

## Exercise 1: Linear Programming

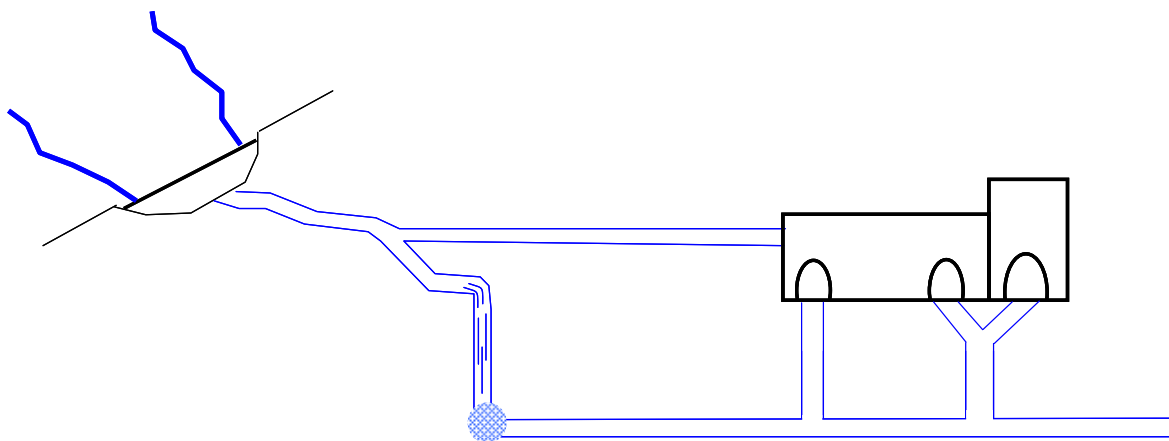
The following (linear) optimization problem has to be solved graphically:

Two power stations are located at a river with a constant discharge of  $18 \text{ m}^3/\text{s}$ . The older one of the two power stations houses two identical turbines, with a maximum capacity of  $4 \text{ m}^3/\text{s}$  each. Due to technical reasons the flow rate through both turbines has to be equal at all times. The new power station houses a single turbine with a higher capacity of  $6 \text{ m}^3/\text{s}$ . The degree of efficiency is equal for all three turbines and independent of the flow rate. Discharge control of the river is realized by a newly built reservoir upstream of the power stations, which began operation together with the new power station.

Not far from the power stations the river has a beautiful waterfall, which is a great tourist attraction for the region. Thus the responsible authority for approval set the condition when the first power station was built that at all times at least the same amount of water has to go down the waterfall as is diverted for power generation. Since the new reservoir allows a constant river discharge the condition has not been extended to the new power station, but it is still valid for the old one.

Because of spatial restrictions the outlet of the new power station has to be combined with the outflow of one of the older turbines. The water from those two turbines flows through a channel with a maximum discharge capacity of  $7.5 \text{ m}^3/\text{s}$ , before it joins the main river with the water from the waterfall and the third turbine again.

What is the maximum flow rate through the three turbines in compliance with the constraints?



Exercise 1 has to be solved arithmetically using the primal simplex algorithm. (The inactive constraint  $x_2 \leq 6$  is left out for faster calculation!)

## 1. Solution

The solution of an optimization problem of LP is always found on the edge of the feasible solutions. To calculate the optimal solution it would be possible to calculate the value of the objective function at each corner of the solution plane. In order to reduce the calculation effort especially for complex problems the simplex algorithm is used. It moves from a starting point along the best path towards the optimal solution.

The formulated problem can be solved with the primal simplex algorithm, for which the following assumptions are valid:

1. objective function has to be maximized
2. right hand side is not negative
3. constraints  $\leq$ , i.e. of the type  $a_1 \cdot x_1 + \dots + a_n \cdot x_n \leq \text{const.}$

In a first step all inequalities have to be transformed into equations by introduction of slack variables.

$$x_1 \leq 4 \quad \Rightarrow$$

$$4 \cdot x_1 + x_2 \leq 18 \quad \Rightarrow$$

$$x_1 + x_2 \leq 7,5 \quad \Rightarrow$$

$$x_1, x_2, y_1, y_2, y_3 \geq 0$$

The objective function has to be transformed as well. As with the constraints, here as well no variable is allowed on the right hand side.

$$Z = 2 \cdot x_1 + x_2 \quad \Rightarrow$$

The next step is to note the equations as matrix in a simplex tableau.

**Table 1: Tableau 1 (start tableau)**

BV	$x_1$	$x_2$	$y_1$	$y_2$	$y_3$	Z	RS

To calculate the values of the five variables (Z is not considered) three equations are available. To solve the system of equations, two variables are set to 0. Those are called the non-basic variables. The remaining variables constitute the basis and thus are called basic variables. They are noted in column BV. In column RS the right side of each equation is noted.

Tableau 1 can be read as follows. All non-basic variables = 0, i.e.  $x_1 = 0$  and  $x_2 = 0$ . The values of  $y_1$  through  $y_3$  can thus directly be read from the table:  $y_1 = 4$ ,  $y_2 = 18$  und  $y_3 = 7.5$ . The value of the objective function is 0.

Tableau 1 represents one feasible basic solution and therefore can be used as start tableau and the actual simplex algorithm can commence.

**Step 1: Identification of the pivot column**

To reach the optimal solution with minimum effort, the variable which exerts the highest influence on the objective function (the one with the highest negative coefficient) has to be introduced into the basis. In the present case this is  $x_1$ . Column  $x_1$  becomes the so-called pivot column.

**Step 2: Calculation of the quotient  $q_i$**

To maximize the value of the objective function  $x_1$  has to be maximized. To achieve this, the variable which limits the objective function the most has to leave the basis. The variable is found by dividing the right hand side by the coefficient in front of  $x_1$  for each row. The smallest positive value of  $q_i$  identifies the variable.

$$y_1 = 4 - x_1 \geq 0 \quad \Rightarrow$$

$$y_2 = 18 - 4 \cdot x_1 \geq 0 \quad \Rightarrow$$

$$y_3 = 7,5 - x_1 \geq 0 \quad \Rightarrow$$

**Step 3: Identification of pivot row and pivot element**

The calculation in step 2 results in  $y_1$  leaving the basis, and row  $y_1$  becomes the pivot row. The pivot element is the intersection of pivot row and pivot column.

**Step 4: Recalculation of the pivot row**

The pivot element has to become 1. Thus the pivot row is divided by the pivot element (just like a transformation in any other system of equations).

**Step 5: Recalculation of the remaining rows**

All other elements in the pivot column have to become 0. This is done by adding or subtracting multiples of the pivot row (again a common transformation in any system of equations). The result is a new tableau (table 2) and a new intermediate solution:  $x_1 = 4$ ,  $x_2 = 0$ ,  $y_1 = 0$ ,  $y_2 = 2$ ,  $y_3 = 3,5$  and  $Z = 8$ .

**Table 2: Tableau 2**

BV	$x_1$	$x_2$	$y_1$	$y_2$	$y_3$	Z	RS

**Step 6: Check if optimal solution is found**

As long as negative coefficients are found in row Z the optimal solution has not yet been reached. The simplex algorithm has to continue with iteration back to step 1.

For the next tableau variable  $x_2$  has to be introduced into the basis and  $y_2$  has to leave it. After transformation the resulting tableau at the end of the second calculation cycle is shown in table 3.

**Table 3: Tableau 3**

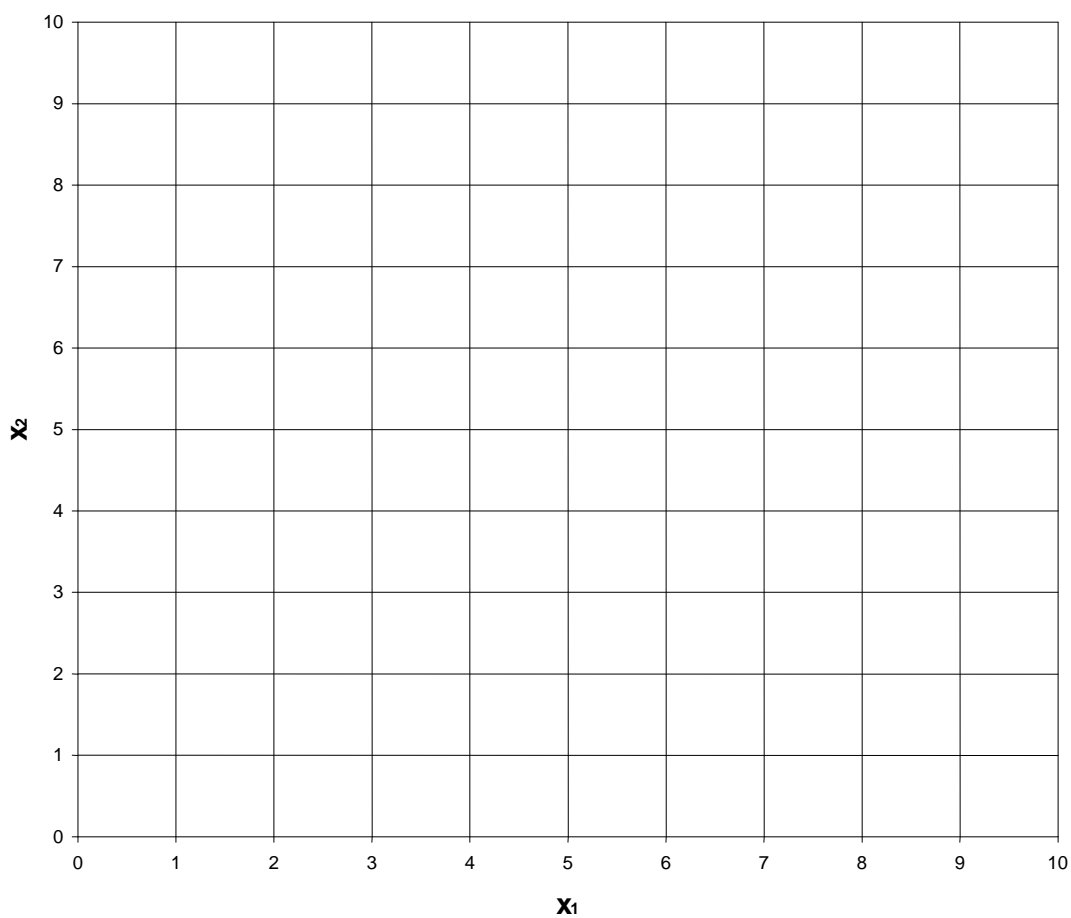
BV	$x_1$	$x_2$	$y_1$	$y_2$	$y_3$	Z	RS
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After the third calculation cycle the optimal solution is found. It can be read from tableau 4.

**Table 4: Tableau 4 (optimal tableau)**

BV	$x_1$	$x_2$	$y_1$	$y_2$	$y_3$	Z	RS

The solution is  $x_1 = 3.5$ ,  $x_2 = 4$ ,  $y_1 = 0.5$ ,  $y_2 = 0$ ,  $y_3 = 0$  und  $Z = 11$ . The pathway to the optimal solution is shown in figure 3.



**Figure 1: Pathway of the simplex algorithm towards the optimal solution**

## Exercise 2: Dynamic Programming

The release  $x_k$  from a reservoir (Fig. 1) has to be optimized using Dynamic Programming in order to maximize the profit from power generation.

The inflow  $QZ_k$  is shown in Tab. 1 and the profit from power generation  $r_k(x_k)$  is shown in Tab. 2. All values for the release  $x_k$ , the storage volume  $s_k$ , and the inflow  $QZ_k$  are given in  $10^7 \text{ m}^3$ . The profit from power generation  $r_k(x_k)$  is given in monetary units (MU).

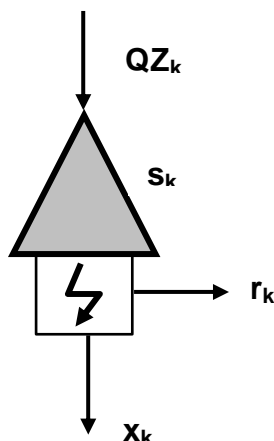


Figure 1: schematic sketch of the system

Table 1: inflow QZ

quarter	1	2	3	4
inflow	2	4	6	4

Table 2: profit from power generation r

release x	quarter			
	1	2	3	4
0	0.0	0.0	0.0	0.0
2	1.9	2.2	1.7	1.6
4	3.9	4.1	3.6	3.4
6	4.8	5.9	5.0	4.5

During reservoir operation the following constraints have to be met:

- maximum release:  $0 \leq x_k \leq 6$
- storage capacity:  $0 \leq s_k \leq 8$
- initial storage volume:  $s_0 = s_4 = 6$



Table 4: backward-moving calculation step 2,  $k=3$ ,  $QZ_3=6$ 

initial storage $s_{k-1}$	total volume $s_{k-1}+QZ_k$	possible releases $x_k$	profit from release $r_k(x_k)$	final storage $s_k$	partial cumul. profit $R^*_k$	cumul. profit $r_k(x_k)+R^*_k$	max. profit $R^*_{k-1}$	opt. release $x^*_k$
<b>S<sub>2</sub></b>	<b>S<sub>2</sub>+QZ<sub>3</sub></b>	<b>X<sub>3</sub></b>	<b>r<sub>3</sub>(X<sub>3</sub>)</b>	<b>S<sub>3</sub></b>	<b>R<sup>*</sup><sub>3</sub></b>	<b>r<sub>3</sub>(X<sub>3</sub>) + R<sup>*</sup><sub>3</sub></b>	<b>R<sup>*</sup><sub>2</sub></b>	<b>X<sup>*</sup><sub>3</sub>(S<sub>2</sub>)</b>
<b>0</b>		0						
		2						
		4						
		6						
<b>2</b>		0						
		2						
		4						
		6						
<b>4</b>		2						
		4						
		6						
<b>6</b>		4						
		6						
<b>8</b>		6						

Table 5: backward-moving calculation step 3,  $k=2$ ,  $QZ_2=4$ 

initial storage $s_{k-1}$	total volume $s_{k-1}+QZ_k$	possible releases $x_k$	profit from release $r_k(x_k)$	final storage $s_k$	partial cumul. profit $R^*_k$	cumul. profit $r_k(x_k)+R^*_k$	max. profit $R^*_{k-1}$	opt. release $x^*_k$
<b>S<sub>1</sub></b>	<b>S<sub>1</sub>+QZ<sub>2</sub></b>	<b>X<sub>2</sub></b>	<b>r<sub>2</sub>(X<sub>2</sub>)</b>	<b>S<sub>2</sub></b>	<b>R<sup>*</sup><sub>2</sub></b>	<b>r<sub>2</sub>(X<sub>2</sub>) + R<sup>*</sup><sub>1</sub></b>	<b>R<sup>*</sup><sub>1</sub></b>	<b>X<sup>*</sup><sub>2</sub>(S<sub>1</sub>)</b>
<b>0</b>	4	0	0,0	4	7,0	7,0		
		2	2,2	2	5,2	7,4		
		4	4,1	0	3,6	7,7	7,7	4
<b>2</b>	6	0	0,0	6	8,4	8,4		
		2	2,2	4	7,0	9,2		
		4	4,1	2	5,2	9,3		
		6	5,9	0	3,6	9,5	9,5	6
<b>4</b>	8	0						
		2						
		4						
		6						
<b>6</b>	10	2	2,2	8	9,5	11,7		
		4	4,1	6	8,4	12,5		
		6	5,9	4	7,0	12,9	12,9	6
<b>8</b>	12	4	4,1	8	9,5	13,6		
		6	5,9	6	8,4	14,3	14,3	6

**Table 6: backward-moving calculation step 4,  $k=1$ ,  $QZ_4=2$**

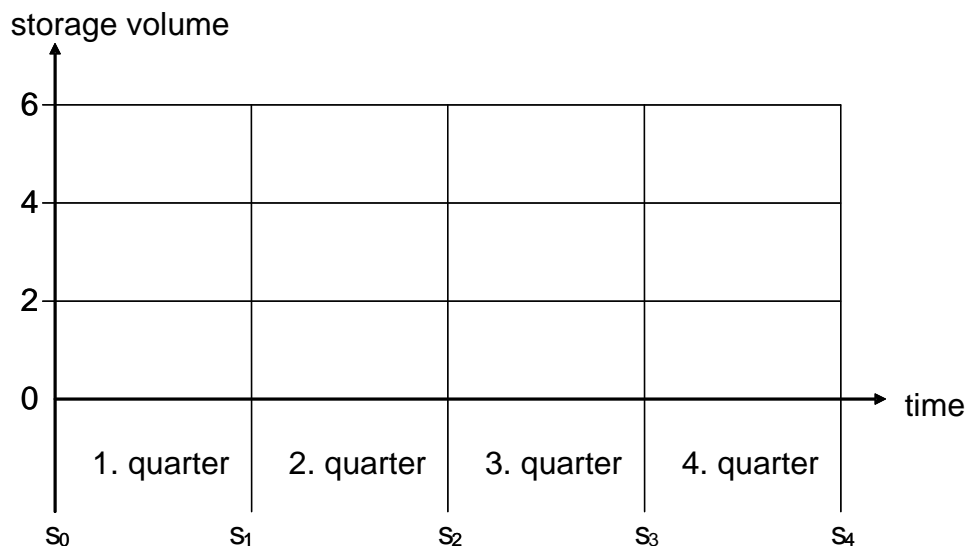
initial storage $s_{k-1}$	total volume $s_{k-1}+QZ_k$	possible releases $x_k$	profit from release $r_k(x_k)$	final storage $s_k$	partial cum. profit $R^*_k$	cumul. profit $r_k(x_k)+R^*_k$	max. profit $R^*_{k-1}$	opt. release $x^*_k$
$s_0$	$s_0+QZ_1$	$x_1$	$r_1(x_1)$	$s_1$	$R^*_1$	$r_1(x_1)+R^*_1$	$R^*_0$	$x^*_1(s_0)$
6	8	0	0,0	8	14,3	14,3	15,0	4
		2	1,9	6	12,9	14,8		
		4	3,9	4	11,1	15,0		
		6	4,8	2	9,5	14,3		

**4. Forward-moving procedure**

Starting from the calculated optimum of the first quarter's objective function value (last step in the backward-moving calculation) the path of best decisions can be determined along the optima of the following quarters (in the backward-moving calculation the previous steps).

**Table 7: Optimal operation of the reservoir**

		solution 1			solution 2		
quarter	step	$s_i$	$x_i$	$s_{i+1}$	$s_i$	$x_i$	$s_{i+1}$
1	4						
2	3						
3	2						
4	1						



**Figure 2: Optimal releases from the forward-moving procedure**