# History of Probabolity

The concepts of chance and uncertainty are as old as civilization itself. People have always had to cope with uncertainty about the weather, their food supply, and other aspects of their environment, and have striven to reduce this uncertainty and its effects. Even the idea of gambling has a long history. By about the year 3500 b.c., games of chance played with bone objects that could be considered precursors of dice were apparently highly developed in Egypt and elsewhere. Cubical dice with markings virtually identical to those on modern dice have been found in Egyptian tombs dating from 2000 b.c. We know that gambling with dice has been popular ever since that time and played an important part in the early development of probability theory.

It is generally believed that the mathematical theory of probability was started by the French mathematicians Blaise Pascal (1623–1662) and Pierre Fermat (1601–1665) when they succeeded in deriving exact probabilities for certain gambling problems involving dice. Some of the problems that they solved had been outstanding for about 300 years. However, numerical probabilities of various dice combinations had been calculated previously by Girolamo Cardano (1501–1576) and Galileo Galilei (1564–1642).

The theory of probability has been developed steadily since the seventeenth century and has been widely applied in diverse fields of study. Today, probability theory is an important tool in most areas of engineering, science, and management. Many research workers are actively engaged in the discovery and establishment of new applications of probability in fields such as medicine, meteorology, photography from satellites, marketing, earthquake prediction, human behavior, the design of computer systems, finance, genetics, and law. In many legal proceedings involving antitrust violations or employment discrimination, both sides will present probability and statistical calculations to help support their cases.

In addition to the many formal applications of probability theory, the concept of probability enters our everyday life and conversation. We often hear and use such expressions as "It probably will rain tomorrow afternoon," "It is very likely that the plane will arrive late," or "The chances are good that he will be able to join us for dinner this evening." Each of these expressions is based on the concept of the probability, or the likelihood, that some specific event will occur. Despite the fact that the concept of probability is such a common and natural part of our experience, no single scientific interpretation of the term *probability* is accepted by all statisticians, philosophers, and other authorities. Through the years, each interpretation of probability that has been proposed by some authorities has been criticized by others. Indeed, the true meaning of probability is still a highly controversial subject and is involved in many current philosophical discussions pertaining to the foundations of statistics. Three different interpretations of probability will be described here. Each of these interpretations can be very useful in applying probability theory to practical problems.

- **Experiment and Event**. An *experiment* is any process, real or hypothetical, in which the possible outcomes can be identified ahead of time. An *event* is a well-defined set of possible outcomes of the experiment
- **Sample Space**. The collection of all possible outcomes of an experiment is called the *sample space* of the experiment.

The sample space of an experiment can be thought of as a *set*, or collection, of different possible outcomes; and each outcome can be thought of as a *point*, or an *element*, in the sample space. Similarly, events can be thought of as *subsets* of the sample space.

**Example: Rolling a Die**. When a six-sided die is rolled, the sample space can be regarded as containing the six numbers 1, 2, 3, 4, 5, 6, each representing a possible side of the die that shows after the roll. Symbolically, we write

$$S = \{1, 2, 3, 4, 5, 6\}.$$

One event $A$ is that an even number is obtained, and it can be represented as the subset $A = \{2, 4, 6\}$. The event $B$ that a number greater than 2 is obtained is defined by the subset $B = \{3, 4, 5, 6\}$.

- **Tossing a Coin.** Suppose that a coin is tossed three times. Then the sample space $S$ contains the following eight possible outcomes $s1, \dots, s8$:

$s1$: HHH, $s2$: THH, $s3$: HTH, $s4$: HHT, $s5$: HTT, $s6$: THT, $s7$: TTH, $s8$: TTT.

In this notation, H indicates a head and T indicates a tail. The outcome $s3$, for example, is the outcome in which a head is obtained on the first toss, a tail is obtained on the second toss, and a head is obtained on the third toss.

- To apply the concepts introduced in this section, we shall define four events as follows: Let $A$ be the event that at least one head is obtained in the three tosses; let $B$ be the event that a head is obtained on the second toss; let $C$ be the event that a tail is obtained on the third toss; and let $D$ be the event that *no* heads are obtained. Accordingly,
- $A = \{s1, s2, s3, s4, s5, s6, s7\}$, $B = \{s1, s2, s4, s6\}$, $C = \{s4, s5, s6, s8\}$, $D = \{s8\}$.

# Classical definition of probability

Let $\mathscr{E}$ be an experiment and let $\mathscr{A}$ be the collection of all possible outcomes of $\mathscr{E}$. Let $S$ be the class of all subsets of $\mathscr{A}$. $S$ is called the *sample space* associated with $\mathscr{E}$. An *event* is any subset of $S$. A *probability* is a set function $P$ defined on $S$ having the following properties:

(1) $P\{A\} \geqslant 0$ for all $A \in S$.

(2) If $E$ is the set containing all possible outcomes then $P\{E\} = 1$.

(3) If $A = \bigcup_{i=1}^{\infty} A_i$, where $A_i \cap A_j = \emptyset \; (i \neq j)$, then $P\{A\} = \sum_{i=1}^{\infty} P\{A_i\}$.

In many combinatorial problems it is very convenient to use the *classical definition of probability*. Suppose that as the result of a trial only one of $n$ pairwise incompatible and equally probable results $E_i$ $(i = 1, 2, \ldots, n)$ can be realized. We shall assume that the event $A$ consists of $m$ elementary results $E_k$. Then, according to the classical definition of probability,

$$P\{A\} = \frac{m}{n}.$$

# The Classical Interpretation of Probability

- The classical interpretation of probability is based on the concept of *equally likely outcomes*. For example, when a coin is tossed, there are two possible outcomes: a head or a tail. If it may be assumed that these outcomes are equally likely to occur, then they must have the same probability. Since the sum of the probabilities must be 1, both the probability of a head and the probability of a tail must be 1/2. More generally, if the outcome of some process must be one of $n$ different outcomes, and if these $n$ outcomes are equally likely to occur, then the probability of each outcome is $1/n$.

- Probability will be most useful when applied to a real experiment in which the outcome is not known in advance, but there are many hypothetical experiments that provide useful tools for modeling real experiments. A common type of hypothetical experiment is repeating a well-defined task infinitely often under similar conditions. Some examples of experiments and specific events are given next. In each example, the words following “the probability that” describe the event of interest.
- 1. In an experiment in which a coin is to be tossed 10 times, the experimenter might want to determine the probability that at least four heads will be obtained.
- 2. In an experiment in which a sample of 1000 transistors is to be selected from a large shipment of similar items and each selected item is to be inspected, a person might want to determine the probability that not more than one of the selected transistors will be defective.
- 3. In an experiment in which the air temperature at a certain location is to be observed every day at noon for 90 successive days, a person might want to determine the probability that the average temperature during this period will be less than some specified value.
- 4. From information relating to the life of Thomas Jefferson, a person might want to determine the probability that Jefferson was born in the year 1741.
- 5. In evaluating an industrial research and development project at a certain time, a person might want to determine the probability that the project will result in the successful development of a new product within a specified number of months.

**Tossing a Coin.** Suppose that a fair coin is to be tossed 10 times, and it is desired to determine (a) the probability $p$ of obtaining exactly three heads and (b) the probability $p'$ of obtaining three or fewer heads.

(a) The total possible number of different sequences of 10 heads and tails is $2^{10}$, and it may be assumed that each of these sequences is equally probable. The number of these sequences that contain exactly three heads will be equal to the number of different arrangements that can be formed with three heads and seven tails. Here are some of those arrangements:

HHHTTTTTTT, HHTHTTTTTT, HHTTHTTTTT, TTHTHTHTTT, etc.

Each such arrangement is equivalent to a choice of where to put the 3 heads among the 10 tosses, so there are $\binom{10}{3}$ such arrangements. The probability of obtaining exactly three heads is then

$$p = \frac{\binom{10}{3}}{2^{10}} = 0.1172.$$

(b) Using the same reasoning as in part (a), the number of sequences in the sample space that contain exactly $k$ heads ($k = 0, 1, 2, 3$) is $\binom{10}{k}$. Hence, the probability of obtaining three or fewer heads is

$$p' = \frac{\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3}}{2^{10}}$$

$$= \frac{1 + 10 + 45 + 120}{2^{10}} = \frac{176}{2^{10}} = 0.1719. \quad ◀$$

**Playing Cards.** Suppose that a deck of 52 cards containing four aces is shuffled thoroughly and the cards are then distributed among four players so that each player receives 13 cards. We shall determine the probability that each player will receive one ace.

The number of possible different combinations of the four positions in the deck occupied by the four aces is $\binom{52}{4}$, and it may be assumed that each of these $\binom{52}{4}$ combinations is equally probable. If each player is to receive one ace, then there must be exactly one ace among the 13 cards that the first player will receive and one ace among each of the remaining three groups of 13 cards that the other three players will receive. In other words, there are 13 possible positions for the ace that the first player is to receive, 13 other possible positions for the ace that the second player is to receive, and so on. Therefore, among the $\binom{52}{4}$ possible combinations of the positions for the four aces, exactly $13^4$ of these combinations will lead to the desired result. Hence, the probability $p$ that each player will receive one ace is

$$p = \frac{13^4}{\binom{52}{4}} = 0.1055.$$

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**29.** A child plays with 10 letters of the alphabet: А, А, А, Е, И, К, М, М, Т, Т. What is the probability that with a random arrangement of the letters in a row he will obtain the word "МАТЕМАТИКА"?

**30.** In the elevator of an 8-story building, 5 persons entered on the first floor. Assume that each of them can, with equal probability, leave on any of the floors, starting with the second. Find the probability that all five will leave on different floors.

**33.** A deck of playing cards contains 52 cards, divided into 4 different suits with 13 cards in each suit. Assume that the deck is carefully shuffled so that all permutations are equiprobable. Draw 6 cards. Describe the space of elementary events.

a) Find the probability that among these cards there will be a king of diamonds.

b) Find the probability that among these cards there will be representatives of all suits.

c) What is the smallest number of cards one must take from the deck so that the probability that among them one encounters at least two cards of the same face value will be greater than $\frac{1}{2}$?

**34.** $n$ friends sit down at random at a round table. Find the probability that:

a) two fixed persons $A$ and $B$ sit together with $B$ to the left of $A$;

b) three fixed persons $A$, $B$ and $C$ sit together with $A$ to the right of $B$ and $C$ to the left of $B$;

c) find these same probabilities in the case when the friends sit in a row on one side of a rectangular table.

**35.** Two numbers are chosen at random from the sequence of numbers $1, 2, \ldots, n$. What is the probability that one of them is less than $k$ and the other is greater than $k$, where $1 < k < n$ is an arbitrary integer?

- **Rolling to dice.** We shall now consider an experiment in which two balanced dice are rolled, and we shall calculate the probability of each of the possible values of the sum of the two numbers that may appear. $P_i$ denote the probability that the sum of the two numbers is *i* for *i* =2, 3, . . . , 12.
- A) Find sample space S for rolling two dice
- B) Find probability of $P_2, P_3, P_4, P_5, P_6, P_7$
- **1.** If two balanced dice are rolled, what is the probability hat the sum of the two numbers that appear will be odd?
- **2.** If two balanced dice are rolled, what is the probability that the sum of the two numbers that appear will be even?
- **3.** If two balanced dice are rolled, what is the probability that the difference between the two numbers that appear will be less than 3?

**5.** If four dice are rolled, what is the probability that each of the four numbers that appear will be different?

**6.** If six dice are rolled, what is the probability that each of the six different numbers will appear exactly once?

**7.** If 12 balls are thrown at random into 20 boxes, what is the probability that no box will receive more than one ball?

**8.** An elevator in a building starts with five passengers and stops at seven floors. If every passenger is equally likely to get off at each floor and all the passengers leave independently of each other, what is the probability that no two passengers will get off at the same floor?