TOSHKENT IRRIGATSIYA VA QISHLOQ XO'JALIGINI MEXANIZATSIYALASH MUHANDISLARI INSTITUTI

FAN: OLIY MATEMATIKA

Mavzu: **LIMITLARNING ASOSIY XOSSALARI. ANIQMASLIKLAR VA ULARNI OCHISH. BIRINCHI VA IKKINCHI AJOYIB LIMITLAR.**

Mavzu:

Limitlarning asosiy xossalari. Aniqmasliklar va ularni ochish. Birinchi va ikkinchi ajoyib limitlar. Funksiya limitining iqtisodiy va taqribiy masalalarga tadbiqlari.

Reja:

1. Funksiya limitining asosiy xossalari.
2. Aniqmasliklarni ochish.
3. 1 va 2-ajoyib limitlar.

Funksiya limitining asosiy xossalari.

1. *f(x) funksiya x⟶a da ko'pi bilan bitta limitga ega bo'lishi mumkin.*
2. *Agar* $\lim_{x \to a} f(x) = b \quad (b \in R)$ *bo'lsa, x=a nuqtaning biror atrofida f(x) funksiya chegaralangan bo'ladi.*
3. *Agar* $\lim_{x \to a} f(x) = b$ *bo'lib, b≠0 bo'lsa, x=a nuqtaning shunday bir atrofi topiladiki, bu atrofdagi barcha x lar uchun (x=a bundan mustasno bo'lishi mumkin) f(x) ning ishorasi b ning ishorasi bilan bir sil bo'ladi.*

Funksiya limitining asosiy xossalari.

4. *Agar* $\lim\limits_{x\to a} f(x) = b$ *bo'lib, x=a nuqtaning biror atrofidagi barcha* $x\neq a$ *lar uchun* $f(x)\geq 0$ ($f(x)\leq 0$) *bo'lsa,* $b\geq 0$ *(mos ravishda,* $b\leq 0$*) bo'ladi.*

5. *Agar x=a nuqtaning biror atrofidagi barcha* $x\neq a$ *larda* $\varphi(x)\leq f(x)\leq g(x)$ *bo'lib,*

$$\lim_{x\to a}\varphi(x) = \lim_{x\to a} g(x) = b \quad bo'lsa \quad \lim_{x\to a} f(x) = b$$

bo'ladi.

6. *O'zgarmasning limiti o'ziga teng:*

$$\lim_{x\to a} c = c$$

Funksiya limitining asosiy xossalari.

7. *O'zgarmas ko'paytuvchini limit belgisidan tashqariga chiqarish mumkin:*

$$\lim_{x \to a}(k \cdot f(x)) = k \lim_{x \to a} f(x).$$

8. *Agar f(x), g(x) funksiyalar x→a da chekli limitga ega bo'lsa, f(x)±g(x), f(x)·g(x) funksiyalar ham x→a da chekli limitga ega va quyidagi tengliklar o'rinli.*

$$\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x),$$

$$\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

Funksiya limitining asosiy xossalari.

9. *Agar f(x), g(x) funksiyalar x⟶a da chekli limitga ega va*

$$\lim_{x\to a} g(x)\neq 0 \ \ bo'lsa, \ \ \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

tenglik o'rinli bo'ladi.

10. *Agar*

$$\lim_{x\to a} g(x)=0 \ \ va \ \ \lim_{x\to a} f(x)=b\neq 0 \quad (b\in R)$$

bo'lib, x=a nuqtaning biror atrofida (x=a nuqtaning o'zi bundan mustasno bo'lishi mumkin) g(x)≠0 bo'lsa, $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\infty \quad bo'ladi.$

BIRINCHI AJOYIB LIMIT

$\frac{\sin x}{x}$ funksiya x=0 da aniqlanmagan, chunki kasrning surat va maxraji nolga aylanadi. Bu funksiyaning x—> 0 dagi limitini topamiz.

Radiusi **1** bo'lgan aylanani qaraymiz. M O B burchakni x bilan belgilaymiz bunda 0<x<$\frac{\pi}{2}$, rasmdan quyidagilar chiqadi:

$$S_{\Delta MOA} < S_{sekMOA} < S_{\Delta COA} \qquad (1)$$

$$S_{\Delta MOA} = \frac{1}{2} \cdot OA \cdot MB = \frac{1}{2} \cdot 1 \cdot \sin x = \frac{1}{2} \sin x\,.$$

$$S_{sekMOA} = \frac{1}{2} \cdot OA \cdot MA = \frac{1}{2} \cdot 1 \cdot x = \frac{1}{2} \mathrm{x}$$

$$S_{\Delta COA} = \frac{1}{2} \cdot OA \cdot AC = \frac{1}{2} \cdot 1 \cdot \operatorname{tg} x = \frac{1}{2} \operatorname{tg} x$$

(1) Tengsizlik $\frac{1}{2}$ ga qisqartirilganidan so'ng quyiagicha yoziladi : $\sin x < x < \mathrm{t}g\, x$. Tengcizlikning hamma hadlarini $sin\, x$ ga bo'lamiz

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad \text{yoki} \quad 1 > \frac{\sin x}{x} > \cos x$$

Biz bu tengsizlikni $x > 0$ da faraz qilib chiqardik. $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$ va $\cos(-x) = \cos x$ ekanligini e'tiborga olib bu tengsizlik $x<0$ da ham to'g'ri degan hulosaga kelamiz . Ammo: $\lim_{x\to 0} \cos x = 1,$ $\lim_{x\to 0} 1 = 1$.

Demak $\frac{sin\, x}{x}$ o'zgaruvchi shunday ikki miqdor oralig'idaki, ularning ikkalasi ham birgina limitga intiladi va u limit 1 ga teng . Shunday qilib yuqorida ko'rilgan teoremalarga asosan $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

Misol: 1) $\lim\limits_{x\to 0}\frac{\mathrm{t}g\, x}{x}=\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=1\cdot\frac{1}{1}=1.$

2)

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{2sin^2\frac{x}{2}}{x}=\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot sin\frac{x}{2}=1\cdot 0=0$$

Ikkinchi ajoyib limit.

Ushbu $(1+\frac{1}{n})^n$ o‘zgaruvchi miqdorni tekshiramiz n bu yerda natural sonlar qatorining 1,2,3,….. qiymatlarini qabul qiladigan o‘suvchi miqdor.

1- teorema. $(1+\frac{1}{n})^n$ o‘zgaruvchi miqdor $n \to \infty$ da 2 bilan 3 orasida yotuvchi limitga ega.

Isbot: Nyuton binomi formulasiga muvofiq bunday yozish mumkin:

$$(1+\frac{1}{n})^n = 1 + \frac{n}{1}\cdot\frac{1}{n} + \frac{n(n-1)}{1*2}\cdot\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{1\cdot2\cdot3}\cdot\left(\frac{1}{n}\right)^3 + \ldots.$$

$$+\frac{n(n-1)(n-2)\cdots[n-(n-1)]}{1\cdot2\cdot\ \ldots\cdot n}\cdot(\frac{1}{n})^n \ . \qquad (1)$$

(1) tenglikni quyidagicha almashtirishimiz mumkin :

$$(1+\ \frac{1}{n})^n = 1+1+\frac{1}{1*2}\left(1-\frac{1}{n}\right)+\frac{1}{1*2*3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$

$$+\frac{1}{1\cdot2\cdot\ \ldots\cdot n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right). \qquad (2)$$

Oxirgi tenglikdan $(1+\frac{1}{n})^n$ o‘zgaruvchi miqdor n o‘sishi bilan o‘sadigan o‘zgaruvchan miqdor ekanligi chiqadi.

Haqiqatdan n qiymatdan n+1 qiymatga o’tganda oxirgi yig’indining har bir qo’shiluvchisi o’sadi. $\frac{1}{1\cdot 2}(1-\frac{1}{n}) < \frac{1}{1\cdot 2}\left(1-\frac{1}{n+1}\right)$ va hokazo. Hamda yana bir had qo’shiladi (yoyilmaning hamma hadlari musbat) Endi (1+ $\frac{1}{n})^n$ o’zgaruvchi miqdor cheklanganligini ko’rsatamiz . $\left(1-\frac{1}{n}\right) < 1; \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) < 1$ va hokazo ekanligini qayd qilib , (2) ifodadan ushbu tengsizlikni hosil qilamiz:

$$(1+\frac{1}{n})^n < 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{1\cdot 2\cdot 3\cdot \dots \cdot n}.$$

So'ngra : $\frac{1}{1\cdot 2\cdot 3} < \frac{1}{2^2}, \frac{1}{1\cdot 2\cdot 3\cdot 4} < \frac{1}{2^3}, \dots \frac{1}{1\cdot 2\cdot \dots \cdot n} < \frac{1}{2^{n-1}}$

ekanligini qayd qilib tengsizlikni buday yozmog'imiz mumkin :

$$(1+\frac{1}{n})^n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}.$$

Bu tengsizlikni o'ng tomonida tagiga chizilgan hadlar maxraji $q=\frac{1}{2}$ va birinchi hadi a=1 bo'lgan geometrik progressiyani hosil qiladi shuning uchun :

$$(1+\frac{1}{n})^n < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}]=$$

$$1+\frac{a-aq^n}{1-q} = 1+\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} =$$

$$1+[2-(\frac{1}{2})^{n-1}] < 3\ .$$

Demak barcha n lar uchun quyidagi tengsizlik kelib chiqadi:

$(1+\frac{1}{n})^n < 3.$ (2) tengsizlikdan esa $(1+\frac{1}{n})^n \geq 2$ kelib chiqadi .

Shunday qilib ushbu tengsizlikni hosil qilamiz :

$$2 \leq (1+\frac{1}{n})^n < 3 \quad (3)$$

Bu bilan $(1+\frac{1}{n})^n$ o’zgaruvchi miqdor cheklanganligi aniqlandi.

Demak $(1+\frac{1}{n})^n$ o’zgaruvchi miqdor o’suvchi va cheklangan shuning uchun u limitga ega. Bu limit e harfi bilan belgilanadi.

Ta’rif: $(1+\frac{1}{n})^n$ o’zgaruvchi miqdorning n→ ∞ dagi limiti e soni deyiladi

$$\mathbf{e} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

e soni $2 < e < 3$ tengsizlikni qanoatlantiradi. e irratsional son bo'lib

uning verguldan keyingi 10 ta ishonchli raqamlar qiymati

e = 2,7182818284…..

2-teorema. x cheksizlikka intilganda $\left(1+\frac{1}{x}\right)^x$ funksiya e limitga intiladi, yani :

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \mathrm{e}$$

Isbot: Agar n butun musbat qiymatlar qabul qilsa n→ ∞ da $\left(1+\frac{1}{n}\right)^n$ →e aniqlangan edi . Endi x kasr qiymatlar ham, manfiy qiymatlar ham qabul qilgan holda cheksizlikka intilsin.

x→ +∞ deylik . Uning har bir qiymati ikkita musbat butun son orasida

n≤ $x < n+1$, bunda quyidagi tengsizliklar bajariladi :

$$\frac{1}{n} \geq \frac{1}{x} > \frac{1}{n+1}, \qquad 1+\frac{1}{n} \geq 1+\frac{1}{x} > 1+\frac{1}{n+1},$$

$$(1+\frac{1}{n})^{n+1} > (1+\frac{1}{x})^{x} > (1+\frac{1}{n+1})^{n}$$

Agar x→ ∞ bo'lsa u xolda n→ ∞ bo'lishi ravshan .

Oralarida $\left(1+\frac{1}{x}\right)^{x}$

o'zgaruvchi yotadigan o'zgaruvchilarning limitlarini topamiz :

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) =$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \mathbf{e}\cdot\mathbf{1} = \mathbf{e}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}} =$$

$$\frac{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)} = \frac{\mathbf{e}}{1} = \mathbf{e}$$

Demak, $$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \mathbf{e} \qquad \textbf{(4)}$$

2) $x\to -\infty$ deylik. Yangi $t=-(x+1)$ yoki $x=-(t+1)$ o'zgaruvchini kiritamiz. $t\to +\infty$ da $x\to -\infty$ bo'ladi. Bunday yoza olamiz :

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{t\to+\infty}\left(1-\frac{1}{t+1}\right)^{-t-1} = \lim_{t\to+\infty}\left(\frac{t}{t+1}\right)^{-t-1} =$$

$$\lim_{t\to+\infty}\left(\frac{t+1}{t}\right)^{t+1} = \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{t+1} =$$

$$\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\cdot\left(1+\frac{1}{t}\right) == e\cdot 1 = e$$

Agar (4) tenglikda $\frac{1}{x}=\alpha$ faraz qilinsa u xolda $x\to\infty\ da \quad \alpha\to 0 \quad (\alpha\neq 0)$ va quyidagi tengsizlik hosil bo'ladi. $\lim_{\alpha\to\infty}(1+\alpha)^{\frac{1}{\alpha}} = e$

Misollar :

1.

$$\left(1+\frac{1}{n}\right)^{n+5} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right.$$

$$\left(1+\frac{1}{n}\right)^{n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{5} =$$

$$\mathrm{e}\cdot 1 = \mathrm{e}$$

$$2.\ \mathrm{e} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\cdot\left(1+\frac{1}{x}\right)^{x}\cdot\left(1+\frac{1}{x}\right)^{x} =$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\cdot \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\cdot\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \mathrm{e}\cdot\mathrm{e}\cdot\mathrm{e} = \mathrm{e}^3$$

$$3\ .\ \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x} = \left|\begin{matrix}\frac{x}{2}=y\\ x=2y\\ x\to\infty\ da\\ y\to\infty\end{matrix}\right| = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{2y} = \mathrm{e}^2$$

$$4.\ \lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x+3} =$$

$$\lim_{x\to\infty}\left(\frac{x-1+4}{x-1}\right)^{x+3} = \lim_{x\to\infty}\left(\mathbf{1}+\frac{\mathbf{4}}{\boldsymbol{x}-\mathbf{1}}\right)^{\boldsymbol{x}+\mathbf{3}} =$$

$$\lim_{x\to\infty}\left(\mathbf{1}+\frac{\mathbf{4}}{\boldsymbol{x}-\mathbf{1}}\right)^{(\boldsymbol{x}-\mathbf{1})+\mathbf{4}} =$$

$$\left|\begin{matrix} x-1=y \\ x\to\infty \\ y\to\infty \end{matrix}\right| = \lim_{y\to\infty}\left(\mathbf{1}+\frac{\mathbf{4}}{\boldsymbol{y}}\right)^{\boldsymbol{y}+\mathbf{4}} =$$

$$\lim_{y\to\infty}\left(\mathbf{1}+\frac{\mathbf{4}}{\boldsymbol{y}}\right)^{\frac{\boldsymbol{y}}{\mathbf{4}}\cdot\mathbf{4}} \cdot$$

$$\lim_{y\to\infty}\left(\mathbf{1}+\frac{\mathbf{4}}{\boldsymbol{y}}\right)^{\mathbf{4}} = \boldsymbol{e}^{\mathbf{4}}\cdot\mathbf{1} = \boldsymbol{e}^{\mathbf{4}}$$

TOSHKENT IRRIGATSIYA VA QISHLOQ XO'JALIGINI MEXANIZATSIYALASH MUHANDISLARI INSTITUTI

E‘TIBORLARINGIZ UCHUN RAXMAT

☎ + 998 71 237 0986