgn}(k-i)(x_i - x_{0i})^2, \ \ \mathrm{sgn}(z) := \begin{cases} 1, & \text{if } z \geq 0, \\ -1, & \text{if } z < 0; \end{cases} \tag{14}$$

$$\xi_k := (\xi_{k1}, \xi_{k2}, ..., \xi_{kn}), \ \ \xi_{kj} = \mathrm{sgn}(k-j) \frac{r_k^2 - r_{kj}^2}{r_k^2}, \tag{15}$$

$$r_{kj}^2 = \mathrm{sgn}(k-j)(x_j + x_{0j})^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \mathrm{sgn}(k-i)(x_i - x_{0i})^2, \ \ k, j = \overline{1, n}.$$

In what follows for brevity, we omit the index (letter) k in the notations u_k, r_k, ξ_{k1}, ..., ξ_{kn}.

Let us calculate the necessary derivatives of the desired solution

$$\frac{\partial u}{\partial x_j} = \omega \frac{\partial P}{\partial x_j} + P \sum_{i=1}^{n} \frac{\partial \omega}{\partial \xi_i} \frac{\partial \xi_i}{\partial x_j}, \ \ j = \overline{1, n},$$

$$\frac{\partial^2 u}{\partial x_j^2} = P\sum_{i=1}^{n}\sum_{l=1}^{n}\frac{\partial \xi_i}{\partial x_j}\frac{\partial \xi_l}{\partial x_j}\frac{\partial^2 \omega}{\partial \xi_i \partial \xi_l} + \sum_{i=1}^{n}\left(2\frac{\partial P}{\partial x_j}\frac{\partial \xi_i}{\partial x_j} + P\frac{\partial^2 \xi_i}{\partial x_j^2}\right)\frac{\partial \omega}{\partial \xi_i} + \omega\frac{\partial^2 P}{\partial x_j^2}\,,\quad j=\overline{1,n}$$

and substitute into equation (2):

$$\sum_{i=1}^{n} A_{ik}\frac{\partial^2 \omega}{\partial \xi_i^2} + \sum_{i=1}^{n}\sum_{l=i+1}^{n} B_{ilk}\frac{\partial^2 \omega}{\partial \xi_i \partial \xi_l} + \sum_{i=1}^{n} C_{ik}\frac{\partial \omega}{\partial \xi_i} + D_k\omega = 0, \tag{16}$$

where

$$A_{ik} = P\sum_{j=1}^{n}\operatorname{sgn}(k-j)\left(\frac{\partial \xi_i}{\partial x_j}\right)^2, \tag{17}$$

$$B_{ilk} = 2P\sum_{j=1}^{n}\operatorname{sgn}(k-j)\frac{\partial \xi_i}{\partial x_j}\frac{\partial \xi_l}{\partial x_j},$$

$$C_{ik} = \sum_{j=1}^{n}\operatorname{sgn}(k-j)\left(2\frac{\partial P}{\partial x_j}\frac{\partial \xi_i}{\partial x_j} + P\frac{\partial^2 \xi_i}{\partial x_j^2} + P\frac{2\alpha_j}{x_j}\frac{\partial \xi_i}{\partial x_j}\right), \tag{18}$$

$$D_k = \sum_{j=1}^{n}\operatorname{sgn}(k-j)\left(\frac{\partial^2 P}{\partial x_j^2} + \frac{2\alpha_j}{x_j}\frac{\partial P}{\partial x_j}\right).$$

Now we consider A_k. Since

$$\xi_{kj} = -\operatorname{sgn}(k-j)\frac{4x_j x_{0j}}{r^2}$$

and

$$\frac{\partial \xi_i}{\partial x_j} = -\operatorname{sgn}(k-i)\frac{4x_{0j}}{r^2}\delta_{ij} - \operatorname{sgn}(k-j)\frac{2\left(x_j - x_{j0}\right)}{r^2}\xi_i, \tag{19}$$

where

$$\delta_{ij} = \begin{cases} 1, & i=j,\\ 0, & i\neq j \end{cases}$$

is the Kronecker delta, we have

$$\left(\frac{\partial \xi_i}{\partial x_j}\right)^2 = \frac{16x_{i0}^2}{r^4}\delta_{ij} + \frac{16\left(x_j - x_{j0}\right)x_{i0}}{r^4}\delta_{ij}\xi_i + \frac{4\left(x_j - x_{j0}\right)^2}{r^4}\xi_i^2, \tag{20}$$

Substituting (20) into (17), we get

$$A_{ik} = -\frac{4Px_{i0}}{x_i r^2}\xi_i\left(1-\xi_i\right). \tag{21}$$

Using the product of the derivatives $\dfrac{\partial \xi_i}{\partial x_j}$ and $\dfrac{\partial \xi_l}{\partial x_j}$ in the form

$$\frac{\partial \xi_i}{\partial x_j}\frac{\partial \xi_l}{\partial x_j} = \left(-\frac{4\left(x_i - x_{i0}\right)}{x_i r^2} - \frac{4\left(i_l - x_{l0}\right)}{x_l r^2} + \frac{8}{r^2}\right)\xi_i\xi_l,\ l \geq i+1,$$

we have

$$B_{ilk} = P\left(\frac{4x_{i0}}{x_i r^2} + \frac{4x_{l0}}{x_l r^2}\right)\xi_i\xi_l. \tag{22}$$

Substituting the following derivatives

$$\frac{\partial P}{\partial x_j} = -2\beta \operatorname{sgn}(k-j) P \frac{x_j - x_{j0}}{r^2},$$

$$\frac{\partial^2 \xi_i}{\partial x_j^2} = -\operatorname{sgn}(k-j)\frac{4\left(x_j - x_{j0}\right)}{x_i r^2}\delta_{ij}\xi_i - \operatorname{sgn}(k-j)\frac{2\xi_i}{r^2} + \frac{8\left(x_j - x_{j0}\right)^2}{r^4}\xi_i$$

and (19) into (18), we get

$$C_{ik} = -P\left(-(\beta+1)\frac{4x_{i0}}{x_i r^2}\xi_i + 2\alpha_i \frac{4x_{i0}}{x_i r^2} - \xi_i \sum_{j=1}^{n} \alpha_j \frac{4x_{j0}}{x_j r^2}\right). \tag{23}$$

Taking into account first derivative (14) of P and its second derivative

$$\frac{\partial^2 P}{\partial x_j^2} = 2\beta \operatorname{sgn}(k-j) P \left[2\left(\beta+1\right)\frac{\left(x_j - x_{j0}\right)^2}{r^4} - \frac{1}{r^2}\right],$$

it is not difficult to find the expression

$$D_k = 4\beta P \sum_{j=1}^{n} \frac{\alpha_j x_{j0}}{x_j r^2}. \tag{24}$$

Now substituting (21)–(24) into equation (16), we obtain the following equation

$$\sum_{i=1}^{n} \frac{x_{i0}}{x_i}\left[\xi_i\left(1-\xi_i\right)\frac{\partial^2 \omega}{\partial \xi_i^2} + \left[2\alpha_i - (\beta+1)\xi_i\right]\frac{\partial \omega}{\partial \xi_i} - \alpha_i \beta \omega\right]$$
$$-\sum_{i=1}^{n}\sum_{l=i+1}^{n}\left(\frac{x_{i0}}{x_i} + \frac{x_{l0}}{x_l}\right)\xi_i\xi_l \frac{\partial^2 \omega}{\partial \xi_i \partial \xi_l} - \sum_{i=1}^{n} \xi_i \frac{\partial \omega}{\partial \xi_i}\sum_{j=1}^{n}\frac{\alpha_j x_{j0}}{x_j} = 0,$$

which is equivalent to the system:

$$\xi_j\left(1-\xi_j\right)\frac{\partial^2 \omega}{\partial \xi_j^2} - \xi_j \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i \frac{\partial^2 \omega}{\partial \xi_i \partial \xi_j} - \alpha_j \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i \frac{\partial \omega}{\partial \xi_i}$$

$$+\left[2\alpha_j-(\beta+\alpha_j+1)\xi_j\right]\frac{\partial\omega}{\partial\xi_j}-\alpha_j\beta\omega=0,\ j=\overline{1,n}. \tag{25}$$

Thus, the multidimensional generalized Euler-Poisson-Darboux equation (2) equivalently reduced to the system (25).

Comparing the system (25) with the system (7) and by virtue of (12), we obtain 2^n particular solutions of equation (2):

$$u_{k,1}\left(x;x_0\right)=\gamma_1 r_k^{-2\beta}F_A^{(n)}\left[\begin{matrix}\beta,\alpha_1,...,\alpha_n;\\2\alpha_1,...,2\alpha_n;\end{matrix}\ \xi_k\right], \tag{26}$$

$$u_{k,2}\left(x;x_0\right)=\gamma_2\frac{\left(x_1x_{01}\right)^{1-2\alpha_1}}{r_k^{2\beta+2-4\alpha_1}}F_A^{(n)}\left[\begin{matrix}\beta+1-2\alpha_1,1-\alpha_1,\alpha_2,...,\alpha_n;\\2-2\alpha_1,2\alpha_2,...,2\alpha_n;\end{matrix}\ \xi_k\right], \tag{27}$$

$$u_{k,3}\left(x;x_0\right)=\gamma_3\frac{\left(x_2x_{02}\right)^{1-2\alpha_2}}{r_k^{2\beta+2-4\alpha_2}}F_A^{(n)}\left[\begin{matrix}\beta+1-2\alpha_2,\alpha_1,1-\alpha_2,\alpha_3,...,\alpha_n;\\2\alpha_1,2-2\alpha_2,2\alpha_3,...,2\alpha_n;\end{matrix}\ \xi_k\right], \tag{28}$$

....

$$u_{k,n}\left(x;x_0\right)=\gamma_n\frac{\left(x_{n-1}x_{0n-1}\right)^{1-2\alpha_{n-1}}}{r_k^{2\beta+2-4\alpha_{n-1}}}F_A^{(n)}\left[\begin{matrix}\beta+1-2\alpha_{n-1},\alpha_1,...,\alpha_{n-2},1-\alpha_{n-1},\alpha_n;\\2\alpha_1,...,2\alpha_{n-2},2-2\alpha_{n-1},2\alpha_n;\end{matrix}\ \xi_k\right], \tag{29}$$

$$u_{k,n+1}\left(x;x_0\right)=\gamma_{n+1}\frac{\left(x_nx_{0n}\right)^{1-2\alpha_n}}{r_k^{2\beta+2-4\alpha_n}}F_A^{(n)}\left[\begin{matrix}\beta+1-2\alpha_n,\alpha_1,...,\alpha_{n-1},1-\alpha_n;\\2\alpha_1,...,2\alpha_{n-1},2-2\alpha_n;\end{matrix}\ \xi_k\right], \tag{30}$$

$$u_{k,n+2}\left(x;x_0\right)=\gamma_{n+2}\frac{\left(x_1x_{01}\right)^{1-2\alpha_1}\left(x_2x_{02}\right)^{1-2\alpha_2}}{r_k^{2\beta+4-4\alpha_1-4\alpha_2}}\times$$
$$\times F_A^{(n)}\left[\begin{matrix}\beta+2-2\alpha_1-2\alpha_2,1-\alpha_1,1-\alpha_2,\alpha_3,...,\alpha_n;\\2-2\alpha_1,2-2\alpha_2,2\alpha_3...,2\alpha_n;\end{matrix}\ \xi_k\right], \tag{31}$$

....

$$u_{k,(n^2+n+2)/2}\left(x;x_0\right)=\gamma_{(n^2+n+2)/2}\frac{\left(x_{n-1}x_{0n-1}\right)^{1-2\alpha_{n-1}}\left(x_nx_{0n}\right)^{1-2\alpha_n}}{r_k^{2\beta+4-4\alpha_{n-1}-4\alpha_n}}\times$$
$$\times F_A^{(n)}\left[\begin{matrix}\beta+2-2\alpha_{n-1}-2\alpha_n,\alpha_1,...,\alpha_{n-2},1-\alpha_{n-1},1-\alpha_n;\\2\alpha_1,...,2\alpha_{n-2},2-2\alpha_{n-1}...,2-2\alpha_n;\end{matrix}\ \xi_k\right], \tag{32}$$

....

$$u_{k,2^n}\left(x;x_0\right)=\gamma_{2^n}\frac{\prod_{j=1}^{n}\left(x_jx_{0j}\right)^{1-2\alpha_j}}{r_k^{2\beta+2n-4\alpha_1-...-4\alpha_n}}F_A^{(n)}\left[\begin{matrix}\beta+n-2\alpha_1-...-2\alpha_n,1-\alpha_1,...,1-\alpha_n;\\2-2\alpha_1,...,2-2\alpha_n;\end{matrix}\ \xi_k\right], \tag{33}$$

where β, r_k and ξ_k are defined in (13), (14) and (15), respectively; $\gamma_i\left(i=\overline{1,2^n}\right)$ are constants to be selected in a special way based on the applied problem under consideration.

5 Some properties of particular solutions

It can be directly shown that the particular solutions (26) – (33) satisfy the equation (2) with respect to the variables $(x_1, ..., x_n)$ and $(x_{01}, ..., x_{0n})$, however, these functions with respect to variables $(x_1, ..., x_n)$ do not satisfy the adjoint equation

$$\sum_{j=1}^{k}\frac{\partial^2 u}{\partial x_j^2}-\sum_{j=1}^{k}\frac{\partial u}{\partial x_j}\left(\frac{2\alpha_j u}{x_j}\right)=\sum_{j=k+1}^{n}\frac{\partial^2 u}{\partial x_j^2}-\sum_{j=k+1}^{n}\frac{\partial u}{\partial x_j}\left(\frac{2\alpha_j u}{x_j}\right),\ x_1>0,...,x_n>0. \quad (34)$$

In this connection, according to Sabitov's proposal [25], we multiply the functions (26) – (33) by $x^{(2\alpha)} := x_1^{2\alpha_1}...x_n^{2\alpha_n}$, since the product $x^{(2\alpha)}u_{k,j}(x;x_0)$ is a solution of equation (34) whenever $u_{k,j}(x;x_0)$ is a solution of the original equation (2). Thus, if $u_{k,j}(x;x_0)$ are functions defined in (26) – (33), then we obtain 2^n particular solutions

$$q_{k,j}(x;x_0)=x^{(2\alpha)}u_{k,j}(x;x_0),\ j=\overline{1,2^n}, \quad (35)$$

which are satisfy the equation (2) and adjoint equation (34) with respect to the variables $(x_{01}, ..., x_{0n})$ and $(x_1, ..., x_n)$, respectively.

Theorem 1. If $0 < 2\alpha_j < 1$ $(j=\overline{1,n})$, then particular solutions (26) – (33) have a singularity of the order $\dfrac{1}{r^{n-2}}$ at $r \to 0$.

Proof. We will consider the first particular solution, the singularity of the remaining solutions is proved in a similar way. By virtue of (26) and (35), we have

$$q_{k,1}(x;x_0)=\gamma_1 x^{(2\alpha)} r_k^{-2\beta} F_A^{(n)}\left[\begin{matrix}\beta,\alpha_1,...,\alpha_n;\\ 2\alpha_1,...,2\alpha_n;\end{matrix}\ \xi_k\right].$$

Taking into account the expression (13) for β, one can rewrite particular solution $q_{k,1}(x;x_0)$ in the form

$$q_{k,1}(x;x_0)=\frac{1}{r^{n-2}}\tilde{q}_{k,1}(x;x_0),$$

where

$$\tilde{q}_{k,1}(x;x_0)=\gamma_1\frac{x^{(2\alpha)}}{r_k^{2\alpha}}F_A^{(n)}\left[\begin{matrix}\beta,\alpha_1,...,\alpha_n;\\ 2\alpha_1,...,2\alpha_n;\end{matrix}\ \frac{4x_1x_{01}}{r_k^2},...,\frac{4x_kx_{0k}}{r_k^2},-\frac{4x_{k+1}x_{0k+1}}{r_k^2},...,-\frac{4x_nx_{0n}}{r_k^2}\right]. \quad (36)$$

Now we show that $\tilde{q}_{k,1}(x;x_0)$ is bounded at $r \to 0$. On the right side (36) we make a replacement $x_j - x_{0j} = \varepsilon t_j$ $(j=\overline{1,n})$, where $t := (t_1, ..., t_n)$ are new variables and $\varepsilon \geq 0$, then

$$\tilde{q}_{k,1}(x;x-\varepsilon t)=\gamma_1\frac{x^{(2\alpha)}\varepsilon^{-2\alpha}}{T_k^{2\alpha}}F_A^{(n)}\left[\begin{matrix}\beta,\alpha_1,...,\alpha_n;\\ 2\alpha_1,...,2\alpha_n;\end{matrix}\ 1-\frac{z_1(\varepsilon)}{\varepsilon^2},...,1-\frac{z_n(\varepsilon)}{\varepsilon^2}\right].$$

where

$$z_j(\varepsilon)=\frac{T_k^2\varepsilon^2+\mathrm{sgn}(k-j)\cdot 4x_j(x_j-\varepsilon t_j)}{T_k^2},$$

$$T_k^2 = \sum_{j=1}^{k} t_j^2 - \sum_{j=k+1}^{n} t_j^2,\ j = \overline{1,n}.$$

Using the limit correlation (11), we have

$$\lim_{\varepsilon\to 0} \tilde{q}_{k,1}\left(x; x-\varepsilon t\right) = \gamma_1 \frac{\Gamma(\beta-\alpha)}{4^{2\alpha}\Gamma(\beta)} \prod_{j=1}^{n} \frac{\Gamma\left(2\alpha_j\right)}{\Gamma\left(\alpha_j\right)} < \infty.$$

Thus the function $\tilde{q}_{k,1}\left(x; x_0\right)$ is bounded, hence the function $q_{k,1}\left(x; x_0\right)$ has the singularity of the order $\dfrac{1}{r^{n-2}}$ at $r \to 0$. Q.E.D.

6 Conclusion

This study focused on constructing particular solutions for the generalized multidimensional Euler-Poisson-Darboux equation. The method utilized for constructing particular solutions was derived from the renowned monograph by Appel and Campet de Ferrier. Each particular solution contains the first hypergeometric Lauricella function. Furthermore, the singularity of the obtained particular solutions at the origin was proved.

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