Probability theory

Repeated independent trials

The content of the presentation

Repeated independent trials

Repeated independent trials.

- If several trials are performed, and the probability of the event A in each trial does not depend on the outcomes of other trials, then such trials are called Repeated independent trials.

- In different independent trials, event A may have either different probabilities or the same probability. We will further consider only such independent trials in which the event A has the same probability.

Repeated independent trials.

Examples:

1. Throwing a die. Dropping a number of points from 1 to 6 occurs with a probability of 1/6 in each of the trials;
2. Acquire n lottery tickets. For each of the lottery tickets, the probability of winning is a constant value;
3. tossed up n times a coin. Heads or tails occur with a probability of ½ on each trial.

Example 1 and examples 2,3 differ from each other in that in the first example, 6 events may occur, and in the second and third, only 2 events may appear: won - did not win, heads - tails, i.e. conditionally we can call such outcomes "success - failure". Such tests are called **Bernoulli trials**.

Repeated independent trials.

Repeated independent trials, each of which may occurrence of event A (success) with constant probability p or non-occurrence of event A (failure) with a constant probability q=1-p, are called **Bernoulli trials** or **Bernoulli scheme.**

Swiss mathematician Jacob Bernoulli(1654-1705).

Repeated independent trials

Bernoulli formula

Bernoulli formula.

Let it be produced n Bernoulli trials. The probability that in these trials event A will occur exactly m times can be found by **Bernoulli's formula:**

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}$$

n-number of trials

P is the probability of occurrence of event A in one trial

q-probability of event A not occurring in one trial

$P_n(m)$ is the probability that event A will occur exactly m times in n trials

Bernoulli formula.

Example. The probability that the electricity consumption during the day will not exceed the established norm is 0.75. Find the probability that in the next week the consumption of electricity for four days will not exceed the norm.

Solution. Denote A - the flow rate will not exceed the norm.

By condition n = 7, m = 4, p = P(A) = 0.75.

According to the Bernoulli formula: $P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}$

$$P_7(4) = C_7^4 \cdot p^4 \cdot q^{7-4} = \frac{7!}{4! \cdot 3!} \cdot 0{,}75^4 \cdot 0{,}25^3 = 35 \cdot 0{,}316 \cdot 0{,}0156 \approx 0.172$$

Answer:the probability that in the next week the consumption of electricity within four days will not exceed the norm is 0.1969

Bernoulli formula

Example. Two equal chess players play chess. What is more likely: one of them will win 2 games out of 4 or 3 games out of 6?

Solution.

1) Find the probability of one of them winning 2 games out of 4:
n=4, m=2, p=1/2, q=1/2. According to the Bernoulli formula:

$$P_4(2) = C_4^2 \cdot p^2 \cdot q^{4-2} = \frac{4!}{2! \cdot 2!} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 = 6 \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{8}$$

2) Find the probability of one of them winning 3 games out of 6:
n=6, m=4, p=1/2, q=1/2. According to the Bernoulli formula:

$$P_6(3) = C_6^3 \cdot p^3 \cdot q^{6-3} = \frac{6!}{3! \cdot 3!} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^3 = 20 \cdot \frac{1}{8} \cdot \frac{1}{8} = \frac{5}{16}$$

Let's compare the obtained results: since 3/8 > 5/16, then one of them is more likely to win 2 out of 4 games.

Bernoulli formula

Example. A study of the incubation of eggs of the Belarus-9 egg cross showed that chickens are hatched on average from 70% of the eggs laid in the incubator. Of the total number of eggs laid in the incubator, 6 are randomly selected and marked. Find the probability that the marked eggs will hatch:

a) less than three chickens $P_6(m<3)$; (0.07047)

b) more than three chickens $P_6(m>3)$; (0.74431)

c) at least three chickens $P_6(m\geq 3)$; (0.92953)

d) no more than three chickens $P_6(m\leq 3)$; (0.25569)

Bernoulli formula

Example. Two light bulbs are connected in parallel in a circuit. The probability that with a certain increase in the voltage in the circuit above the nominal value, only one bulb will burn out is 0.18. find the probabilities of burning out for each of these bulbs, if it is known that these probabilities exceed 0.7 and are equal to each other.

Solution. The test consists in checking the operation of an electric light bulb. Total number of testsn= 2.

A - when the voltage rises, the bulb will not burn out.

By condition $P_2(1)=0,18$.

It is required to find the probability p of the occurrence of event A in each trial.

$$P_2(1) = C_2^1 \cdot p^1 \cdot q^{2-1} = 2 \cdot p \cdot (1-p) = 0.18 \Rightarrow p^2 - p + 0.09 = 0$$

This equation has two roots: p=0.9 and p=0.7. According to the condition p > 0.7. Therefore, p=0.7 does not satisfy the condition of the problem.

Answer: The probability that each of the bulbs will not burn out is p=0.9.

Repeated independent trials.

The most likely number of occurrences of the event.

The most likely number of occurrences of the event.

Example. The probability of manufacturing a standard part on an automatic machine is 0.8. Find the probabilities of the possible number of defective parts among 5 selected ones.

Solution. Probability of manufacturing a defective part
P \u003d 1 - 0.8 \u003d 0.2.
We find the desired probabilities using the Bernoulli formula:

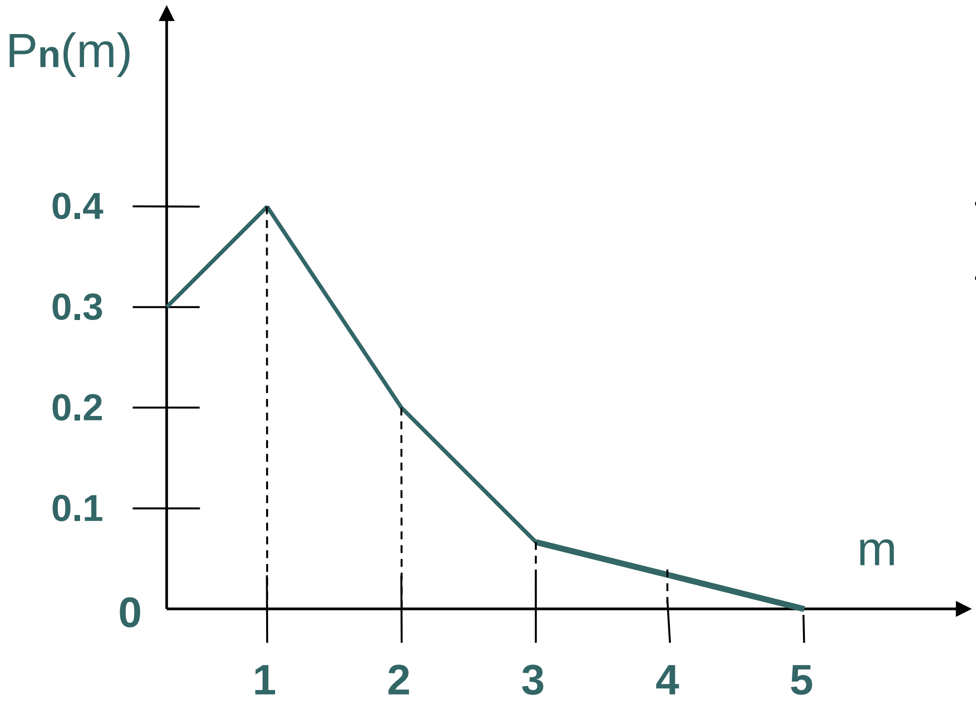
$P_5(0)=0.32768$; $P_5(3)=0.0512$;

$P_5(1)=0.4096$; $P_5(4)=0.0064$;

$P_5(2)=0.2048$; $P_5(5)=0.00032$.

The obtained probabilities will be represented graphically by points with coordinates $(m,P_n(m))$. Connecting these points, we get**a polygon, or polygon, of a probability distribution.**

The most likely number of occurrences of the event.



Considering the polygon of the probability distribution, we see that there are such valuesm(in this case, one m_0=1), which have the highest probability $P_n(m)$.

The most likely number of occurrences of the event.

Number m_0 occurrence of event A in n independent trials is called **most likely**, if the probability of the occurrence of this event $P_n(m_0)$ at least not less than the probabilities of other events $P_n(m)$ for any m.

For finding m_0 double inequality is used:

$$n \cdot p - q \leq m_0 \leq n \cdot p + p$$

The most likely number of occurrences of the event.

Since the most probable number can only be **whole**, That:

a) If the bounds are fractional, then m_0 can take only one value;

b) If the boundaries are integer (differ by 1), then m_0 can take two values equal to the boundary. Then, to determine the most probable number, you need to compare the probabilities at the boundaries.

The most likely number of occurrences of the event.

Example. As a result of long-term observations, the probability of rain on July 21 in the city N is 0.3. Find the most likely number of rainy days on July 21 for the next 30 years.

Solution. By condition: $p=0.3$, $q=0.7$, $n=30$.

$n \cdot p - q \leq m_0 \leq n \cdot p + p$

$0.3 \cdot 30 - 0.7 \leq m_0 \leq 0.3 \cdot 30 + 0.3$

$8.3 \leq m_0 \leq 9.3$

$m_0 = 9$

Answer: The most likely number of rainy days on July 21 for the next 30 years is 9.

most likely 9 times in 30 years July 21 will be rainy.

The most likely number of occurrences of the event.

Example. How many times does it take to roll a die for the most likely roll of a three of a kind to be 10?

Solution. By condition: $p=1/6$, $q=5/6$, $m_0=10$.

$n \cdot p - q \le m_0 \le n \cdot p + p$

$n \cdot 1/6 - 5/6 \le 10 \le n \cdot 1/6 + 1/6$ (multiply by 6)

$n - 560 \le n+1$ (we write in the form of two inequalities)

$$\begin{cases} n - 5 \le 60 \\ n+1 \ge 60 \end{cases} \Rightarrow \begin{cases} n \le 65 \\ n \ge 59 \end{cases}$$

Therefore, $59 \le n \le 65$.

Answer: For the most likely roll of a 3 to be 10, the die must be rolled 59, 60, 61, 62, 63, 64, or 65 times.

The most likely number of occurrences of the event.

Task 1. Warehouses of seed potatoes before planting are checked for the absence of foci of decay. In the checked warehouse there were 20% of tubers with spots. Find:

a) the most likely number of tubers without spots among 9 tubers selected at random; (m_0=7 And m_0=8)

b) the probability of the most probable number of tubers without spots. ($P_9(8) = P_9(7) \approx 0.3020$)

Task 2. The probability that event A will occur in eachnindependent tests is equal to 0.7. How many such trials must be performed so that the most probable number of occurrence of event A in these trials would be 20? (28or 29 trials)

Repeated independent trials..

- **Homework**

1. The probability of winning one lottery ticket is 1/7. What is the probability of winning with 7 tickets?
   a) on two tickets;
   b) three tickets?

2. In a certain field, 15% of seedling mint plants were damaged by herbicides. Find the most likely number of herbicide-damaged mint plants among 20 plants randomly selected from this field.

3. Sberbank clients are served by two branches. The first branch served 120 clients during the working day, the second - 140 clients. The probability that these customers took money from the accounts is 0.94 and 0.8, respectively. Find the most probable number of customers who have withdrawn money from their accounts. Which branch serves more customers?

Repeated independent trials..

Local Laplace theorem.

Local Laplace theorem.

Use the Bernoulli formula for large valuesndifficult enough, since the formula requires operations on huge numbers. For example, if

n= 50,m=30, p=0.1, then to find the probability$P_{30}(50)$you need to evaluate the expression

$$P_{50}(30) = C_{50}^{30} \cdot 0{,}1^{30} \cdot 0{,}9^{20}$$

Is it possible to calculate the probability of interest to us without resorting to the formula Bernoulli? It turns out you can. **Local Laplace theorem** and gives an asymptotic formula that allows you to approximately find the probability of an event occurring exactlymonce antrials if the number of trials is large enough.

Local Laplace theorem.

Laplace Pierre Simon

(03/23/1749 - 03/05/1827), Normandy

"What we know is so insignificant compared to what we don't know."

Local Laplace theorem.

Local Laplace theorem. If the probability p of the occurrence of event A in each trial is constant and different from zero and one, then the probability $P_n(m)$ that event A will appear in n tests exactly m times, approximately equal (the more accurate, the more n)

$$P_n(m) \approx \frac{\phi(x)}{\sqrt{n \cdot p \cdot q}},$$

Where

$$\phi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}},$$

$$x = \frac{m - n \cdot p}{\sqrt{n \cdot p \cdot q}}$$

Local Laplace theorem.

Comment. For a particular case, namely for p=1/2, the asymptotic formula was found in 1730 by Moivre.

In 1783, Laplace generalized Moivre's formula for arbitrary p other than 0 and 1. Therefore, the theorem in question here is sometimes called the Moivre-Laplace theorem.

Abraham de Moivre

(05/26/1667 - 11/27/1754), France.

According to legend, De Moivre accurately predicted the day of his own death. Finding that the duration of his sleep began to increase in arithmetic progression, he easily calculated when it would reach 24 hours, and, as always, he was not mistaken.

Local Laplace theorem.

To simplify calculations related using the formula

$$\phi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}},$$

drawn up function value table. $\phi(x)$

When using this table, keep in mind **function properties**: $\phi(x)$

1. The function $\phi(x)$ is even, i.e. . $\phi(-x) = \phi(x)$
2. The function $\phi(x)$ is monotonically decreasing for positive x values, and at $x \to \infty, \quad \phi(x) \to 0.$

(In practice, we can assume that already at x > 5 $\phi(x) \approx 0$).

The Moivre-Laplace theorem is used when n·p·q≥ 10.

Local Laplace theorem. Solution algorithm

1. We find n·p·q. If n·p·q≥ 10, then the Moivre-Laplace theorem can be applied.
2. We calculate x using the formula $$x = \frac{m - n \cdot p}{\sqrt{n \cdot p \cdot q}}$$
3. From the table we find $$\phi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}},$$
4. Calculate the probability $$P_n(m) \approx \frac{\phi(x)}{\sqrt{n \cdot p \cdot q}},$$

Local Laplace theorem.

Example. The probability of failure of the combination lock within a month is 2%. What is the probability that in a batch of 600 locks installed by the company, 20 locks will fail within a month.

Solution. By condition n=600, m=20, p=0.02, q=0.98 . $P_{600}(20)$ needs to be found. n·p·q=600·0.02·0.98=11.76, therefore, the local Laplace theorem can be applied.

1. ; $\sqrt{npq} = \sqrt{11.76} \approx 3.43$
2. ; $x = \dfrac{m - n \cdot p}{\sqrt{n \cdot p \cdot q}} = \dfrac{20 - 600 \cdot 0{,}02}{3{,}43} \approx 2{,}33$
3. $\phi(x) = \dfrac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}},$ ⇨ according to the table we find; $\phi(2{,}33) \approx 0{,}026$
4. . $P_{600}(20) = \dfrac{\phi(x)}{\sqrt{npq}} \approx \dfrac{0.026}{3.43} \approx 0.00758$

Local Laplace theorem.

Task1. Find the probability that event A occurs exactly 80 times in 400 trials if the probability of this event occurring in each trial is 0.2.

Task2. The probability of hitting the target by the shooter with one shot is $p = 0.75$. Find the probability that with 10 shots the shooter will hit the target 8 times.

Local Laplace theorem.

Example. In some area, out of every 100 families, 80 have refrigerators. Find the probability that out of 400 families 300 have refrigerators.
Solution. The probability that a family has a refrigerator is
R= 80/100 = 0.8;*n*= 400, *m*= 300,*q*=0.2.

1. *npq*=400· 0.8 · (1-0.8) = 64 >10, therefore, the local de Moivre-Laplace formula can be applied.
2. ; $\sqrt{npq} = \sqrt{64} = 8$
3. ; $x = \dfrac{m - n \cdot p}{\sqrt{n \cdot p \cdot q}} = \dfrac{300 - 400 \cdot 0{,}8}{8} \approx -2{,}5$
4. According to the table we find; $\phi(-2{,}5) = \phi(2{,}5) \approx 0{,}0175$
5. .

$$P_{400}(300) = \frac{\phi(x)}{\sqrt{npq}} \approx \frac{0{,}0175}{8} \approx 0.0022$$

Local Laplace theorem.

Let, in the conditions of the previous example, it is necessary to find the probability that from 300 to 360 families (inclusive) have refrigerators. In this case, according to the addition theorem, the probability of the desired event:

$$P_{400}(300 \le m \le 360) = P_{400}(300) + P_{400}(301) + \cdots + P_{400}(360)$$

In principle, each term can be calculated using the local Moivre–Laplace formula, but a large number of terms makes the calculation very cumbersome. In such cases, use**Laplace integral theorem**.

Independent retests.

Laplace integral theorem

Laplace integral theorem

Moivre-Laplace integral theorem. If the probability *p of* event A in each trial is constant and different from 0 and 1, then the probability that the number *m* occurrence of event A in *n* independent tests concluded ranging from *a to b* (inclusive), for a sufficiently large number *n* approximately equal to

$$P_n(a \le m \le b) \approx \Phi(x_2) - \Phi(x_1),$$

Where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_0^x e^{-t^2/2} \mathrm{dt},$$

$$x_1 = \frac{a - n \cdot p}{\sqrt{n \cdot p \cdot q}}, \quad x_2 = \frac{b - n \cdot p}{\sqrt{n \cdot p \cdot q}}$$

Laplace integral theorem

Function*F(x)*called**the Laplace function.**

Function Properties*F(x)*:

1. Function *F(x) is* odd, i.e. *F(-x) = - F(x).*
2. The function Φ(x) is monotonically increasing, and at $x \to \infty, \Phi(x) \to 0.5,$

(in practice, we can assume that already at *x*> 5 *F(x)* ≈ 0.5).

Laplace's integral theorem is used for n·p>10.

For the Laplace function, there are alsostatistical and mathematical tables.

Laplace integral theorem

Example. In some area, out of every 100 families, 80 have refrigerators. It is necessary to find the probability that out of 400 families from 300 to 360 families (inclusive) have refrigerators.

Solution. *R*= 80/100 = 0.8; *n*= 400, *q*=0.2, *a*= 300, *b*= 360.

1. *np*= 0.8·400 = 320>10, so we can apply Laplace's integral theorem.
2. $x_1 = \frac{a - n \cdot p}{\sqrt{n \cdot p \cdot q}} = \frac{300 - 400 \cdot 0.8}{\sqrt{400 \cdot 0.8 \cdot 0.2}} = \frac{-20}{8} = -2.5$ $\quad x_2 = \frac{b - n \cdot p}{\sqrt{n \cdot p \cdot q}} = \frac{360 - 400 \cdot 0.8}{\sqrt{400 \cdot 0.8 \cdot 0.2}} = \frac{40}{8} = 5$
3. *F*(-2.5)=-*F*(2.5)≈-0.4938, *F*(5)≈0.499997;
4. $\cdot P_{400}(300 \leq m \leq 360) \approx \Phi(x_2) - \Phi(x_1) = 0{,}499997 - (-{,}4938) = 0{,}993793$

Answer: the probability that between 300 and 360 families (inclusive) have refrigerators is 0.993793.

Laplace integral theorem

According to the repair shop, an average of 12% of kinescopes fail during the warranty period. What is the probability that out of 50 randomly selected kinescopes the warranty period will last:

a) 47 kinescopes;

6) at least 47 kinescopes;

c) less than 47 kinescopes;

d) more than 47 kinescopes;

e) no more than 47 kinescopes;

e) 50 kinescopes?

Laplace integral theorem

When crossing two varieties of lupine in the second generation, the expected ratio of alkaloid plants to non-alkaloid plants is the ratio of 9:7. Find the probability that among the obtained 150 hybrid plants

a) half of the plants will be alkaloid?

b) More than half of the plants will be alkaloid?

Laplace integral theorem

1. Find the probability that among 1000 newborn children of boys there will be:

a) at least half;

b) less than half.

Assume that the probability of having a boy is 0.51.

2. When harvesting potatoes, an average of 10% of tubers are damaged. Find the probability that in a random sample of 200 tubers 15 to 50 tubers are damaged.

Independent retests.

Poisson formula.

Poisson formula.

- If the number of independent trials n is large enough, and the probability of occurrence of an event in each trial is different from 0 and 1 and is small (p is close to 0), so $n \cdot p \leq 10$, then to calculate the probability of occurrence of an event k once applied **Poisson formula.**

Poisson Simeon

(06/21/1781 - 04/25/1840)

French scientist, member of the Paris Academy of Sciences, honorary member of the St. Petersburg Academy of Sciences. Poisson's works relate to theoretical and celestial mechanics, mathematics and mathematical physics.

Poisson formula.

Theorem. If the probability p of the occurrence of event A in each trial is constantly close to zero, the number of independent trials n is large enough, then the probability that event A occurs m times in n independent trials is approximately equal to

$$P_n(m) \approx \frac{\lambda^m}{m!} \cdot e^{-\lambda},$$

Where $\lambda = n \cdot p$

Poisson's formula can be applied to $\lambda \leq 10$.

There are statistical and mathematical tables for the Poisson distribution.

Poisson formula.

Example. There are 1825 students at the faculty. What is the probability that September 1 is the birthday of four faculty members at the same time?

Solution. The probability that a student's birthday is September 1 is p = 1/365. Since p = 1/365 is small, n= 1825 - great and

λ=np=1825 • (1/365) = 5 < 10, then we apply the Poisson formula:

$$P_{1825}(4) \approx \frac{\lambda^m}{m!} \cdot e^{-\lambda} = \frac{5^4}{4!} \cdot e^{-5} = \frac{625}{24 \cdot e^5} = \frac{625}{24 \cdot 2.7^5} \approx \frac{625}{3443.7377} \approx 0.18$$

According to the tables, you can more accurately and quickly find P (m,λ).
So for this example $P_{1825}(4) = P(m,\lambda) = P(4,5) \approx 0.17547$.

Answer: the probability that September 1 is the birthday of four faculty students at the same time is 0,17547.

Poisson formula.

Task 1. Some electronic device fails if a certain chip fails. The probability of its failure within 1 hour of operation of the device is 0.004. What is the probability that in 1000h the operation of the device will have to change the microcircuit five times? $(R_{1000}(5) \approx 0.1563)$

Task 2. The telephone exchange serves 2000 subscribers. For each subscriber, the probability of calling within an hour is 0.0025. Find the probability that the switch will call within an hour:

a) three subscribers; $(R_{2000}(3) \approx 0.1404)$

b) at least four subscribers.

Independent retests. Scheme

Most Likely Number

$$n \cdot p - q \le m_0 \le n \cdot p + p$$

Independent retests

n is small, p (or q) is not very small

Bernoulli formula

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}$$

$npq < 10$

n is great, p (or q) is not very small

Laplace formula

$$P_n(m) \approx \frac{\phi(x)}{\sqrt{n \cdot p \cdot q}}$$

$$x = \frac{m - n \cdot p}{\sqrt{n \cdot p \cdot q}}$$

$npq >= 10$

n is great, p (or q) is very small

Poisson formula

$$P_n(m) \approx \frac{\lambda^m}{m!} \cdot e^{-\lambda},$$

$$\lambda = n \cdot p$$

$np < 10$

Table forφ(x)
Table for F(x)
Poisson function table

Independent retests. Problem solving.

Task 3. Based on the results of inspections by tax inspectorates, it was found that, on average, every second small business in the region has a violation of financial discipline. Find the probability that out of 1000 small businesses registered in the region have violations of financial discipline:
a) 480 enterprises; b) the most probable number of enterprises;
c) not less than 480; d) from 480 to 520.

Task 4. Probability of a small business going bankrupt over timetequals 0.2. Find the probability that out of six small enterprises during time t: a) two will remain; b) more than two.

Task 5. 4000 packages of banknotes were sent to the bank. The probability that the package contains an insufficient or excess number of banknotes is equal to 0.0001. Find the probability that the check will reveal: a) three erroneously completed packets; b) no more than three packages.

Homework

1) Using an anti-aircraft gun, they fire at a target. The probability of hitting the target is 0.7. What is the probability that out of 80 shots fired at a staff exercise, the following will reach the target: a) 75 shots; b) at least 75 shots; c) less than 75 shots; d) no more than 75 shots; e) more than 75 shots; e) all shots?
2) It is known that the probability of producing a drill with increased brittleness (marriage) is 0.02. Drills are packed in boxes of 100 pieces. What is the minimum number of drills that must be put in a box so that, with a probability of at least 0.9, there are at least 1000 good ones in it?
3) How many raisins, on average, should high-calorie buns contain in order for the probability of having at least one raisin in a bun to be at least 0.99?

4. A coin has been tossed 5 times. Find the probability of occurring head:

a) less than two times: (3/16)

b) at least two times (13/16)

5. When an event A occurs at least four times, an event B will occur. If the probability of occurring of the event A is equal to 0.8 in each test. Find the probability of occurring of the event B. (0,74)

6. The probability of hitting the target when one shot is fired is equal to 0.8. Find the probability of the hitting target successfully exactly 75 times in 100 times. (0,01565)

7. A department of technical control are checking a party which consists of 10 items. The probability of being standard item is 0,75. Find the most likely number of occurrences of the items which is standard. (8)