Groundwater Hydraulics

Institute for Fluid Mechanics and Environmental Physics in Civil Engineering, Universität Hannover

Continuity equation

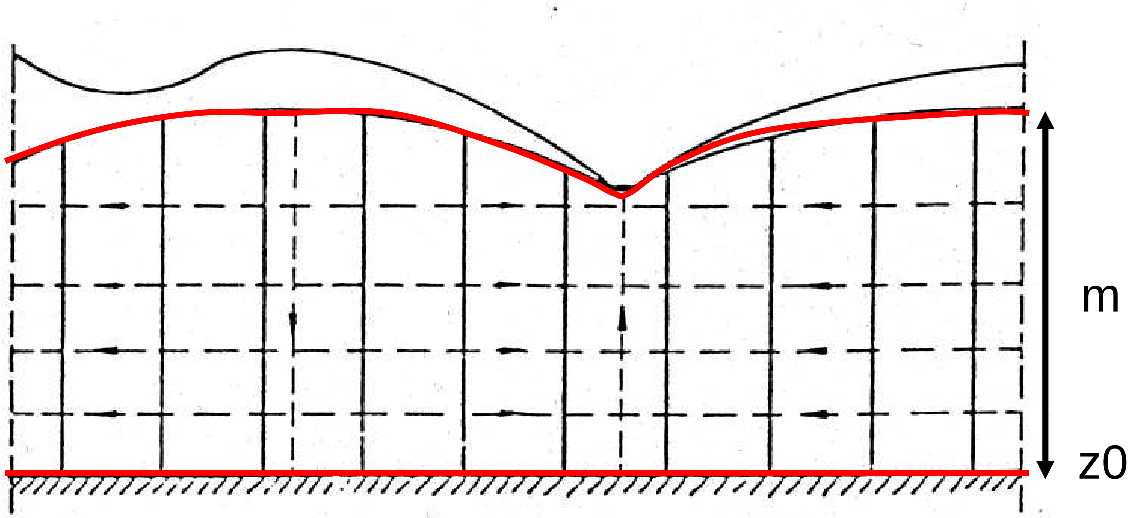
Groundwater flow equation in general

$$S_0 \frac{\partial h}{\partial t} - \vec{\nabla} \cdot \mathbf{K}_f \vec{\nabla} h = W_0$$

Continuity equation

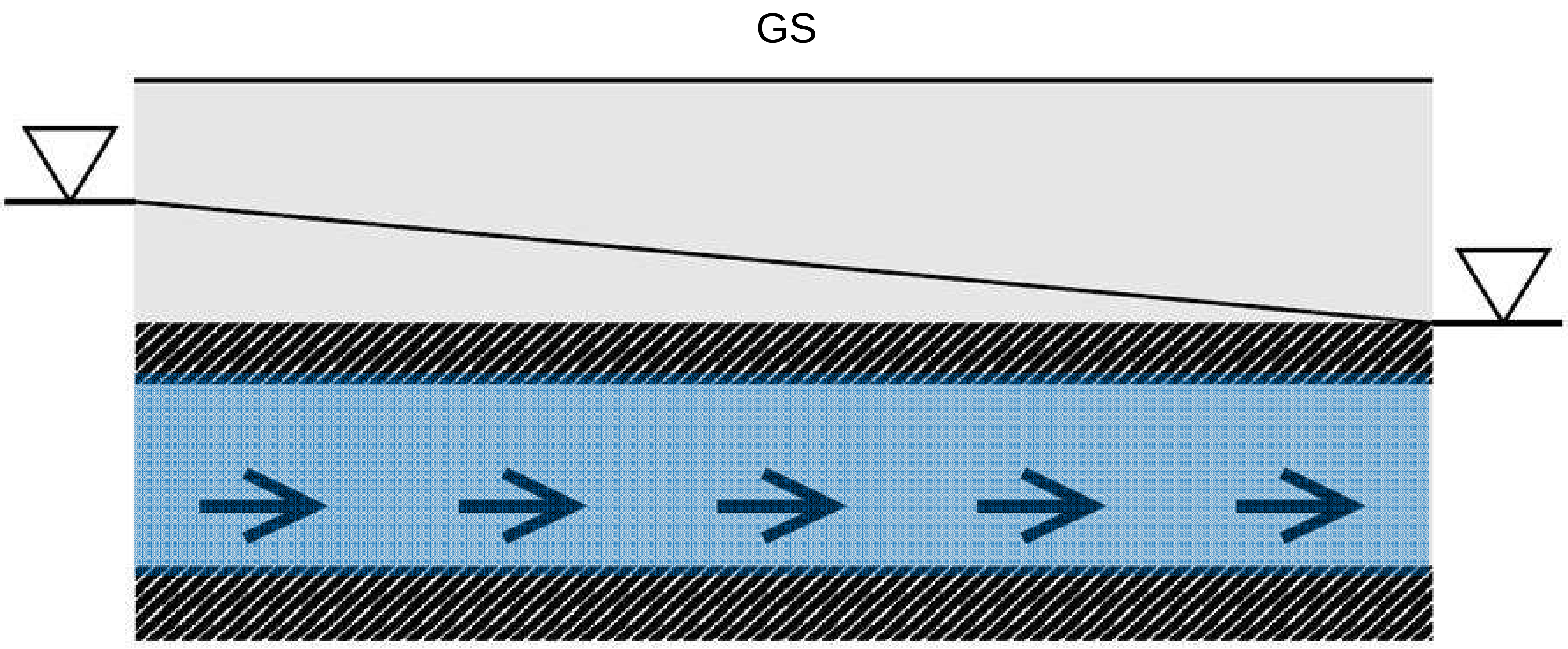
Averaging over depth

$$\int_{z_0}^{z_0+m} \left(S_0 \frac{\partial h}{\partial t} - \vec{\nabla} \cdot \left(k_f \vec{\nabla} h \right) - W_0 \right) dz = 0$$



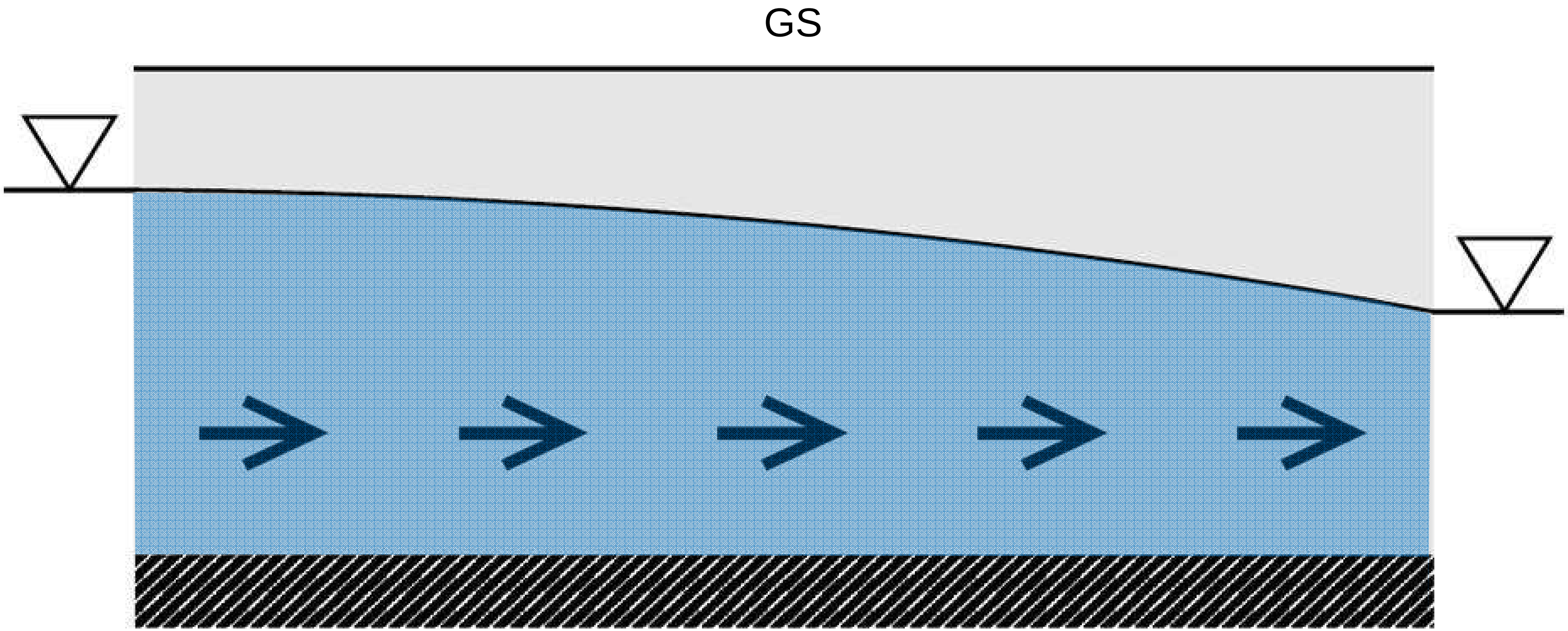
Continuity equation

Reminder aquifer types, confined aquifer:



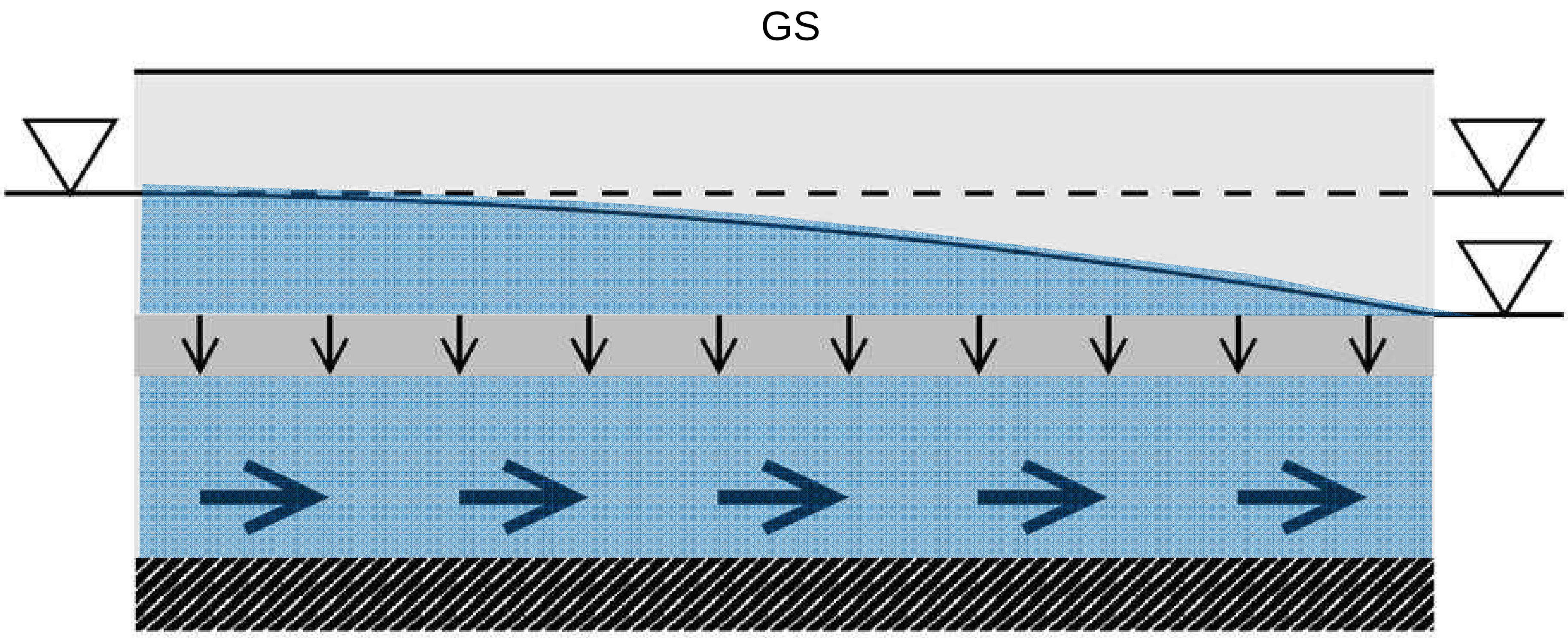
Aquifer types

Reminder aquifer types, unconfined (phreatic) aquifer:



Continuity equation

Reminder aquifer types, leaky (semi-confined) aquifer:



Continuity equation

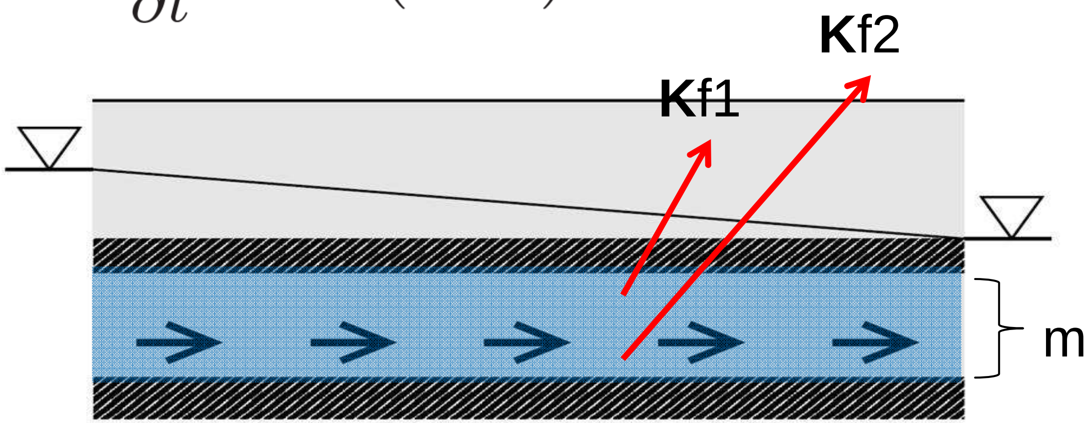
Integration of S0, Kf:

General groundwater flow equation

$$S_0 \frac{\partial h}{\partial t} - \vec{\nabla} \cdot \left(K_f \vec{\nabla} h \right) = W_0$$

becomes:

$$S \frac{\partial h}{\partial t} - \vec{\nabla} \cdot \left(T \vec{\nabla} h \right) = W$$



Storage coefficient S:
depth-integrated S_0

$$S = \int_0^m S_0(x,y)dz$$

Transmissivity Tensor **T**:
depth-integrated **K**f

$$\boldsymbol{T} = \int_0^m \boldsymbol{K}(x,y)dz$$

Source/sink W:
depth-integrated W_0

$$W = \int_0^m W_0 dz$$

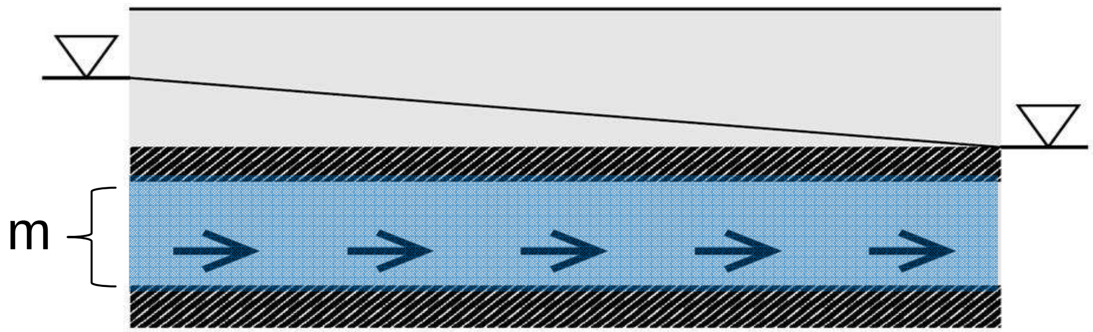
Continuity equation

Integrated groundwater flow equation: Confined aquifers

m constant and independent of h:

$$S = \overline{S_0} \cdot m$$

$$\boldsymbol{T} = \overline{K} \cdot m$$



ie, homogeneous, isotropic aquifer with constant depth m:

$$S\frac{\partial h}{\partial t} - \vec{\nabla} \cdot \left(T\vec{\nabla}h\right) = 0$$

$$S\frac{\partial h}{\partial t} - T\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial x^2}\right) = 0$$

Continuity equation

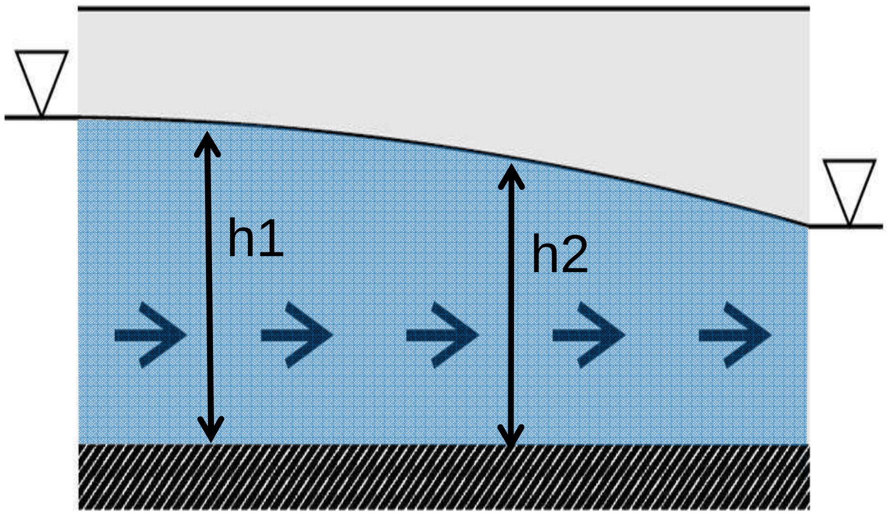
Integrated groundwater flow equation: Unconfined aquifers

m NOT constant, depends on h:

$$S = \varphi_f$$

$$\boldsymbol{T} = \overline{K} \cdot m = \overline{K} \cdot h$$

$$W = N$$



$$\varphi_f \frac{\partial h}{\partial t} - \frac{1}{2} \vec{\nabla} \cdot \left(\overline{K_f} \vec{\nabla} h^2 \right) = N$$

Continuity equation

Integrated groundwater flow equation: Leaky aquifers

m constant and independent of h,
extended source/sink term:

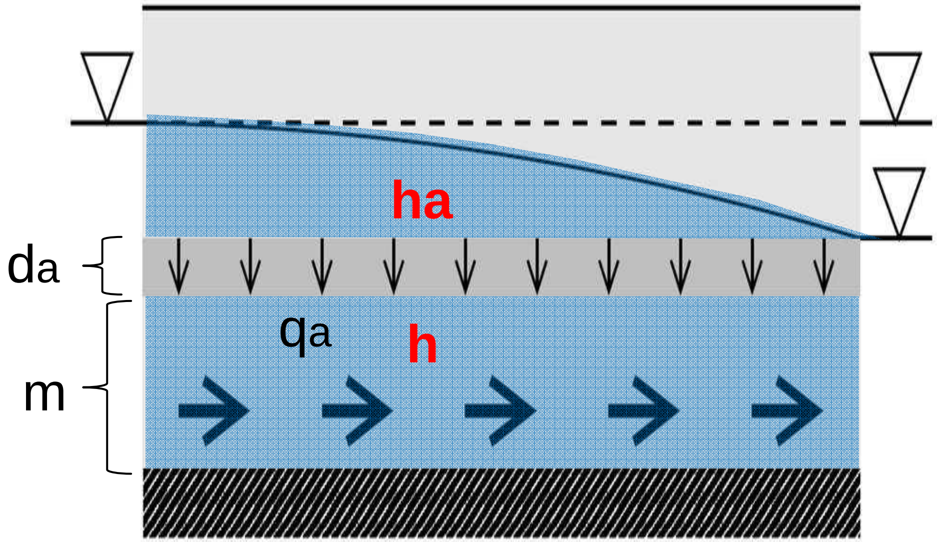
$$\boldsymbol{S} = \overline{S_0} \cdot m$$

$$\boldsymbol{T} = \overline{K} \cdot m$$

$$W = q_a + q_b$$

ie, with

$$q_a = -K_a \frac{h_a - h}{d_a}$$



Continuity equation

Integrated groundwater flow equation: Leaky aquifers

m constant and independent of h, extended source/sink term:

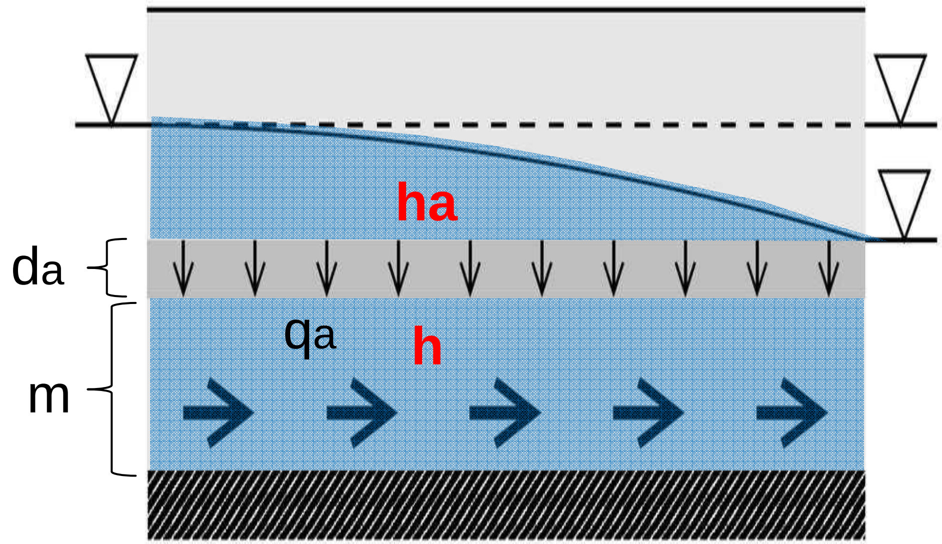
$$S = \overline{S_0} \cdot m$$

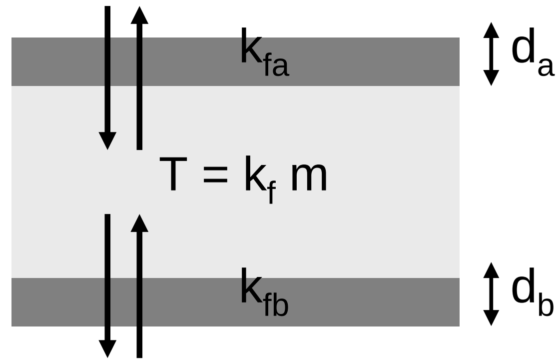
$$T = \overline{K} \cdot m$$

$$W = q_a + q_b$$

ie, with

$$q_a = -K_a \frac{h_a - h}{d_a} \qquad q_b = -K_b \frac{h_b - h}{d_b}$$

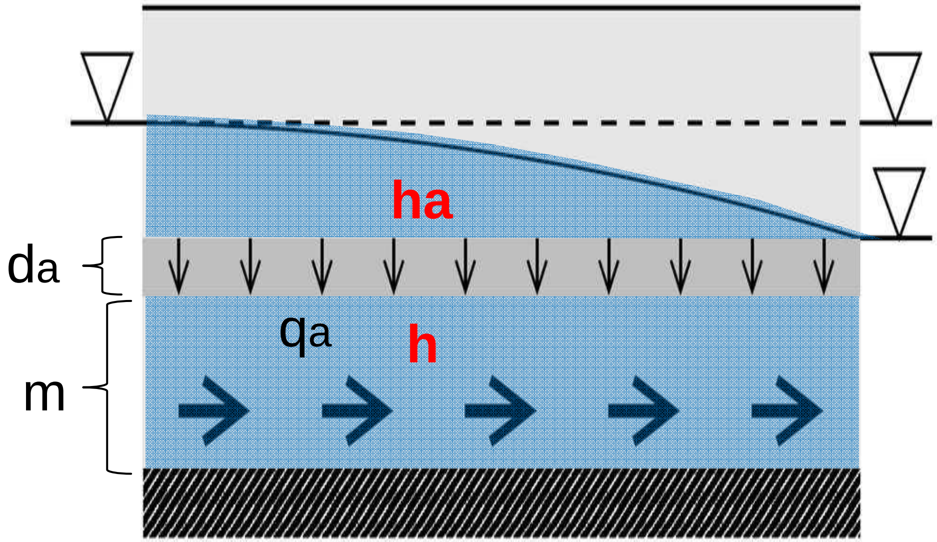




Continuity equation

Integrated groundwater flow equation: Leaky aquifers

m constant and independent of h, extended source/sink term:



$$S = \overline{S_0} \cdot m$$

$$T = \overline{K} \cdot m$$

$$W = q_a + q_b$$

$$\frac{S}{T}\frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} - \frac{\partial^2 h}{\partial y^2} - \frac{h_a - h}{\lambda_a^2} - \frac{h_b - h}{\lambda_b^2} = 0 \qquad \lambda_a = \sqrt{\frac{T d_a}{k_{f,a}}}$$

Continuity equation

Integrated groundwater flow equation summary:

Confined aquifers:

$$S\frac{\partial h}{\partial t} - \vec{\nabla} \cdot \left(T\vec{\nabla}h\right) = 0$$

Unonfined aquifers:

$$n_f\frac{\partial h}{\partial t} - \frac{1}{2}\vec{\nabla} \cdot \left(\overline{K_f}\vec{\nabla}h^2\right) = N$$

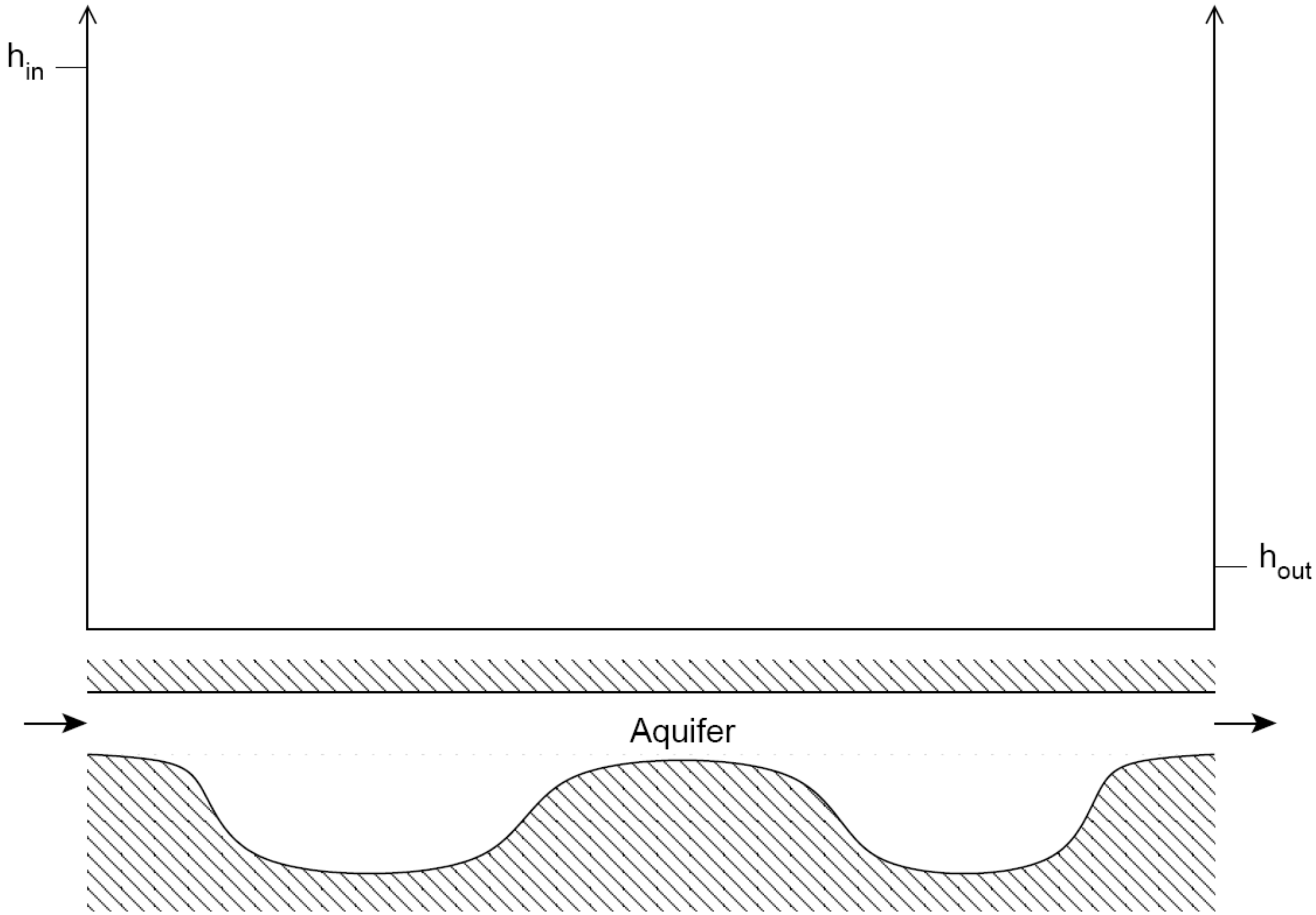
Leaky aquifers:

$$\frac{S}{T}\frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} - \frac{\partial^2 h}{\partial y^2} - \frac{h_a - h}{\lambda_a^2} - \frac{h_b - h}{\lambda_b^2} = 0$$

Continuity equation

Exercise #5

The figure shows an aquifer with an uneven bottom and an impermeable top layer. The hydraulic heads at the left- and right-hand side boundaries are given. Interpolate qualitatively the distribution of hydraulic head in between, assuming that the hydraulic conductivity K and the total discharge Q are uniform.



1d solutions of groundwater flow

Goal:

- Estimation of simple scenarios
- Estimation of time scales

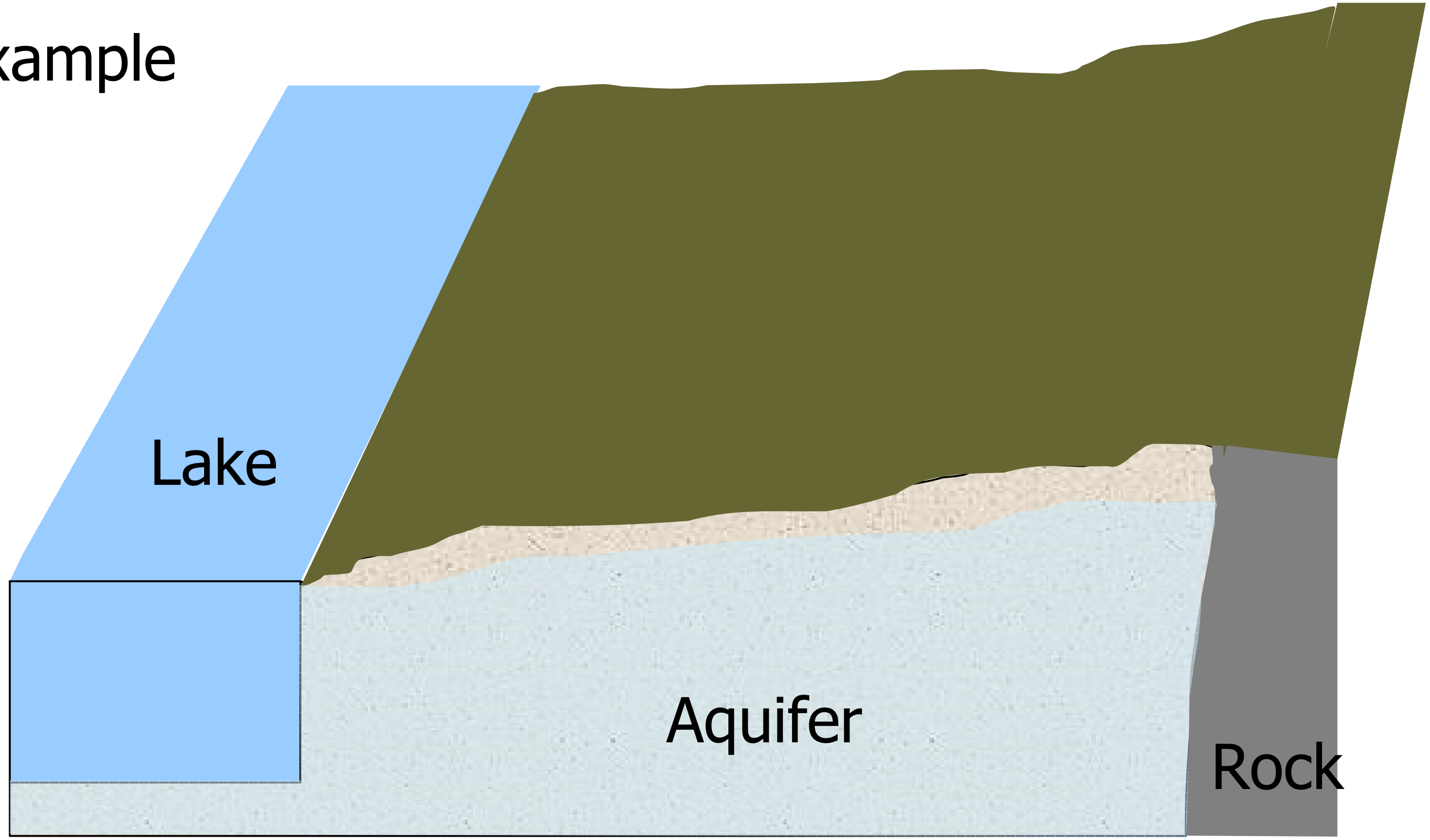
Procedure:

- Simplify as much as possible -> 2d auf 1d
- Classification of the aquifer
- Boundary conditions
- Parameters
- Solution of the problem

1d Solutions of groundwater flow

1d Systems, steady state flow

For example



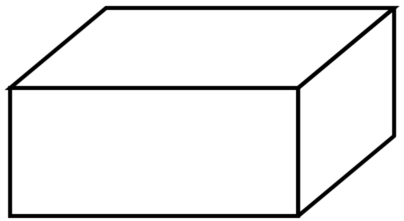
1d Solutions of groundwater flow

1d System

-> Consider only x-direction

- no time-dependent flow
- isotropic aquifer
- homogeneous aquifer

Boundary condition



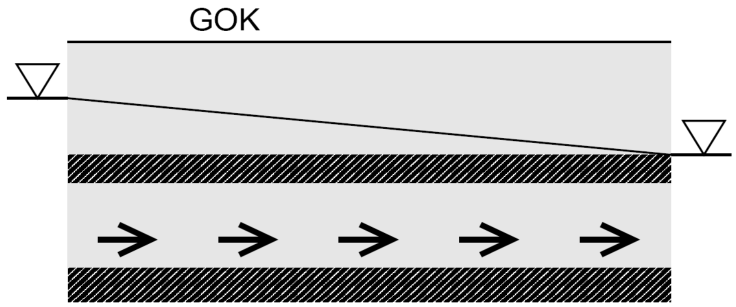
Boundary condition

General solution -> Boundary condition -> Specific solution

1d Solutions of groundwater flow

Confined aquifer

$$S\frac{\partial h}{\partial t} - \vec{\nabla}_{(x,y)} \cdot \left(T\vec{\nabla}_{(x,y)}h\right) = 0$$



$$T\frac{\partial^2 h}{\partial x^2} = 0,$$

$$q = -\frac{T}{m}\frac{\partial h}{\partial x}$$

Boundary conditions: $h(x=0) = h_0,\ h(x=L) = h_L$

1d Solutions of groundwater flow

Simplification of the equation:

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) = 0 \qquad \frac{\partial^2 h}{\partial x^2} = 0$$

General solution: $$\frac{\partial h}{\partial x} = K_1 \qquad h = K_1 x + K_2$$

Boundary conditions:

$$h(x = 0) = K_2 = h_0$$

$$h(x = L) = K_1 L + h_0 = h_L \qquad K_1 = \frac{h_L - h_0}{L}$$

1d Solutions of groundwater flow

Sepcific solution: confined aquifer

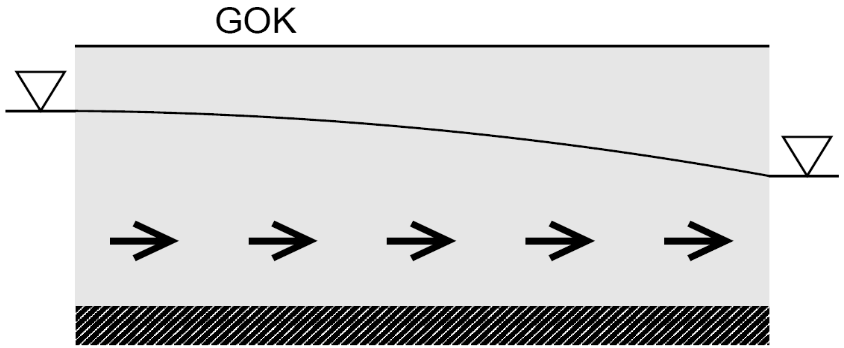
$$h(x) = h_0 + \frac{h_L - h_0}{L} x$$

$$Q(x) = -T \frac{h_L - h_0}{L}$$

1d Solutions of groundwater flow

Phreatic aquifer with recharge

$$n_f \frac{\partial h}{\partial t} - \frac{1}{2} \vec{\nabla}_{(x,y)} \cdot \left(\overline{k_f} \vec{\nabla}_{(x,y)} h^2 \right) = N$$



$$k_f \frac{\partial^2 h^2}{\partial x^2} = -2N,$$

$$q = -k_f \frac{\partial h}{\partial x}$$

Boundary conditions: $h(x=0) = h_0, \; h(x=L) = h_L$

1d Solutions of groundwater flow

$$k_f \frac{\partial^2 h^2}{\partial x^2} = -2N$$

General solution:

$$\frac{\partial h^2}{\partial x} = -\frac{2N}{k_f} x + K_1$$

$$h^2 = -\frac{N}{k_f} x^2 + K_1 x + K_2$$

1d Solutions of groundwater flow

Boundary conditions:

$$h(x=0) = \sqrt{K_2} = h_0$$

$$h(x=L) = \sqrt{-\frac{N}{k_f}L^2 + K_1 L + h_0^2} = h_L$$

$$K_1 = \frac{h_L^2 - h_0^2}{L} + \frac{N}{k_f}L$$

1d Solutions of groundwater flow

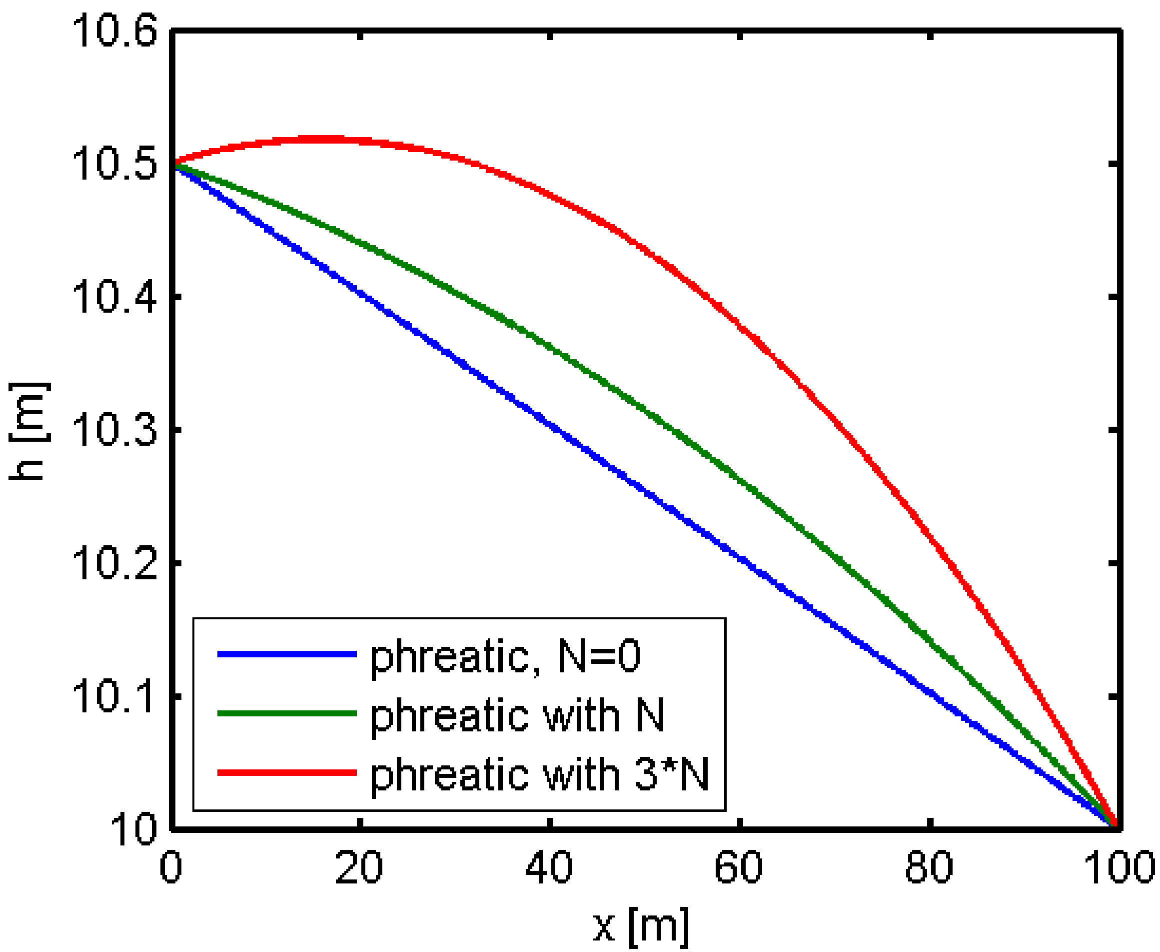
Specific solution: phreatic aquifer with recharge

$$h(x) = \sqrt{h_0^2 + \frac{N}{k_f} x(L - x) + (h_L^2 - h_0^2)\frac{x}{L}}$$

$$Q(x) = \frac{k_f (h_0^2 - h_L^2)}{2L} + N\left(x - \frac{L}{2}\right)$$

1d Solutions of groundwater flow

Piezometric head



1d Solutions of groundwater flow