Mavzu: Aniq integralning tadbiqlari.

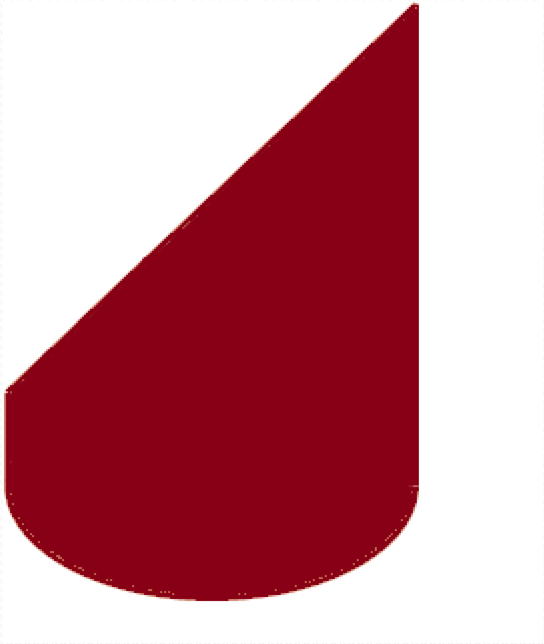
Reja:

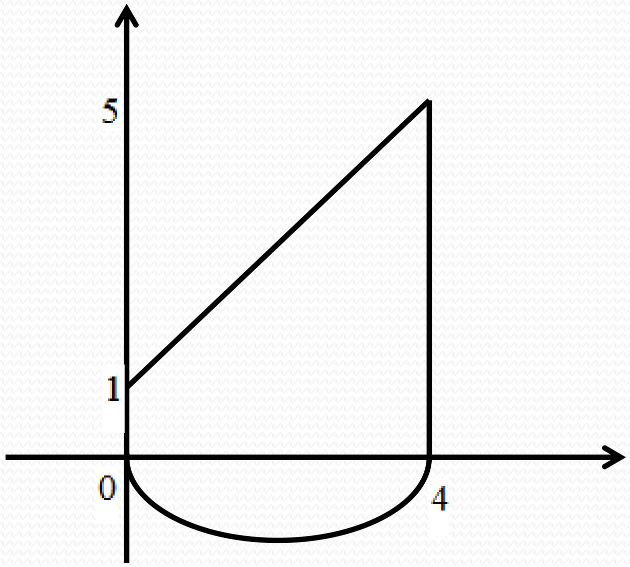
- Aniq integral yordamida tekis figuralarning yuzalarini hisoblash
- Aniq integral yordamida yoy uzunligini hisoblash
- Aniq integral yordamida aylanma jismlarning hajmini hisoblash


Aniq integralning tadbiqlari
Yuza hisoblash
Funksiya oshkor ko'rinishida berilgan yuzani hisoblash
Funksiya parametrik ko'rinishda berilganda yuza hisoblash
Qutb koordinatalar sistemasida egri chiziqli sektorning yuzasini hisoblash
Yoy uzunligini hisoblash
To'g'ri chiziqli kordinatalar sistemasida egri chiziqli yoy uzunligini hisoblash
Egri chiziq tenglamasi parametrli ko'rinishda berilganda yoy uzunligini hisoblash
Qutb koordinatalarda egri chiziq yoyining uzunligini hisoblash
Hajm hisoblash
Aylanma jism hajmini hisoblash
Aylanma jism hajmini parallel kesimlar yuzalari bo'yicha hisoblash

KEYS-1

Xonadon hovlisi maydoni (tuzilishi rasmda keltirilgan) uchun soliq to’lashi lozim. Soliq miqdorini aniqlash uchun hovli maydonining yuzasini bilish kerak. Hovli maydonining yuzasi topilsin.



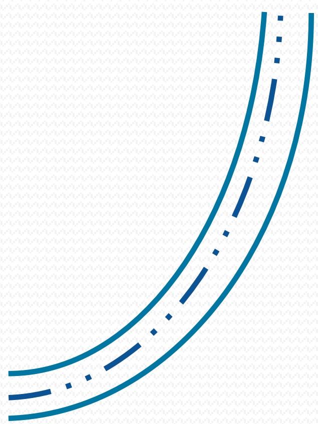


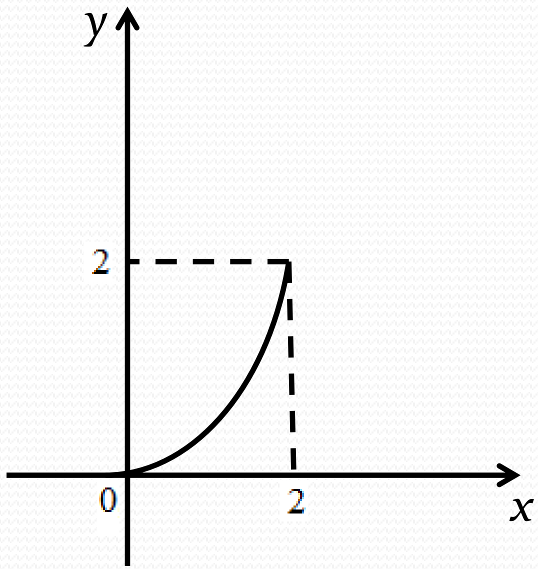
KEYS-2

Uzunligi 4 metr va diametri 2 metr bo’lgan sisternali suv tashiydigan mashina yordamida, o’lchovlari 6;8;3,5 metrga teng bo’lgan hovuzni yuqori chegarasiga 0,5 metr qolgunicha to’ldirishi uchun necha marta suv tashib keltirilishi kerak.

KEYS-3

Akbar akaga, uning fermer xo’jaligiga qarashli yerdan o’tgan zovur kanalini tozalash vazifasi qo’yildi. Akbar aka ish hajmini hisoblab chiqishi uchun zovur kanalining uzunligini bilishi kerak. Zovur kanalining uzunligini toping.





Tekis figuralar yuzalarini hisoblash.

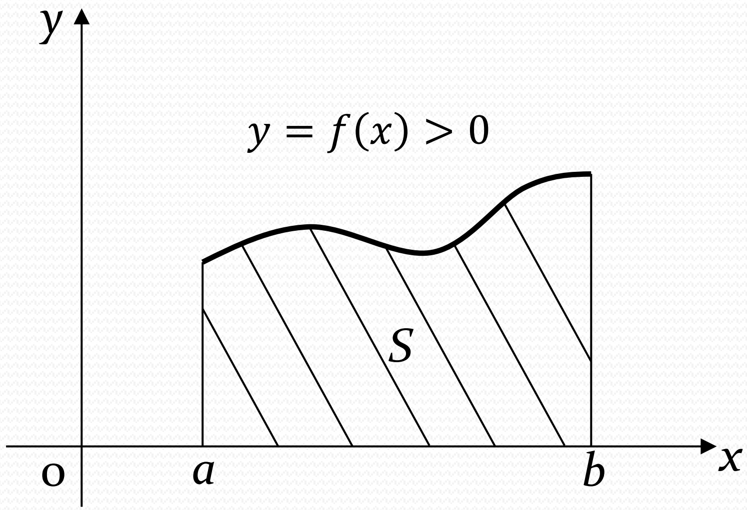
Aniq integral mavzusida ko'rildiki, agar $[a;b]$ kesmada $f(x) \geq 0$ bo'lsa, u holda $y = f(x)$ egri chiziq, *Ox* o'qi va *x*=*a* hamda *x*=*b* to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzi $S = \int_a^b f(x)dx$ ga teng bo'ladi. Agar $[a;b]$ kesmada $f(x) \leq 0$ bo'lsa, u holda aniq integral $\int_a^b f(x)dx \leq 0$ bo'ladi.

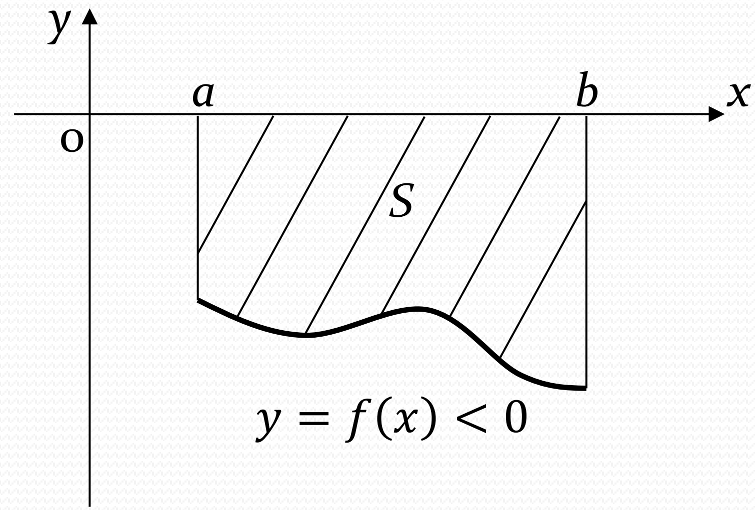
Absolyut kattaligiga ko'ra u tegishli trapetsiyaning yuziga teng:

$$\mathrm{S} = \left|\int_a^b f(x)dx\right|.$$

Agar *f(x)* funksiya $[a;b]$ kesmada o'z ishorasini chekli son marta almashtirsa, u holda butun kesma bo'yicha olingan

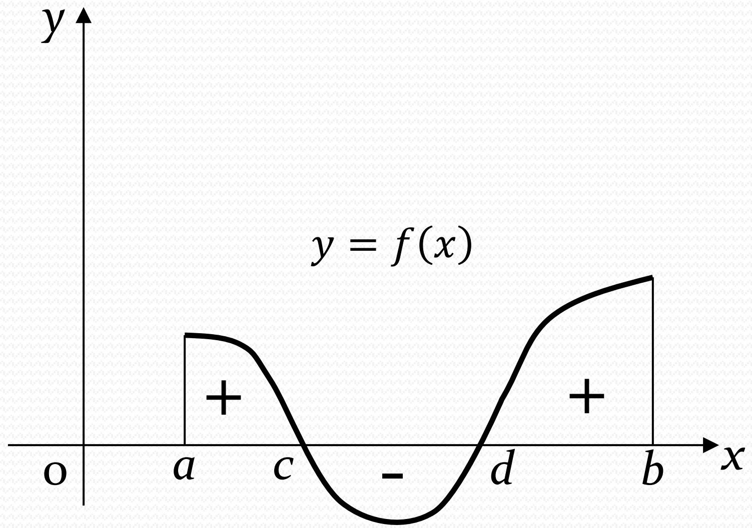
integralni xususiy kesmalar bo'yicha olingan integrallar yig'indisiga ajratamiz.

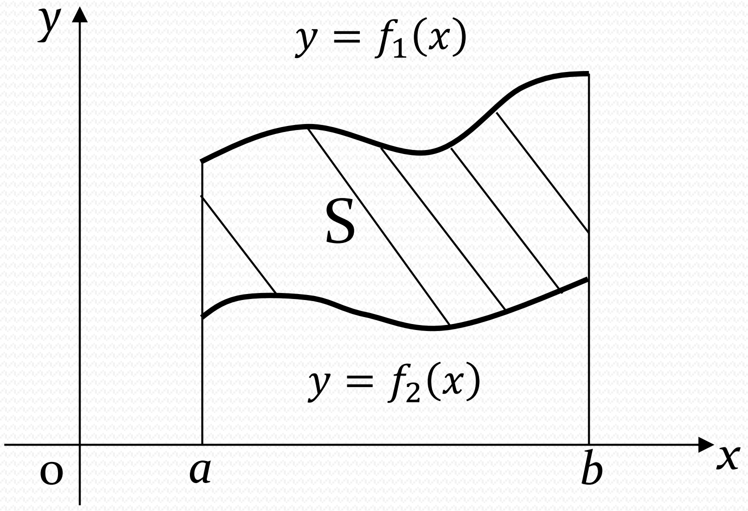




$f(x) > 0$ bo'lgan kesmalarda integral musbat, $f(x) < 0$ bo'lgan kesmalarda manfiy bo'ladi. Butun kesma bo'yicha olingan integral *Ox* o'qidan yuqorida va quyida yotuvchi yuzalarning tegishli algebraik yig'indisini beradi:

$$S = \int_a^c f(x)dx + \left|\int_c^d f(x)dx\right| + \int_d^b f(x)dx .$$





Agar $y_1 = f_1(x)$ va $y_2 = f_2(x)$ egri chiziqlar hamda $x=a$ va $x=b$ to'g'ri chiziqlar bilan chegaralangan figuraning yuzini hisoblash kerak bo'lsa, u holda $f_1(x) \geq f_2(x)$ shart bajarilgan figuraning yuzi quyidagiga teng:

$$S = \int_a^b f_1(x)dx - \int_a^b f_2(x)dx = \int_a^b [f_1(x) - f_2(x)]dx.$$

1 keysning yechimini ko’ramiz. Yechish. Chizmada koordinata tekisligini shunday joylashtiramizki, chizmaning bir qismi *Ox* o’qidan pastda joylashadi, ya’ni bu yerda $f(x) < 0$ bo’ladi.

$f_1(x) = x + 1$. $f_2(x)$ funksiyani topamiz. Bu funksiya uchi (2;-1) nuqtada bo’lgan parabola bo’lib, uning umumiy ko’rinishi $x^2 = 2py, p = 2. f_2(x) = \frac{x^2}{4} - x$.

$$S = \int_0^4 [f_1(x) - f_2(x)]dx = \int_0^4 \left(x + 1 - \frac{x^2}{4} + x\right) dx =$$

$$\int_0^4 \left(2x + 1 - \frac{x^2}{4}\right) dx = \left(x^2 + x - \frac{x^3}{12}\right)\Big|^4 = 16 + 4 - \frac{64}{12} = 14\frac{2}{3}.$$

Aylanma jism hajmini hisoblash

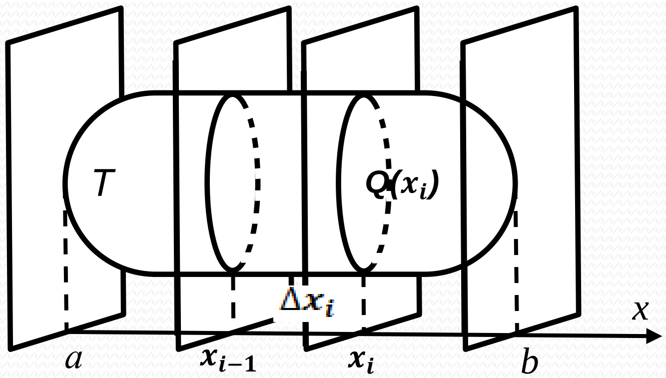
Biror jismning V hajmini hisoblaymiz. Bu jismning Ox o'qiga perpendikulyar tekislik bilan kesimining yuzi ma'lum bo'lsin. Bu yuza kesuvchi tekislikning vaziyatiga bog'liq bo'ladi, ya'ni x ning funksiyasi bo'ladi: $S=S(x)$. Faraz qilaylik, $S(x)$ uzluksiz funksiya bo'lsin. Berilgan jismning hajmini hisoblash uchun quyidagicha ish bajaramiz. $[a; b]$ kesmani

$$a = x_0, x_1, x_2, \dots, x_{i-1}, x_i, \dots, x_n = b$$

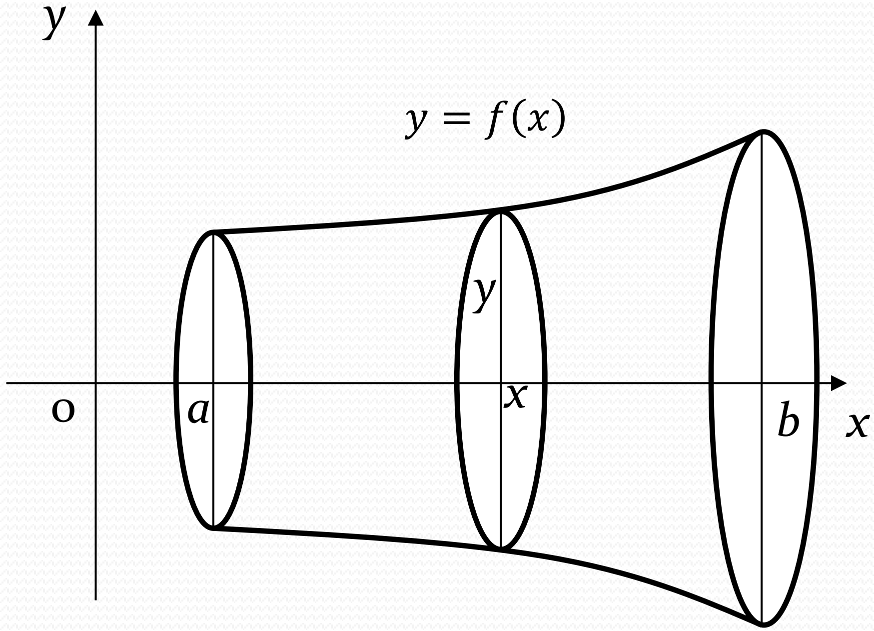
Nuqtalar bilan ixtiyoriy bo'lakka bo'lamiz va bu nuqtalar orqali Ox o'qiga perpendikulyar tekisliklar o'tkazamiz. Bu tekisliklar jismni n ta qatlamga agratadi, ularning hajmlarini

$$\Delta V_1, \Delta V_2, \dots, \Delta V_i, \dots, \Delta V_n$$

bilan belgilaymiz.



T
Q(x_i)
Δx_i
x
a
x_{i−1}
x_i
b



y
y = f(x)
y
o
a
x
b
x

U holda $V = \sum_{i=1}^{n} \Delta V_i$ bo’ladi, x_{i-1} va x_i abssissali kesimlar hosil qilgan qatlamlardan birini qarab chiqamiz. Uning ΔV_i hajmi, balandligi $\Delta x_i = x_i - x_{i-1}$, asosi biror ξ_i abssissali jismning kesimi bilan mos tushadigan to’g’ri tsilindrning hajmiga taqriban teng, bunda $x_{i-1} \le \xi_i \le x_i$ va shuning uchun ham $S(\xi_i)$ yuzaga ega bo’ladi. Bunday tsilindrning hajmi $S(\xi_i)\Delta x_i$ ga teng. Shunday qilib $\Delta V_i \approx S(\xi_i)\Delta x_i$. Shuning uchun butun jismning hajmi uchun quyidagi taqribiy tenglikni hosil qilamiz:

$$V \approx \sum_{i=1}^{n} S(\xi_i)\Delta x_i .$$

Jism hajmining aniq qiymati $\Delta x_i \to 0$ da shu yig’indining limitiga teng bo’ladi. Lekin bu yig’indi $[a; b]$ kesmada *S(x)* funksiya uchun integral yig’indi bo’ladi, shuning uchun $max\Delta x_i \to 0$ da uning limiti $\int_a^b S(x)dx$ aniq integral bo’ladi. Demak, jismning *V* hajmi ham son jihatdan shu aniq integralga teng bo’ladi:

$$V = \int_a^b S(x)dx .$$

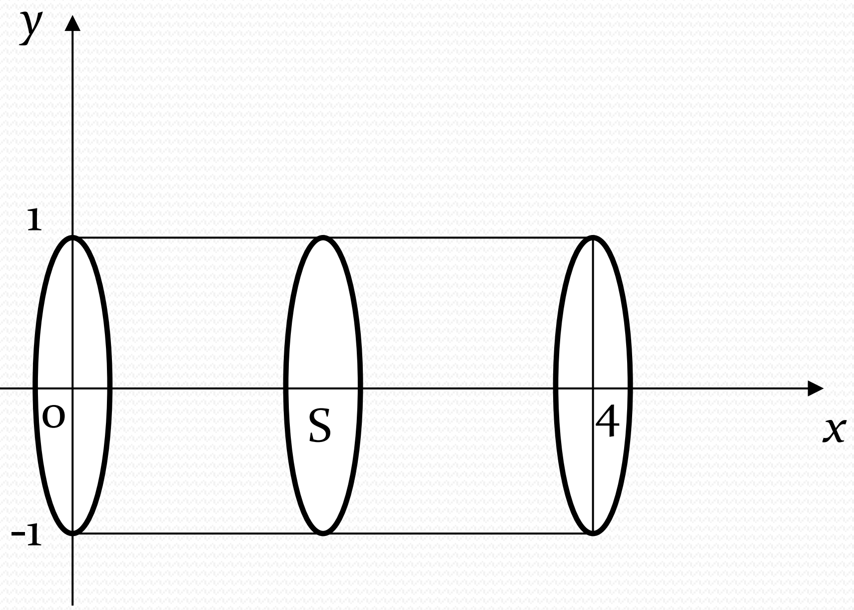
Agar qaralayotgan jism *y=f(x)* chiziq bilan chegaralangan egri chiziqli trapetsiyaning *Ox* o'qi atrofida aylanishidan hosil bo'lsa, *Ox* o'qiga perpendikulyar *x* abssissali kesim doiradan iborat bo'lib, uning radiusi *y=f(x)* ordinataga mos keladi.

Bu holda $S(x) = \pi y^2$ yoki $S(x) = \pi[f(x)]^2$ va *Ox* o'qi atrofida aylanayotgan jismning hajmi formulasiga kelamiz: $V = \pi \int_a^b y^2 dx$ yoki $V = \pi \int_a^b [f(x)]^2 dx$.

Oy o'qi atrofida aylanayotgan jismning hajmi formulasi ham xuddi shunday hosil qilinadi:

$V = \pi \int_c^d x^2 dy$ yoki $V = \pi \int_c^d [\varphi(y)]^2 dy$, bunda $x = \varphi(y)$ aylanish jismini hosil qiluvchi chiziqning tenglamasi, $c \leq y \leq d$.

2 keysning yechilishi.



$$y{=}1,\ V=\pi\int_0^4 y^2dx\,\pi\int_0^4 1dx=\pi x\Big|_0^4=4\pi\approx 4\cdot 3=12$$

$$V_{hovuz}=6\cdot 8\cdot 3=144$$

$144:12=12\,marta.$

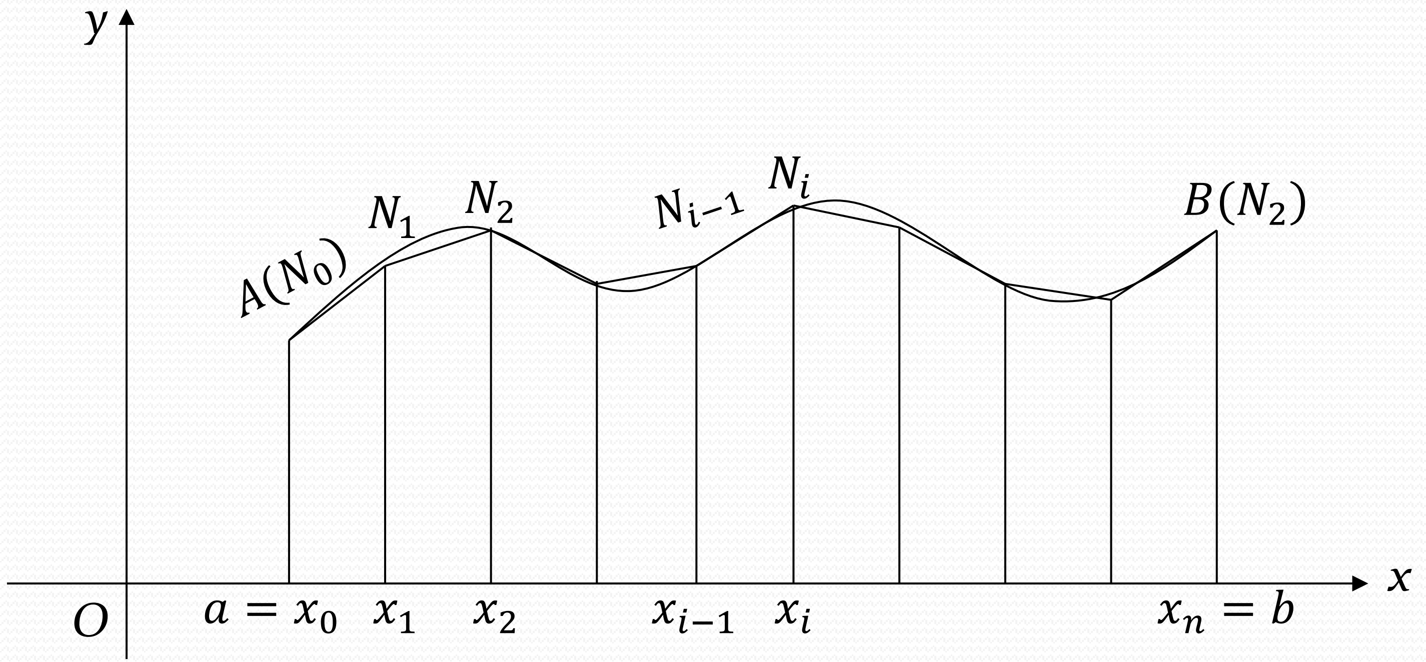
Yassi egri chiziq yoy uzunligini hisoblash

AB yassi egri chiziq berilgan bo'lsin. Uni

$$A = N_0, N_1, \dots, N_{i-1}, N_i, \dots, N_n = B$$

nuqtalar bilan ixtiyoriy n bo'lakka bo'lamiz. Qo'shni bo'linish nuqtalarini kesmalar bilan tutashtirib AB yoyga ichki chizilgan siniq chiziqni hosil qilamiz. Bu siniq chiziq $AN_1, N_1N_2, \dots, N_{i-1}N_i, \dots, N_{n-1}B$ bo'g'inlardan iborat bo'ladi, bu bo'g'inlarni $\Delta l_1, \Delta l_2, \dots, \Delta l_i, \dots, \Delta l_n$ bilan belgilaymiz.

U holda siniq chiziqning perimetri quyidagiga teng bo'ladi: $l_n = \sum_{i=1}^{n} \Delta l_i$.



Egri chiziqning bo'g'inlari soni n ning ortishi va bu bo'g'inlari uzunligi Δl_i ning kamayishi bilan bu perimetrning limiti AB egri chiziqning uzunligiga yaqinlashadi.

Ta’rif. AB egri chiziqning l uzunligi deb AB egri chiziqqa ichki chizilgan siniq chiziq perimetrining siniq chiziq bo’g’inlari soni cheksiz ortganda va eng katta bo’inning uzunligi nolga intilgandagi limitiga aytiladi:

$$l = \lim_{max\Delta l_i \to 0} \sum_{i=0}^{n} \Delta l_i \qquad (1)$$

Bunda bu limit mavjud va u ichki chizilgan siniq chiziqning tanlanishiga bog’liq emas deb, faraz qilinadi.

(1) limitga ega bo’lgan egri chiziqlar *to’g’rilanuvchi egri chiziqlar* deyiladi.

AB egri chiziq *y=f(x)* tenglama bilan berilgan bo’lsin, bu yerda $x \in [a; b]$. Agar *f(x)* funksiya $f'(x)$ funksiya bilan birga $[a; b]$ kesmada uzluksiz bo’lsa, u holda AB egri chiziqning *l* uzunligi quyidagi formula bilan ifodalanadi: $l = \int_a^b \sqrt{1 + [f'(x)]^2} dx$ (2)

Bu formulaning isbotini aylanma jismning hajmini ko'ndalang kesim yuzalari orqali chiqarganimiz kabi isbotlashimiz mumkin.

3 keysning echilishini ko'ramiz. Koordinata tekisligini chizmaga nisbatan shunday joylashtiramizki, chiziqning boshlang'ich nuqtasi koordinata boshi bilan ustma-ust tushadi. Bunda chiziq (2;2) nuqtadan o'tadi. Chiziqning tenglamasini chiqaramiz. Chiziq tenglamasining umumiy ko'rinishi $x^2 = 2py$ bo'lib, p ni topamiz.$4 = 2p \cdot 2,\ p = 1.\ x^2 = 2y,\ y = \frac{x^2}{2},\ y' = x.$

$$l = \int_0^2 \sqrt{1+x^2}\,dx = \left| \begin{matrix} x = tgt \\ dx = \dfrac{dt}{cos^2 t} \\ t_1 = 0, t_2 = arctg2 \end{matrix} \right| = \int_0^{arctg2} \sqrt{1+tg^2 t}\,\frac{dt}{cos^2 t}$$

$$= \int_0^{arctg2} \frac{dt}{cos^3 t} = \left[\frac{sint}{2cos^2 t} + \frac{1}{2} ln\left|tg\left(\frac{t}{2}+\frac{\pi}{4}\right)\right|\right]\Bigg|_0^{arctg2}$$

$$= \sqrt{5} + ln\sqrt{\sqrt{5}-2}$$

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