TOSHKENT IRRIGATSIYA VA QISHLOQ XO'JALIGINI MEXANIZATSIYALASH MUHANDISLARI INSTITUTI

FAN: OLIY MATEMATIKA

Mavzu

KO`P O`ZGARUVCHILI FUNKTSIYA. ANIQLANISH SOHASI. XUSUSIY VA TO`LA ORTTIRMA. XUSUSIY XOSILALAR, TO’LA DIFFERENSIAL. TO’LA DIFFERENSIAL YORDAMIDA TAQRIBIY HISOBLASH. MURAKKAB VA OSHKORMAS FUNKSIYANING XUSUSIY HOSILALARI. YO’NALISH BO’YICHA XOSILA, GRADIYENT. IKKI O’ZGARUVCHILI FUNKSIYANING EKSTREMUMI.

❖ Ko`p o`zgaruvchili funktsiyalar haqida umumiy tushunchalar

Tabiat va jamiyatda juda ko‘p masalalar borki o‘zgaruvchi miqdorlar bog‘lanishlarida bittasining sonli qiymati boshqa bir nechasining qiymati bilan aniqlanadi. Masalan, tomonlarining uzunliklari x va y dan iborat bo‘lgan to‘g‘ri to‘rtburchakning yuzi, uning tomonlarining uzunliklari o‘zgarishi bilan o‘zgarib boradi;

parallelepipedning hajmi uning uchala o‘lchovining o‘zgarishi bilan o‘zgaradi; biror yer maydonidan olinayotgan hosildorlik yerning tuzilishiga, unga o‘g‘it berishga, sug‘orishga, dehqonning malakasiga va boshqa juda ko‘p faktorlarga bog‘liq. Bunday misollarni istalgancha keltirish mumkin.

Bunday bog‘lanishlarni tekishirish uchun ko‘p o‘zgaruvchili (argumentli) funksiyalar tushunchasini kiritamiz va ularni tekshirish apparati amallarini o‘rganamiz.

1-ta'rif. Agar D to'plamning har bir (x, y) haqiqiy sonlari juftligiga biror qoidaga ko'ra E to'plamdagi yagona z haqiqiy son mos qo'yilgan bo'lsa, u holda D to'plamda ikki x va y o'zgaruvchilarning funksiyasi z aniqlangan deyiladi.

Ikki o'zgaruvchining funksiyasi simvolik tarzda quyidagicha belgilanadi: $z = f(x, y)$, $z = z(x, y)$ (funksiya U yoki y bilan o'zgaruvchilar mos ravishda x, t yoki x_1, x_2 lar bilan belgilangan bo'lsa $U = f(x,t)$ yoki $y = f(x_1, x_2)$ tarzda ifodalanishi ham mumkin va h.k.). Bunda x va y o'zgaruvchilarga erkli o'zgaruvchilar yoki argumentlar, z ga erksiz o'zgaruvchi yoki funksiya deb ataladi.

D to‘plamga funksiyaning aniqlanish sohasi, E to‘plamga o‘zgarish yoki qiymatlar sohasi deyiladi. Har bir juft haqiqiy songa biror tayin koordinat sistemasida bitta M nuqta va bitta nuqtaga bir juft haqiqiy son mos kelganligi uchun ikki argumentli funksiyani M nuqtaning funksiyasi ham deb qaraladi, hamda $y = f(x_1, x_2)$ o‘rniga $y = f(M)$ ham deb yozish mumkin.

Argumentning tayin $x = x_0$ va $y = y_0$ qiymatlarida $z = f(x, y)$ funktsiya qabul qiladigan z_0 hususiy qiymat $z_0 = z\Big|_{\substack{y=y_0 \\ x=x_0}}$ yoki $z_0 = f(x_0, y_0)$ kabi yoziladi.

Masalan, $z = x^2 - y^2$ funktsiya uchun $z_0 = z\Big|_{\substack{y=1 \\ x=-2}} = (-2)^2 - 1^2 = 3.$

Geometrik nuqtai-nazardan to`g`ri burchakli Dekart koordinatalar sistemasida haqiqiy sonlarning har bir (x, y) juftiga Oxy tekislikning yagona

$P(x,y)$ nuqtasi mos keladi va aksincha, tekislikning har bir $P(x,y)$ nuqtasiga haqiqiy sonlarning yagona (x,y) jufti mos keladi. Shu sababli ikki o`zgaruvchining funktsiyasini $P(x,y)$ nuqtaning funktsiyasi deb qarash va $z=f(x,y)$ yozuvni $f(P)$ kabi yozish mumkin. Bu holda ikki o`zgaruvchi funktsiyasining aniqlanish sohasi Oxy tekislik nuqtalarining biror to`plamidan yoki butun tekislikdan iborat bo`ladi.

(x, y, z) haqiqiy sonlar uchligining D to`plamida uch o`zgaruvchining $u = u(x, y, z)$ funktsiyasiga shu kabi ta`rif beriladi. To`rt o`zgaruvchining va umuman $n$ o`zgaruvchi funktsiyasining aniqlanish sohasi n ta haqiqiy sonlarning $(x_1, x_2, ..., x_n)$ sistemasidan tuzilgan D to`plam bo`ladi. n o`garuvchining funktsiyasi quyidagicha belgilanadi:

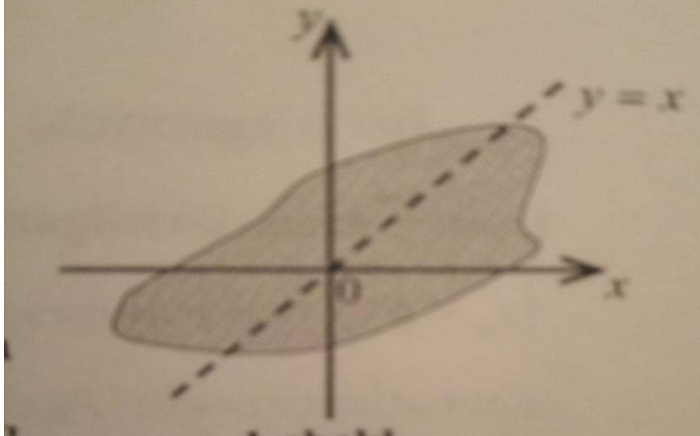
$y = f(x_1, x_2, ..., x_n)$, $y = y(x_1, x_2, ..., x_n)$ va hokazo.

To`rtta va undan ortiq o`zgaruvchiga bog`liq funktsiyalarning aniqlanish sohasini chizmalarda ko`rgazmali namoyish qilib bo`lmaydi. Shu sababli, bundan keyin bir necha o`zgaruvchining funktsiyasi deganda ikki (ayrim hollarda uch) o`zgaruvchining funktsiyasini nazarda tutamiz.

Bir necha o‘zgaruvchining funksiyasi turli usullarda berilishi mumkin. Biz quyida funktsiya berilishining analitik usulidan foydalanamiz. Bu usulda funktsiya formula yordamida beriladi va funktsiyaning aniqlanish sohasi bu formula ma`noga ega bo`ladigan barcha nuqtalar to`plamidan iborat bo`ladi.

1-misol. $z = \dfrac{3x + y}{x - y}$ funktsiyaning aniqlanish sohasini toping.

Funktsiya $x - y = 0$ yoki $y = x$ shartda aniqlanmagan. Bu shart geometrik nuqtai-nazardan shu funktsiyaning aniqlanish sohasi ikkita yarim tekislikdan tashkil topishini bildiradi. Bunda birinchi yarim tekislik $y = x$ to`g`ri chiziqdan yuqorida, ikkinchisi esa bu to`g`ri chiziqdan pastda yotadi (1-shakl).



1-shakl.

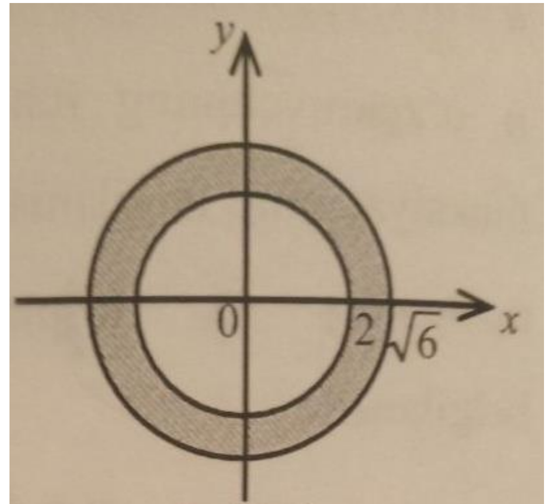
2-misol. $z=\sqrt[4]{9-x^2-y^2}$ Funktsiyaning aniqlanish sohasini toping.

Funktsiya $9-x^2-y^2\geq 0$ yoki $x^2+y^2\leq 9$ shartda haqiqiy qiymatlar qabul qiladi. Demak, funktsiya aniqlanish sohasi markazi koordinatalar boshida bo`lgan radiusi uchga teng bo`lgan doiradan iborat.

3-misol. $z=\arcsin\left(x^2+y^2-5\right)$ funktsiyaning aniqlanish sohasini toping.

Funktsiya $-1\le x^2+y^2-5\le 1$ shartda aniqlangan. Bu shart $4\le x^2+y^2\le 6$ shartga teng kuchli. Funktsiya aniqlanish sohasining chegaraviy chiziqlari $x^2+y^2=4$ va

$x^2+y^2=6$ aylanalar bo`lib, aylana nuqtalari ham bu sohaga tegishli. Demak, funktsiyaning aniqlanish sohasi markazi koordinatalar boshida bo`lgan, radiuslari mos ravishda 2 va $\sqrt{6}$ ga teng ayalanalar orasida va bu aylanalarda yotuvchi barcha nuqtalardan iborat (2-shakl).



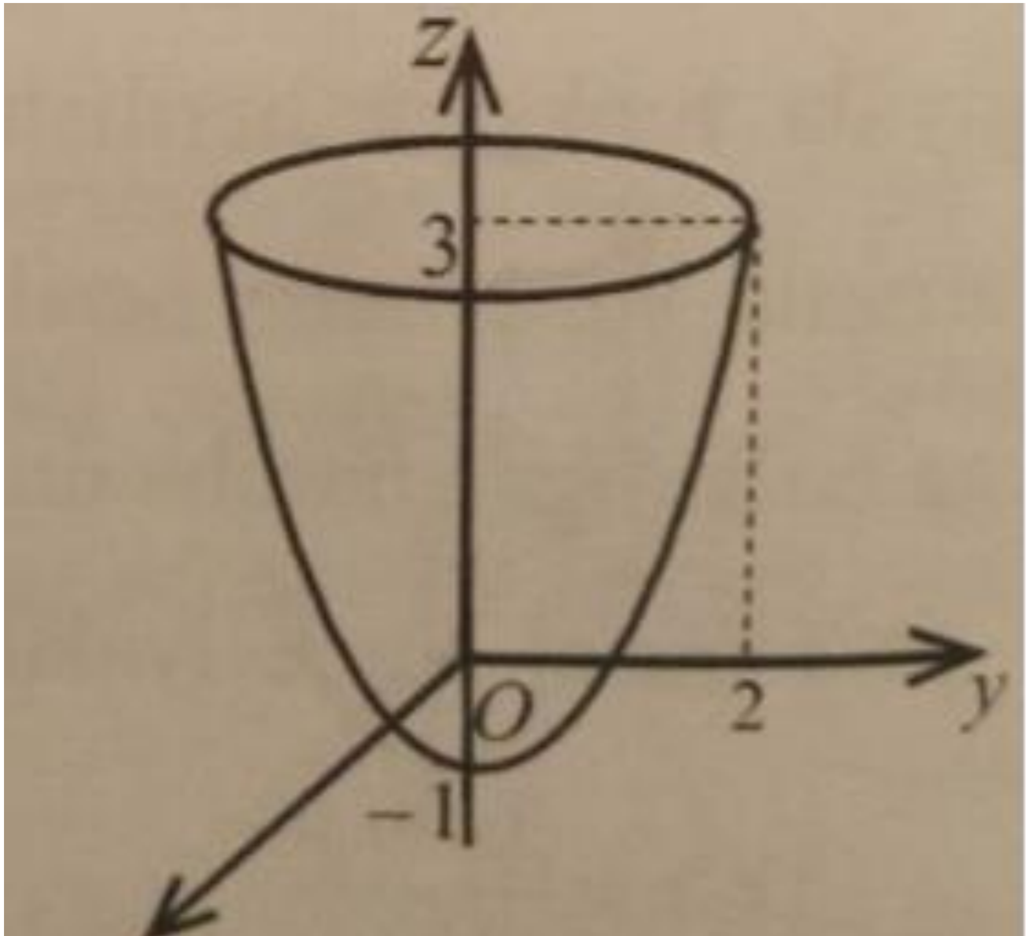
2-shakl.

4-misol. Funktsiya (x, y, z) uchlikning bir vaqtda $x \geq 0,\ y \geq 0,\ z \geq 0$ shartni qanoatlantiruvchi qiymatlarida aniqlangan. Demak, funktsiyaning aniqlanish sohasi $Oxyz$ koordinatalar fazosining birinchi oktantdagi qismidan iborat.

Ikki o`zgaruvchi funktsiyasining geometric tasviri uch o`lchovli fazodagi sirtdan iborat bo`ladi.

Masalan, markazi koordinatalar boshida bo`lgan, radiusi to`rtga teng sferaning yuqori qismi $z = \sqrt{16 - x^2 - y^2}$ funktsiyaning grafigi bo`ladi.

3-shaklda tasvirlangan aylanish paraboloidi $x^2 + y^2 - 1 = z$ funktsiyaning grafigidir.



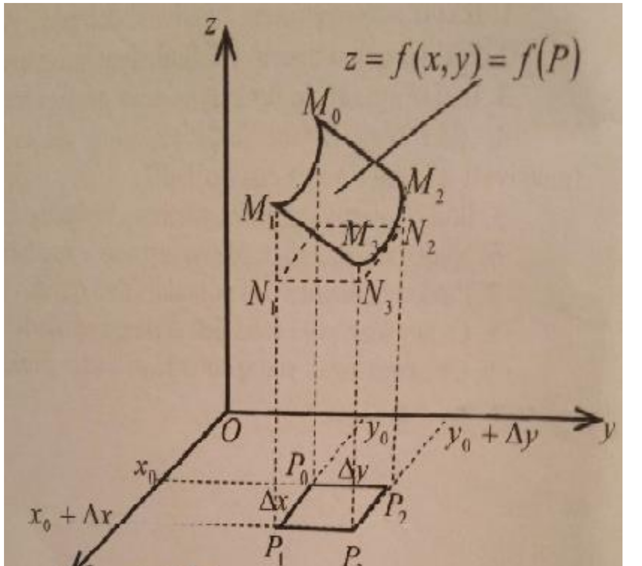
3-shakl.

❖ Funktsiyaning xususiy va to`la orttirmasi

Fazoda biror sirtni ifodalovchi ikki o`zgaruvchining $z = f(x, y) = f(P)$ funktsiyasini qaraymiz (6-shakl).

x o`zgaruvchiga $P_0(x_0, y_0)$ nuqtada $\Delta x$ orttirma beramiz va $y$ o`zgaruvchini o`zgarishsiz qoldirib $P_1(x_0 + \Delta x; y_0)$ nuqtani hosil qilamiz. $P_0$ va $P_1$ nuqtalarga sirtda $M_0(x_0; y_0; z_0)$ va $M_1(x_0 + \Delta x; y_0; z_1)$ nuqtalar mos keladi, bu yerda $z_1 = f(P_1) - f(P)$ nuqtadagi $x$ o`zgaruvchi bo`yicha xususiy orttirmasi deb ataladi (shaklda $\Delta_x z = N_1 M_1$ kesma).

Shu kabi, agar y o`zgaruvchiga $\Delta y$ orttirma berilsa va $x$ o`zgarishsiz qoldirilsa $P_2(x_0; y_0+\Delta y)$ nuqta hosil bo`ladi. Bu nuqtaga sirtda $M_2(x_0; y_0+\Delta y; z_2)$ nuqta mos keladi, bu yerda $z_2 = f(P_2) = f(x_0, y_0+\Delta y)$ (shaklda $z_2 = P_2M_2$).



6-shakl

$\Delta_y z = f(P_2) - f(P)$ yoki $\Delta_y z = f(x_0, y_0 + \Delta y) - f(x_0, y_0)$ ayirmaga $z = f(x, y)$ funktsiyaning $P_0(x_0, y_0)$ nuqtadagi y o`zgaruvchi bo`yicha xususiy orttirmasi deyiladi(shakldа $\Delta_y z = N_2 M_2$).

Endi ikkala x va y o`zgaruvchiga mos ravishda $\Delta x$ va $\Delta y$ orttirma beramiz. U holda $P_0(x_0, y_0)$ nuqta $P_3(x_0 + \Delta x,\ y_0 + \Delta y)$ nuqtaga o`tadi. Bu nuqtaga sirtda $M_3(x_0 + \Delta x,\ y_0 + \Delta y, z_3)$ nuqta mos keladi, bu yerda $z_3 = f(P_3) = f(x_0 + \Delta x,\ y_0 + \Delta y)$.

$\Delta z = f(P_3) - f(P)$ yoki $\Delta z = f(x_0 + \Delta x,\ y_0 + \Delta y) - f(x_0, y_0)$ ayirmaga $z = f(x, y)$ funktsiyaning $P(x, y)$ nuqtadagi to`liq orttirmasi deyiladi (shaklda $\Delta z = N_3 M_3$).

6-misol. $z = x^2 - xy + y^2$ funktsiyaning to'la orttirmasini toping.

$$\Delta z = (x+\Delta x)^2 - (x+\Delta x)(y+\Delta y) + (y+\Delta y)^2 - x^2 + xy - y^2 =$$
$$= x^2 + 2x(\Delta x) + (\Delta x)^2 - xy - x(\Delta y) - y(\Delta x) - (\Delta x)(\Delta y) + y^2 + 2y(\Delta y) + (\Delta y)^2 -$$
$$-x^2 + xy - y^2 = 2x(\Delta x) - x\Delta y + 2y\Delta y - y\Delta x + (\Delta x)^2 - \Delta x\Delta y + (\Delta y)^2 =$$
$$= (2x - y)\Delta x + (2y - x)\Delta y + (\Delta x)^2 - (\Delta x)(\Delta y) + (\Delta x)^2$$

❖ Xususiy hosila

$z = f(x, y)$ funktsiyaning x o`zgaruvchi bo`yicha $\Delta_x z$ orttirmasining shu o`zgaruvchi Δx orttirmasiga nisbati

$$\frac{\Delta_x z}{\Delta x} = \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

ni qaraymiz.

**1-Ta`rif.** Agar $\frac{\Delta_x z}{\Delta x}$ nisbatining $\Delta x \to 0$ dagi limiti bo`lsa, u holda bu limitga $z = f(x, y)$ funktsiyaning $P_0(x_0, y_0)$ nuqtadagi x o`zgaruvchi bo`yicha xususiy hosilasi deyiladi va ushbu

$$\left(\frac{\partial z}{\partial x}\right)_{P_0}, \quad \left(\frac{\partial f}{\partial x}\right)_{P_0}, \quad z'_x(x_0, y_0), \;\; f'_x(x_0, y_0)$$

Ko`rinishlarda belgilanadi.

Demak, ta`rifga ko`ra

$$f'_x(x_0,y_0)=\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

$z=f(x,y)$ funktsiyaning $P_0(x_0,y_0)$ nuqtadagi y o`zgaruvchi bo`yicha xususiy hosilasi ham shu kabi ta`riflanadi:

$$f'_y(x_0,y_0)=\lim_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y}=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

Ta`riflardan $f'_x(x_0,y_0)$ ni topishda $y$ o`zgarishsiz qolishi, $f'_y(x_0,y_0)$ ni topishda esa x o`zgarishsiz qolishi kelib chiqadi.

Demak, ikki o`zgaruvchi funktsiyasining xususiy hosilasi bu o`zgaruvchilardan biri funktsiyasining hosilasi sifatida topiladi. Shu sababli bir o`zgaruvchi funktsiyasining hosilalari uchun keltirib chiqarilgan barcha differensiallash formulalari va qoidalari bir necha o`zgaruvchi funktsiyasining xususiy hosilalari uchun ham saqlanadi. Bunda biror argument bo`yicha xususiy hosilaning qoida va formulalarini qo`llashda ikkinchi argument o`zgarmas deb hisoblanishini yodda tutish lozim. Uch va undan ortiq o`zgaruvchi funktsiyasining xususiy hosilalari shunga o`xshash ta`riflanadi va topiladi.

1-misol. $z = \ln(x^2 + e^{-y})$ funktsiyaning xususiy hosilalari shunga o`xshash ta`riflanadi va topiladi.

Yechish. y ni o`zgarmas deb, $\frac{\partial z}{\partial x}$ xususiy hosilani topamiz:

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 + e^{-y}} \cdot 2x = \frac{2x}{x^2 + e^{-y}}$$

x ni o`zgarmas hisoblab, $\frac{\partial z}{\partial y}$ xususiy hosilani topamiz:

$$\frac{\partial z}{\partial y} = \frac{1}{x^2 + e^{-y}} \cdot (-e^{-y}) = -\frac{e^{-y}}{x^2 + e^{-y}}$$

2-misol. $u = e^{xyz} + y^3 - 5z^4$ funktsiyaning xususiy hosilalarini toping.

Yechish. y va z larni o`zgarmas deb, $\frac{\partial u}{\partial x}$ xususiy hosilani topamiz:

$$\frac{\partial u}{\partial x} = e^{xyz} \cdot (yz) = yze^{xyz}$$

Shu kabi topamiz:

$$\frac{\partial u}{\partial y} = e^{xyz} \cdot (xz) + 3y^2 = xze^{xyz} + 3y^2, \quad \frac{\partial u}{\partial z} = e^{xyz} \cdot (xy) - 20z^3 = xye^{xyz} - 20z^3.$$

❖ To`la differensial

**2-Ta`rif.** Agar $z = f(x,y)$ funktsiyaning $P(x,y)$ nuqtadagi to`liq ortirmasini

$$\Delta z = A\Delta x + B\Delta y + \gamma(\Delta x, \Delta y) \qquad (1)$$

ko`rinishda ifodalash mumkin bo`lsa, u holda bu funktsiya $P(x,y)$ nuqtada differensiallanuvchi deyiladi. Bu yerda $A, B - \Delta x, \Delta y$ ga bog`liq bo`lmagan sonlar, $\gamma(\Delta x, \Delta y) - \Delta x \to 0,\ \Delta y \to 0$ da cheksiz kichik funktsiya, ya`ni, $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \gamma(\Delta x, \Delta y) = 0.$

1-Teorema. Agar $z = f(x,y)$ funksiya $P(x,y)$ nuqtada differensiallanuvchi bo`lsa, u holda u shu nuqtada $f'_x(x,y)$ va $f'_y(x,y)$ xususiy hosilalarga ega bo`ladi, shu bilan birga $A = f'_x(x,y),\ B = f'_y(x,y)$.

2-Teorema. Agar $z = f(x, y)$ funksiya $P(x, y)$ nuqtaning biror atrofida uzluksiz xususiy hosilalarga ega bo`lsa, u holda u shu nuqtada differensiallanuvchi bo`ladi.

3-Teorema. Agar $z = f(x, y)$ funksiya $P(x, y)$ nuqtada differensiallanuvchi bo`lsa, u holda u shu nuqtada uzluksiz bo`ladi.

$z = f(x, y)$ funksiya $P(x, y)$ nuqtada differensiallanuvchi bo`lsin.

**3-Ta`rif.** $\Delta z$ to`liq orttirmaning $\Delta x, \Delta y$ larga nisbatan chiziqli bo`lgan bosh qismi $A\Delta x + B\Delta y$ ga $z = f(x, y)$ funktsiyaning $P(x, y)$ nuqtadagi to`liq differensiali deyiladi va u dz bilan belgilanadi.

Demak, ta`rifga ko`ra $dz = A\Delta x + B\Delta y$ yoki 1-teoremaga binoan

$$dz = f'_x(x, y)\Delta x + f'_y(x, y)\Delta y.$$

Shunday qilib, funksiyaning to`liq differensiali xususiy hosilalarning mos argumentlar orttirmasiga ko`paytmasining yig`indisiga teng.

To`liq differensialni argumentlarning orttirmalari va differensiallarning tengligi, ya`ni $\Delta x = dx,\ \Delta y = dy$ ni inobatga olib, quyidagicha yozish mumkin:

$$dz = f'_x(x,y)dx + f'_y(x,y)dy \qquad (2)$$

3-misol. $z = x^2 y$ funktsiyaning to`liq differensialini toping.

Yechish. Xususiy hosilalarni topamiz: $f'_x(x,y) = 2xy,\ f'_y(x,y) = x^2.$

Bu hosilalar Oxy tekislikda uzluksiz. Yuqoridagi 2-teoremaga ko`ra $Oxy$ tekislikning har bir nuqtasida $z = x^2 y$ funktsiya differensiallanuvchi hamda $dz$ to`liq differensial mavjud. Bundan

$$dz = 2xydx + x^2 dy.$$

❖ To`la differensialning taqribiy hisobga tadbiqlari

Ko`pchilik masalalarni yechishda $z = f(x, y)$ funksiyaning $P_0(x, y)$ nuqtadagi to`liq orttirmasi funksiyaning shu nuqtadagi differensialiga taqriban tenglashtiriladi, ya`ni $\Delta y \approx dy$ deb olinadi.

Demak,

$$f(x_0 + \Delta x,\ y_0 + \Delta y) - f(x_0, y_0) \approx f'_x(x_0, y_0) + f'_y(x_0, y_0)\Delta y$$

yoki

$$f(x, y) \approx f(x_0, y_0) + f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y \qquad (3)$$

Bu almashtirish yordamida qandaydir A kattalikni taqribiy hisoblash algoritmi quyidagicha bo`ladi:

1. A ni biror $f(x,y)$ funktsiyaning $P(x,y)$ nuqtadagi qiymatiga tenglashtiriladi, ya`ni $A = f(x,y)$ deb olinadi;

2. $P_0(x_0, y_0)$ nuqta $P(x,y)$ nuqtaga yaqin va $f(x_0, y_0)$ ni hisoblash qulay qilib tanalanadi;

3. $f(x_0, y_0)$ hisoblanadi;

4. $f'_x(x,y),\ f'_y(x,y)$ lar topilib, $f'_x(x_0,y_0),\ f'_y(x_0,y_0)$ lar hisoblanadi;

5. $x,\ y,\ x_0,\ y_0,\ f(x_0,y_0),\ f'_x(x_0,y_0),\ f'_y(x_0,y_0)$ qiymatlar (3) formulaga qo`iladi.

4-misol. $\ln(0,09^2+0,99^3)$ ni taqribiy hisoblang.

Yechish. 1. $A=\ln(0,09^2+0,99^3),\ f(x,y)=\ln(x^2+y^3)$ deymiz. U holda $f(x,y)=A,\ x=0,09,\ y=0,99;$

2. $x_0=0, y_0=1,$ ya`ni $P_0(0;1)$ deb olamiz;

3. $f(0,1)=\ln(0^2+1^3)=\ln 1=0$;

4. $f'_x(x,y)=\dfrac{2x}{x^2+y^3},\ f'_y(x,y)=\dfrac{3y^2}{x^2+y^3}$ va $f'_x(0;1)=0,\ f'_y(0,1)=3;$

5. $f(x,y)\approx f(x_0,y_0)+f'_x(x_0,y_0)\Delta x+f'_y(x_0,y_0)\Delta y$ formuladan $\ln(0,09^2+0,99^3)=0+0\cdot(0,09-0)+3\cdot(0,99-1)=-0,03.$ natijaga ega bo`lamiz.

❖ Murakkab va oshkormas funktsiyalarning hosilasi

Quyida murakkab va oshkormas funktsiyalarning hosilasini olishni qaraymiz

4-Teorema. Biror D to`plamda $z = f(x, y)$ (bu yerda $x = x(u,v)$, $y = y(u,v)$) murakkab funktsiya aniqlangan va $x = x(u,v)$, $y = y(u,v)$ funktsiyalar $Q_0(u_0, v_0) \in D$ nuqtaning biror atrofida uzluksiz xususiy hosilalarga ega hamda $f(x, y)$ funktsiya $P_0(x_0, y_0)$ (bu yerda $x = x_0(u_0, v_0)$, $y = y_0(u_0, v_0)$) murakkab funktsiya $Q_0(u_0, v_0)$ nuqtada differensiallanuvchi bo`ladi va ushbu

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \tag{4}$$

tenglik o`rinli bo`ladi.

Bu tengliklar $\frac{\partial z}{\partial u}$ va $\frac{\partial z}{dv}$ xususiy hosilalar $Q_0(u_0, v_0)$ nuqtada uzluksiz bo`lishini ko`rsatadi. Bundan $z = f(x(u,v), y(u,v))$ murakkab funktsiyaning differensiallanuvchanligi kelib chiqadi.

Endi ikki o`zgaruvchining $z = f(x, y)$ funktsiyasi berilgan, shu bilan birga bu funktsiyaning argumentlari bitta $t$ o`zgaruvchining funktsiyasi, ya`ni $x = x(t),\ y = y(t)$ bo`lsin deymiz. Bunda (4) tengliklarning birida (masalan birinchisida) u o`zgaruvchi sifatida $t$ o`zgaruvchini olish mumkin. U holda u o`zgaruvchi bo`yicha xususiy hosilalar bir o`zgaruvchi t ning hosilalarini beradi. Shunday qilib,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \qquad (5)$$

Endi $y = y(x)$ shartda $z = f(x, y)$ funktsiyani qaraymiz. Bu yerda z o`zgaruvchi bitta $x$ o`zgaruvchining funktsiyasi, ya`ni $z = f(x, y(x))$. U holda (5) formulaga ko`ra

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

yoki

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x} \qquad (6)$$

(4), (5), (6) ifodalar istalgan chekli sondagi argumentlarning murakkab funktsiyalari uchun umumlashtirilishi mumkin.

Masalan, uch o`zgaruvchining $F = F(x, y, z)$ funktsiya uchun:

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial u}, \qquad \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial v}$$

$$\frac{\partial F}{\partial w} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial w} \qquad (7)$$

bu yerda $x = x(u, v, w), \; y = y(u, v, w), \; z = z(u, v, w);$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial t} \qquad (8)$$

bu yerda $x = x(t), \; y = y(t), \; z = z(t);$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} \qquad (9)$$

bu yerda $y = y(x), \; z = z(x).$

5-misol. Agar $z = x^3 y,\ \ x = u\sin v,\ \ y = v\cos u$ bo`lsa, $\frac{\partial z}{\partial u}$ va $\frac{\partial z}{\partial v}$ larni toping.

Yechish. Xususiy hosilalarni (4) tengliklardan foydalanib topamiz:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = 3x^2 y\cdot\sin v - x^3 v\sin u = x^2(3y\sin v - xv\sin u);$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = 3x^2 y\cdot u\cos v + x^3\cos u = x^2(3y\cos v + x\cos u).$$

6-misol. $F=\ln(x^2+y^2+z^2),\ x=\sin t,\ y=t+\cos t,\ z=t^2$ bo`lsa, $\dfrac{\partial F}{\partial t}$ ni

toping.

Yechish.

$$\frac{\partial F}{\partial t}=\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial t}=$$

$$=\frac{2x}{x^2+y^2+z^2}\cdot\cos t+\frac{2y}{x^2+y^2+z^2}\cdot(1-\sin t)+\frac{2z}{x^2+y^2+z^2}\cdot 2t=$$

$$=\frac{2}{x^2+y^2+z^2}(x\cos t+y-y\sin t+2xzt).$$

Ikkita x va y o`zgaruvchining ushbu $F(x,y)=0$ (10) tenglamasi berilgan bo`lsin. Agar x ning biror to`plamidagi har bir qiymatiga $x$ bilan birgalikda (10) tenglamani qanoatlantiruvchi yagona $y$ qiymat mos kelsa, u holda (10) tenglama bu to`plamda $y=\varphi(x)$ oshkormas funktsiyani aniqlaydi.

Quyida oshkormas funktsiyalarning hosilasini olish qoidalarini keltiramiz:

Ikki o`zgaruvchili $F(x,y)=0$ ko`rinishdagi oshkormas funktsiyaning hosilasi quyidagi tenglik bilan aniqlanadi.

$$\frac{dy}{dx}=-\frac{F_x'(x,y)}{F_y'(x,y)} \qquad (11)$$

Agar uch o`zgaruvchining $F(x,y,z)=0$ tenglamasi $z=\varphi(x,y)$ oshkormas funktsiyani aniqlasa va $F'_z(x,y,z)\neq 0$ bo`lsa, u holda $F(x,y,z)$ funktsiyaning x va y o`zgaruvchilar bo`yicha xususiy hosilalari shu kabi topiladi.

$$\frac{dz}{dx}=-\frac{F'_x(x,y,z)}{F'_z(x,y,z)}, \quad \frac{dz}{dy}=-\frac{F'_y(x,y,z)}{F'_z(x,y,z)} \qquad (12)$$

7-misol. $y\cos x-\sin(x-y)=0$ tenglama bilan oshkormas ko`rinishda berilgan $y(x)$ funktsiyaning hosilasini toping.

Yechish. Tenglamaning chap tomonini $F(x,y)$ orqali belgilaymiz va uning hususiy hosilalarini topamiz:

$$F'_x(x,y)=-y\sin x-\cos(x-y);\ F'_y(x,y)=\cos x+\cos(x-y).$$

demak,

$$\frac{dy}{dx}=-\frac{F'_x(x,y)}{F'_y(x,y)}=\frac{y\sin x+\cos(x-y)}{\cos x+\cos(x-y)}$$

8-misol. $\cos(x+z)+\frac{xy}{z}=0$ tenglama bilan oshkormas ko`rinishda berilgan $z(x,y)$ funktsiyaning hususiy hosilalarini toping.

Yechish. Tenglamaning o`ng tomonini $F(x,y,z)$ bilan belgilab, topamiz:

$$F'_x(x,y,z)=-\sin(x+z)+\frac{y}{z}=\frac{y-z\sin(x+z)}{z};$$

$$F'_y(x,y,z)=\frac{x}{z};\quad F'_z(x,y,z)=-\sin(x+z)-\frac{xy}{z^2}=-\frac{z^2\sin(x+z)+xy}{z^2}$$

Demak,

$$\frac{\partial z}{\partial x}=-\frac{F'_x(x,y,z)}{F'_z(x,y,z)}=\frac{(y-z\sin(x+z))z}{xy+z^2\sin(x+z)};\quad \frac{\partial z}{\partial y}=-\frac{F'_y(x,y,z)}{F'_z(x,y,z)}=\frac{xz}{xy+z^2\sin(x+z)}.$$

❖ Yuqori tartibli xususiy hosila va differensial

$P(x;y)$ nuqtada va uning biror atrofida aniqlangan $z = f(x,y)$ funktsiya shu atrofda x va y o`zgaruvchilar bo`yicha $\frac{\partial z}{\partial x} = f'_x(x,y), \frac{\partial z}{\partial y} = f'_y(x,y)$ xususiy hosilalarga ega bo`lsin. Ular birirnchi tartibli xususiy hosilalar deyiladi.

Bu hosilalar x va y o`zgaruvchilarning funktsiyalarini ifodalaydi. Shu sabali, bu funktsiyalar uchun xususiy hosilalarni toppish masalasini qo`yish mumkin.

Birinchi tartibli xususiy hosilalardan x va y o`zgaruvchilar bo`yicha olingan xususiy hosilalarga, agar ular mavjud bo`lsa, ikkinchi tartibli xususiy hosilalar deyiladi va quyidagicha belgilanadi:

$$\frac{\partial^2 z}{\partial x^2} \text{ yoki } f''_{x^2}(x,y),\ \frac{\partial^2 z}{\partial x \partial y} \text{ yoki } f''_{xy}(x,y),\ \frac{\partial^2 z}{\partial y \partial x} \text{ yoki } f''_{yx}(x,y),\ \frac{\partial^2 z}{\partial y^2} \text{ yoki } f''_{y^2}(x,y).$$

Shunday qilib to`rtta ikkinchi tartibli xususiy hosilalarga ega bo`ldik. Shu kabi sakkizta uchinchi tartibli hosilalar (ikkinchi tartibli xususiy hosilalardan olingan xususiy hosilalar)ni hosil qilish mumkin.

1-misol. $z = x^3 y^3$ funktsiyaning $\frac{\partial^3 z}{\partial x \partial y^2}$ xususiy hosilasini toping.

Yechish. $\frac{\partial z}{\partial x} = 3x^2 y^3,\ \frac{\partial^2 z}{\partial x \partial y} = (3x^2 y^3)'_y = 9x^2 y^2,\ \frac{\partial^3 z}{\partial x \partial y^2} = (9x^2 y^2)'_y = 18x^2 y.$

2-misol. $z = \sin(xy + y^2)$ funktsiyaning $\frac{\partial^2 z}{\partial x \partial y}$ va $\frac{\partial^2 z}{\partial y \partial x}$ xususiy

hosilalarni toping.

Yechish.

$$\frac{\partial z}{\partial x} = \cos(xy + y^2) \cdot y, \qquad \frac{\partial^2 z}{\partial x \partial y} = \left(y \cos(xy + y^2) \right)'_y =$$

$$= \cos(xy + y^2) - y \sin(xy + y^2) \cdot (x + 2y) = \cos(xy + y^2) - y(x + 2y) \sin(xy + y^2);$$

$$\frac{\partial z}{\partial y} = \cos(xy + y^2) \cdot (x + 2y), \qquad \frac{\partial^2 z}{\partial y \partial x} = \left((x + 2y) \cos(xy + y^2) \right)'_x =$$

$$= \cos(xy + y^2) - (x + 2y) \sin(xy + y^2) \cdot y = \cos(xy + y^2) - y(x + 2y) \sin(xy + y^2);$$

2-misolda $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ tenglikka ega bo`ldik. $\frac{\partial^2 z}{\partial x \partial y}$ va $\frac{\partial^2 z}{\partial y \partial x}$ hosilalarga ikkinchi tartibli aralash hususiy hosilalar deyiladi. Ikkinchi tartibli aralash hususiy hosilalar haqidagi quyidagi teoremani isbotsiz keltiramiz.

1-Teorema. Agar $z = f(x, y)$ funktsiyaning ikkinchi tartibli aralash xususiy hosilalari $P(x, y)$ nuqtaning biror atrofida mavjud va shu nuqtada uzluksiz bo`lsa, u holda ular shu nuqtada teng bo`ladi, ya`ni

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Bunday teorema istalgan tartibli hususiy hosilalar uchun ham o`rinli bo`ladi.

Masalan, uzluksiz uchinchi tartibli xususiy hosilalar uchun

$$\frac{\partial^3 z}{\partial x \partial y \partial z} = \frac{\partial^3 z}{\partial^2 y \partial x} = \frac{\partial^3 z}{\partial y \partial^2 x}.$$

Ma`lumki, $z = f(x, y)$ funktsiyaning $P(x, y)$ nuqtadagi to`liq differensiali ushbu

$$dz = f'_x(x, y)dx + f'_y(x, y)dy$$

ko`rinishda bo`ladi. U birirnchi tartibli to`liq differensial deb ataladi.

Ikkinchi tartibli to`liq differensial deb birinchi tartibli to`liq differensialdan olingan differensialga aytiladi va d^2z kabi belgilanadi.

Shunday qilib, ta`rifga ko`ra

$$d(dz) = d^2z = d\left[f'_x(x,y)dx + f'_y(x,y)dy \right] = \left[f'_x(x,y)dx + f'_y(x,y)dy \right]'_x dx +$$

$$+\left[f'_x(x,y)dx + f'_y(x,y)dy \right]'_y dy =$$

$$= f''_{x^2}(x,y)(dx)^2 + f''_{yx}(x,y)dydx + f''_{xy}(x,y)dxdy + f''_{y^2}(dy)^2.$$

Agar $f''_{xy}(x,y)$ *va* $f''_{yx}(x,y)$ aralash xususiy hosilalar uzluksiz bo`lsa,

u holda

$$d^2z = f''_{x^2}(x,y)(dx)^2 + 2f''_{xy}(x,y)dydx + f''_{y^2}(dy)^2 \quad (1)$$

Bu yerda $dx^2 = (dx)^2,\ dy^2 = (dy)^2.$

Uchinchi tartibli to`liq differensial shu kabi ta`riflanadi va aniqlanadi:

$$d^3z = f'''_{x^3}(x,y)dx^3 + 3f'''_{x^2y}(x,y)dx^2dy + 3f'''_{xy^2}(x,y)dxdy^2 + f'''_{y^3}(x,y)dy^3. \qquad (2)$$

Huddi shunday quyida n chi tartibli to`liq differensialni ta`riflanishini keltiramiz:

$$d^nz = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n z \qquad (3)$$

3-misol. $z = x^2 \sin y$ funktsiyaning d^2z va d^3z differensiallarni toping.

Yechish. $f(x, y) = x^2 \sin y$ deb, dastlab barcha ikkinchi tartibli xususiy hosilalarni topamiz:

$$f'_x(x, y) = 2x\sin y, \quad f''_{x^2} = 2\sin y, \; f''_{xy} = 2x\cos y,$$

$$f'_y(x, y) = x^2 \cos y, \; f'_{y^2}(x, y) = -x^2 \sin y$$

Demak, $d^2z = 2\sin y dx^2 + 4x\cos y dxdy - x^2 \sin y dy^2$.

Endi barcha uchinchi tartibli xususiy hosilalarni topamiz:

$$f'''_{x^3} = 0, \; f'''_{x^2y} = 2\cos y, \; f'''_{xy^2} = -2x\sin y, \; f'''_{y} = -x^2 \cos y.$$

Demak, $d^3z = 6\cos y dx^2 dy - 6x\sin y dxdy^2 - x^2 \cos y dy^3$.

❖ Yo`nalish bo`yicha hosila va gradiyent

$u = u(x, y, z)$ funktsiyaning $M(x, y, z)$ nuqtadagi $\vec{s}$ vektor yo`nalishi bo`yicha olingan hosilasi

$$\frac{\partial u}{\partial S} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma$$

formulaga asosan topiladi. Bu yerda $\cos\alpha, \cos\beta, \cos\gamma$ larga $\vec{s}$ vektorning yo`naltiruvchi kosinuslari deyiladi. Gradiyent

$$grad\,\vec{u} = \vec{i}\,\frac{\partial u}{\partial x} + \vec{j}\,\frac{\partial u}{\partial y} + \vec{k}\,\frac{\partial u}{\partial z}$$

formulaga ko`ra aniqlanadi. $\vec{i}, \vec{j}, \vec{k}$ larga ortlar deyiladi.

4-misol. $u = xy^2z^3$ funktsiyaning $M(3;2;1),\ N(5;4;2)$ dagi $\overrightarrow{MN}$ vektor yo`nalishi bo`yicha hosilasini toping.

$$\left.\frac{\partial u}{\partial x}\right|_M = 4, \quad \left.\frac{\partial u}{\partial y}\right|_M = 12, \quad \left.\frac{\partial u}{\partial z}\right|_M = 36.$$

$$\overrightarrow{MN} = \{2;2;1\}, \quad \left|\overrightarrow{MN}\right| = 3.\ \cos\alpha = \frac{2}{3},\ \cos\beta = \frac{2}{3},\ \cos\gamma = \frac{1}{3}.$$

$$\frac{\partial u}{\partial \overrightarrow{MN}} = 4\cdot\frac{2}{3} + 12\cdot\frac{2}{3} + 36\cdot\frac{1}{3} = \frac{68}{3} = 22\frac{2}{3}.$$

5-misol. $z = x^2y$ funktsiyaning $M(1;1)$ nuqtadagi gradiyentini toping.

$\left.\frac{\partial z}{\partial x}\right|_M = 2,\ \left.\frac{\partial z}{\partial y}\right|_M = 1$ larga ko`ra $grad\,\overline{z} = 2\vec{i} + \vec{j}$.

❖ Sirtga o`tkazilgan urinma va normal

Sirt tenglamasi $F(x,y,z)=0$ va unda yotgan $M(x;y;z)$ nuqta berilgan. $M(x;y;z)$ nuqtaga o`tkazilgan urinma tekislik tenglamasi:

$$\frac{\partial F}{\partial x}\left(X-x\right)+\frac{\partial F}{\partial y}\left(Y-y\right)+\frac{\partial F}{\partial \gamma}\left(X-x\right)=0$$

sirtga normal tenglamasi esa

$$\frac{X-x}{\frac{\partial F}{\partial x}}=\frac{Y-y}{\frac{\partial F}{\partial y}}=\frac{Z-z}{\frac{\partial F}{\partial z}}$$

ko`rinishga ega bo`ladi.

Bu yerda X, Y, Z lar urinma tekislik va normal uchun o`zgaruvchi koordinatalar. Agar biror nuqtada $\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0$ bo`lsa, u nuqta maxsus nuqta deyiladi. Bu nuqtada urinma tekislik ham, sirtga normal mavjud emas.

6-misol. $F(x,y,z)=x^2+2y^2-z$ ning $M(1;1;3)$ nuqtadagi urinma va normal tenglamalarini yozing.

$$\left.\frac{\partial F}{\partial x}\right|_M=2, \quad \left.\frac{\partial F}{\partial y}\right|_M=4, \quad \left.\frac{\partial F}{\partial z}\right|_M=-1$$

$2(X-x)+4(Y-y)-(Z-z)=0$ urinma tekislik tenglamasi.

$\frac{X-x}{2}=\frac{Y-y}{4}=\frac{Z-z}{-1}$ normal tekislik tenglamasi.

❖ Ikki o`zgaruvchili funktsiyaning ekstremumi

**1-Ta`rif.** $z = f(x, y)$ funktsiya va $P_0(x_0; y_0)$ nuqtaning biror atrofida va shu nuqtada uzluksiz bo`lsin. Agar bu atrofning barcha $P(x, y)$ nuqtalarida $f(x, y) < f(x_0, y_0)$ $\left(f(x, y) > f(x_0, y_0)\right)$ tengsizlik bajarilsa, u holda $P_0(x_0; y_0)$ nuqtaga $f(x, y)$ funktsiyaning maksimum (minimum) nuqtasi deyiladi.

Funktsiyaning maksimum va minimum nuqtalari ekstremum nuqtalar deb aytiladi.

Funktsiyaning ekstremum nuqtadagi qiymatiga funktsiyaning ekstremumi deyiladi.

2-Teorema. (*ekstremum mavjud bo`lishining zaruriy sharti*). Agar $z = f(x, y)$ funktsiya $P_0(x_0, y_0)$ nuqtada ekstremumga ega bo`lsa, u holda bu nuqtada $\frac{\partial z}{\partial x}$ *va* $\frac{\partial z}{\partial y}$ xususiy hosilalar nolga teng bo`ladi yoki ulardan aqalli bittasi mavjud bo`lmaydi.

7-misol. $f(x, y) = (x-2)^2 + (y-1)^2 - 1$ funktsiyaning ekstremumlarini toping.

Yechish. Xususiy hosilalarni topib, nolga tenglaymiz:

$f'_x(x,y)=2(x-2)=0,\ \ f'_y(x,y)=2(y-1)=0.$ Bundan $x_0=2,\ y_0=1$ ya`ni $P_0(2;1)$ nuqta ekstremum nuqta va $f(2;1)=-1.$ $(x;y)\neq(2;1)$ nuqtalarda $(x-2)^2+(y-1)^2>0$ va $f(x,y)>-1.$

Demak, $P_0(2;1)$ nuqta minimum nuqta va $f_{\min}(x,y)=-1.$

$P_0(x_0;y_0)$ nuqta $z=f(x,y)$ funktsiyaning ekstremum nuqtasi bo`lsin. U holda $f'_x(x_0,y_0)=0,\ f'_y(x_0,y_0)=0$ bo`ladi. Bu hosilalarni $z=f(x,y)$ tenglama bilan berilgan sirtga $P_0(x_0;y_0)$ nuqtada o`tkazilgan urinma tekislikning ushbu

$$z-z_0=f'_x(x_0,y_0)(x-x_0)+f'_y(x_0,y_0)(y-y_0)$$

Tenglamasiga qo`ysak, $z-z_0=0$ yoki $z=z_0$ kelib chiqadi.

Bundan ekstremum nuqtalarida sirtga o`tkazilgan urinma tekislik $Oxy$ koordinata tekisligiga parallel bo`ladi degan xulosa kelib chiqadi. Bu hulosa ikki o`zgaruvchi funktsiyasi ekstremumi zaruriy shartining geometrik ma`nosini bildiradi.

Bundan tashqari funktsiya differensiallanuvchi bo`lmagan nuqtalar ham uzluksiz funktsiyaning ekstremum nuqtalari bo`lishi mumkin. Masalan, grafigi markazi koordinatalar boshida yotgan va o`qi $Oz$ o`q bilan ustma-ust tushuvchi doiraviy konusdan iborat bo`lgan $z=\sqrt{x^2+y^2}$ funktsiya $O(0;0)$ nuqtada minimumga ega, ammo u bu nuqtada differensiallanuvchi emas.

Xususiy hosilalar nolga teng bo`ladigan nuqtalarga kritik nuqtalar deyiladi.

Ekstremum nuqta hamma vaqt kritik nuqta bo`ladi, ammo har qanday kritik nuqta ham ekstremum nuqta bo`lmasligi mumkin.

Shunday qilib, hususiy hosilalarning nolga teng bo`lishi ekstremum mavjud bo`lishining zaruriy sharti bo`ladi. Kritik nuqta ekstremum nuqta bo`lishi uchun ekstremum mavjud bo`lishining yetarli sharti bajarilishi lozim.

3-Teorema. (*ekstremum mavjud bo`lishining yetarli sharti*). $z = f(x,y)$ funktsiya $P_0(x_0, y_0)$ nuqtaning biror atrofida birirnchi va ikkinchi tartibli uzluksiz xususiy hosilalarga ega bo`lib, bunda $f'_x(x_0, y_0) = 0,\ f'_y(x_0, y_0) = 0$ hamda $f''_{x^2}(x_0, y_0) = A,\ f''_{xy}(x_0, y_0) = B,\ f''_{y^2}(x_0, y_0) = C$ bo`lsin. U holda

a) agar $\Delta = AC - B^2 > 0$ bo`lsa, $z = f(x, y)$ nuqtada ekstremumga ega bo`lib, bunda $A < 0$ (yoki $C < 0$) bo`lganda $P_0(x_0, y_0)$ nuqta maksimum nuqta, $A > 0$ (yoki $C > 0$) bo`lganda $P_0(x_0, y_0)$ nuqta minimum nuqta bo`ladi;

b) agar $\Delta = AC - B^2 < 0$ bo`lsa, $P_0(x_0, y_0)$ nuqtada ekstremum mavjud bo`lmaydi;

c) agar $\Delta = AC - B^2 = 0$ bo`lsa, $P_0(x_0, y_0)$ nuqtada ekstremum mavjud bo`lishi ham, mavjud bo`lmasligi ham mumkin.

Teoremani isbotini keltirmasdan, bu teoremaga (hamda 2-teoremaga) asoslangan algoritm bilan tanishamiz.

$z = f(x, y)$ funktsiyani ekstremumga tekshirish algoritmi:

1. $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ xususiy hosilalar topiladi;

2. Kritik nuqtalar aniqlanadi;

3. $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ xususiy hosilalar topiladi;

4. Ikkinchi tartibli hosilalarning kritik nuqtalardagi

$A=\frac{\partial^2 z}{\partial x^2},\ C=\frac{\partial^2 z}{\partial y^2}$ va B= $\frac{\partial^2 z}{\partial x \partial y}$ qiymatlari hisoblanadi;

5. Har bir kritik nuqtada $\Delta = AC - B^2$ ning qiymati hisoblanadi va 3-teorema asosida xulosa chiqariladi.

8-misol. $z = x^4 + y^4 - 4xy$ funktsiyani ekstremumga tekshiring.

Yechish. Funktsiya Oxy tekislikda aniqlangan.

1. $\frac{\partial z}{\partial x} = 4x^3 - 4y, \ \frac{\partial z}{\partial y} = 4y^3 - 4x;$

2. $\begin{cases} 4x^3 - 4y = 0, \\ 4y^3 - 4x = 0 \end{cases}$ sistemani yechib, kritik nuqtalarni topamiz.

Ular uchta $P_1(0;0),\ P_2(1;1),\ P_3(-1;-1);$

3. $\frac{\partial^2 z}{\partial x^2} = 12x^2, \ \frac{\partial^2 z}{\partial y^2}, \ \frac{\partial^2 z}{\partial x \partial y} = -4;$

4. Har bir kritik nuqtada ikkinchi tartibli hususiy hosilalarni hisoblaymiz:

a) $P_1(0;0)$ nuqtada $A_1 = 0,\ C_1 = 0,\ B_1 = -4;$

b) $P_2(1;1)$ nuqtada $A_2 = 12,\ C_2 = 12,\ B_2 = -4;$

c) $P_3(-1;-1)$ nuqtada $A_3 = 12,\ C_3 = 12,\ B_3 = -4.$

5. Har bir kritik nuqtada $\Delta = AC - B^2$ diskreminantni hisoblaymiz va 3-teorema asosida xulosa chiqaramiz:

a) $\Delta_1 = A_1C_1 - B_1^2 = -16 < 0.$ Demak, $P_1(0;0)$ nuqtada ekstremum mavjud emas;

b) $\Delta_2 = A_2C_2 - B_2^2 = 144 - 16 = 128 > 0$, bunda $A_2 > 0.$ Demak, $P_2(1;1)$ nuqta minimum nuqta va $z_{\min} = 1^4 + 1^4 - 4 \cdot 1 \cdot 1 = -2;$

c) $\Delta_3 = A_3C_3 - B_3^2 = 144 - 16 = 128 > 0$, bunda $A_3 > 0.$ Demak, $P_3(-1;-1)$ nuqta minimum nuqta va

$z_{\min} = (-1)^4 + (-1)^4 - 4 \cdot (-1) \cdot (-1) = -2.$

❖ Ikki o`zgaruvchili funktsiyaning eng katta va eng kichik qiymati

$z = f(x, y)$ funktsiya chegaralangan yopiq D sohada differensiallanuvchi bo`lsin. U holda funktsiya bu sohada eng katta va eng kichik qiymatlarga ega bo`lib, bu qiymatlarga yoki sohaning ichidagi kritik nuqtalarda yoki uning chegarasida erishadi.

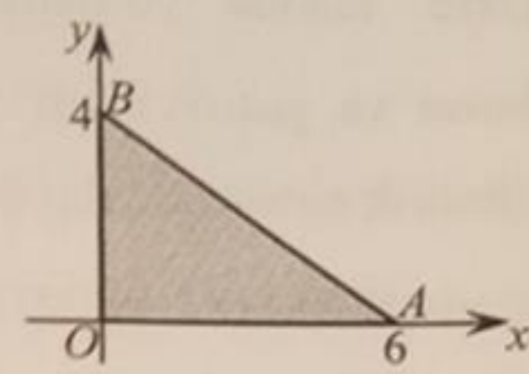
Shunday qilib, chegaralangan yopiq D sohada differensiallanuvchi $z = f(x, y)$ funktsiyaning eng katta va eng kichik qiymatlari quyidagi algoritm bilan topiladi:

1. Sohaning ichida va uning chegarasida yotgan barcha kritik nuqtalar topiladi;
2. Funktsiyaning bu nuqtalaridagi va soha chegarasining qismlari tutashgan nuqtalardagi qiymatlari hisoblanadi;
3. Hisoblangan qiymatlar orasidan eng katta va eng kichigi ajratiladi. Bu qiymatlar $z = f(x, y)$ funktsiyaning chegaralangan yopiq D sohadagi eng katta va eng kichik qiymatlari bo`ladi.

9-misol. $z = x^2 - xy + y^2 - 4x$ funktsiyaning $x = 0,\ y = 0$ va $2x + 3y - 12 = 0$ to`g`ri chiziqlar bilan chegaralangan sohadagi eng katta va eng kichik qiymatlarini toping.

Yechish. D sohada OAB uchburchakdan iborat (1-shakl).

1. $$\begin{cases} \dfrac{\partial z}{\partial x} = 2x - y - 4 = 0 \\ \dfrac{\partial z}{\partial y} = 2y - x = 0 \end{cases}$$ sistemadan soha ichidagi kritik nuqtalarni topamiz.



1-shakl.

Bundan $x=\frac{8}{3},\ y=\frac{4}{3}$. Demak, $P_0\left(\frac{8}{3};\frac{4}{3}\right)$.

Soha chegarasidagi kritik nuqtalarni topamiz:

a) OA to\`g\`ri chiziqda $y=0$ va $z=x^2-4x.$ U holda $z'_x=2x-4=0.$

Bundan $x=2.$ Demak, $P_1(2;0)$;

b) AB to\`g\`ri chiziqda $y=\frac{12-3x}{3}$ va $z=\frac{1}{9}(19x^2-120x+144).$

U holda $z'_x=38x-120=0.$ Bundan $x=\frac{69}{19}$. Demak, $P_2\left(\frac{69}{19};\frac{36}{19}\right)$;

c) BO to`g`ri chiziqda $x=0$ va $z=y^2$. U holda $z'_y=2y=0$. Bundan $O(0;0)$.

2. Funktsiyaning kritik nuqtalaridagi va soha chegarasining qismlari tutashgan nuqtalardagi qiymatlarini hisoblaymiz;

$$z_0=f(P_0)=f\left(\frac{8}{3},\frac{4}{3}\right)=-\frac{16}{3};\quad z_1=f(P_1)=f(2,0)=-4;$$

$$z_2=f(P_2)=f\left(\frac{60}{19},\frac{36}{19}\right)=-\frac{96}{19};$$

$$z_3=f(O)=f(0;0)=0;\ z_4=f(A)=f(6;0)=12;\ z_5=f(B)=f(0;4)=16.$$

3. Funktsiyaning hisoblangan qiymatlarini taqqoslaymiz.

Demak, $z_{\text{eng katta}}=f(B)=16\ va\ z_{\text{eng kichik}}=f(P_0)=-\dfrac{16}{3}.$

❖ Foydalanilgan adabiyotlar

1. Jurayev T va boshqalar. Oliy matematika asoslari, 1995 y.
2. Piskunov M.S. Differensial va integral hisob, 1-tom, 1972 y.
3. Rajabov F va boshqa. Oliy matematika, 2007 y.
4. Fayziboyev E.F., Suleymenov Z, Xudayarov B. Oliy matematikadan misol va masalalar to`plami. 2005 y.
5. Soatov Yo.U. Oliy matematika. Toshkent, O`zbekistan 1996 y.