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Equations of motion of mechanical systems with nonlinear nonholonomic servoconstraints

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Abstract. The proposed paper is devoted to the derivation of equation of motion for the systems with nonholonomic servoconstraints, taking into account the release from servoconstraints containing only tangential components of reaction force of servoconstraints taken as control parameters. Chaplygin's problems and a gyroscope in gimbal mount are considered.

1. Introduction

The concept of servoconstraints was first introduced by A. Begen [1]. A feature of A. Begen's systems with servoconstraints is that for such systems the way to realize the constraints is necessarily taken into account, and the elementary work of servoconstraints reactions under virtual displacements allowed by the constraints is not zero.

The aim of the paper is to control the motion of the system under consideration with the reactions of servoconstraints, realized by servomotors. A servomotor is a simple operating element that is part of industrial equipment. A servomotor is a special electric motor with negative feedback that is designed for use in machines with a digital control program (DCP), in production lines, and many other designs. Servomotors have a fairly high speed characteristics, as well as high positioning accuracy. With proper operation, the servomotor is able to operate 24 hours a day [2].

2. Literature review

Modern servomotors have combined all the achievements of scientific and technological innovative progress, therefore they are able to develop tremendous rotational speeds at very high capacity A wide range of adjustment of shaft rotation of servomotor by software at significant acceleration or braking makes this equipment simply indispensable for use in machines or production lines and many other designs.

The studies by A.G. Azizov [3 - 5] were devoted to the development of methods of analytical dynamics of systems with servoconstraints. He developed a constructive method for determining the reaction forces of servoconstraints, containing mismatches and allowing restoring the broken constraints. To study the systems with servoconstraints, he used the theory of parametric release, developed for a class of mechanical systems by N.G. Chetaev [6] and V.I. Kirgetov [7]. In [8], the dynamics of systems with servoconstraints was discussed, when the constraints were realized by controlling the inertial properties of the system. It was shown that the presence of symmetries allowed reducing dynamic equations to a closed system of differential equations with quadratic right-hand sides. In [9], the methods to determine the reaction forces of servoconstraints at given displacements, at which the work of reaction force of servoconstraints is equal to zero, are shown, i.e. servoconstraints are ideal for these displacements.

In the monograph [10], the state of nonholonomic systems with high-order constraints, including systems with servoconstraints, was considered. The concept of a Hertzian representation point and the tangential space to the variety of all possible positions of a mechanical system at a given time was accepted as a mathematical tool; it allows us to consider from a single position the general issues of nonholonomic mechanics of material point systems and an arbitrary mechanical system. Further, the obtained mathematical tool was developed in the classical theory of motion of nonholonomic systems with constraints of any order, used in the study of a number of problems in control theory. However, in that work, explicit forms of the tangential and normal components of servoconstraints were not distinguished. Please follow these instructions as carefully as possible so all articles within a conference have the same style to the title page. This paragraph follows a section title so it should not be indented.

3. Discussion

In this paper, in contrast to the aforementioned ones, we consider the problem of deriving the extended equations of motion of mechanical systems with non-ideal nonholonomic servoconstraints taking into account the release from servoconstraints; the work of the servoconstraints reactions on virtual displacements is nonzero for both non-released and parametric servo-free system. A system of equations is obtained for such systems; it contains only tangential components of the reaction forces of servoconstraints.

Let the position of mechanical system be determined by generalized coordinates $q_1, q_2, ..., q_n$. Assume that the system is superimposed with d of nonlinear non-ideal nonholonomic servoconstraints of the form

$$\psi_\sigma(q_i, \dot{q}_i, t) = 0 \ ; \quad (\sigma = 1,2,..., d), \tag{1}$$

Introduce kinematic characteristics by the relationships

$$e_\nu = e_\nu(q_i, \dot{q}_i, t) \ , \quad (\nu = 1,2,..., k; \quad k = n - d) \tag{2}$$

which are identically satisfied by the relationships

$$\dot{q}_i = \dot{q}_i(q_j, e_\nu, t), \ (i, j = 1,2,..., n; \quad \nu = 1,2,..., k; \quad k = n - d) \tag{3}$$

In order to parametrically release the system from servoconstraints, introduce the independent parameters

$$\xi_\sigma = \psi_\sigma(q_i, \dot{q}_i, t); \quad (\sigma = 1,2,..., d \ ; \ \mathrm{i} = 1,2,..., \mathrm{n})$$

characterizing the deviation of the system from servoconstraints.

Assume that the work of servoconstraints reactions on virtual displacements satisfying the conditions

$$\sum_i \frac{\partial \psi_\sigma}{\partial \dot{q}_i} \delta q_i = 0$$

are nonzero for both non-released and parametric servo-free systems.

Denoting $e_{k+\sigma} = \xi_\sigma$, instead of relations (3), we can write

$$\dot{q}_i = \dot{q}_i(q_j, e_\nu, e_{k+\sigma}, t) \tag{4}$$

$$(i, j = 1,2,...,n \;\; ; \;\; \nu = 1,2,...,\mathrm{k} \;\; ; \;\; \mathrm{k = n - d})$$

For variations (4), we obtain the dependences

$$\delta q_i = \sum_\nu \frac{\partial \dot{q}_i}{\partial e_\nu}\delta\pi_y + \sum_\sigma \frac{\partial \dot{q}_i}{\partial e_{k+\sigma}}\delta\pi_{k+\sigma} \quad , \tag{5}$$

where $\delta\pi_{k+\sigma}$ are the variations corresponding to the release parameters.

Substituting relations (5) into the general equation of dynamics in generalized coordinates

$$\sum_i \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i - R_i \right]\delta q_i = 0 \tag{6}$$

or

$$\sum_i \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \right]\delta q_i = \sum_i \left[Q_i + \sum_\sigma \lambda_\sigma \frac{\partial \psi_\sigma}{\partial \dot{q}_i} + \sum_s \mu_s \frac{\partial \dot{q}_i}{\partial e_s} \right]\delta q_i \, .$$

we obtain differential equations of motion of a system of the Chaplygin type equations:

$$\frac{d}{dt}\cdot\frac{\partial \tilde{T}}{\partial e_\nu} - \frac{\partial \tilde{T}}{\partial \pi_\nu} - \sum \frac{\partial T}{\partial \dot{q}_l}\left(\frac{d}{dt}\cdot\frac{\partial \dot{q}_l}{\partial e_\nu} - \frac{\partial \dot{q}_l}{\partial \pi_\nu} \right) = \tilde{Q}_\nu + \sum_s \mu_s A_{\mu s} \tag{7}$$

$$\frac{d}{dt}\cdot\frac{\partial \tilde{T}}{\partial e_{k+\sigma}} - \frac{\partial \tilde{T}}{\partial \pi_{k+\sigma}} - \sum_i \frac{\partial T}{\partial \dot{q}_l}\left(\frac{d}{dt}\cdot\frac{\partial \dot{q}_l}{\partial e_{k+\sigma}} - \frac{\partial \dot{q}_l}{\partial \pi_{k+\sigma}} \right) = \tilde{Q}_{k+\sigma} + \lambda_{k+\sigma} + \sum_s \mu_s B_{k+\sigma,s}$$

in which the left parts coincide with those obtained in [11].

Here

$$\tilde{T} = \tilde{T}\left(q_j, e_\nu, e_{k+\sigma}, t\right), \quad \frac{\partial}{\partial \pi_y} = \sum_i \frac{\partial \dot{q}_i}{\partial e_\nu}\frac{\partial}{\delta q_i} \quad ,$$

$$\frac{\partial}{\partial \pi_{k+\sigma}} = \sum_i \frac{\partial \dot{q}_i}{\partial e_{k+\sigma}}\cdot\frac{\partial}{\partial q_i} \quad , \quad \tilde{Q}_\nu = \sum_i Q_i \frac{\partial \dot{q}_i}{\partial e_\nu} \quad ,$$

$$\tilde{Q}_{k+\sigma} = \sum_i Q_i \frac{\partial \dot{q}_i}{\partial e_{k+\sigma}} \;, \quad A_{\nu s} = \sum_i \frac{\partial \dot{q}_i}{\partial e_\nu} \cdot \frac{\partial \dot{q}_i}{\partial e_s}, \quad B_{k+\sigma,s} = \sum_i \frac{\partial \dot{q}_i}{\partial e_s} \cdot \frac{\partial \dot{q}_i}{\partial e_{k+\sigma}} \;,$$

$$(\nu, s = 1,2,...,k \;; \quad \sigma = 1,2,...,\mathrm{d} \;)$$

Equations (7) are reduced to Voronets-Hamel type equations:

$$\frac{d}{dt} \cdot \frac{\partial \tilde{T}}{\partial e_\nu} - \frac{\partial \tilde{T}}{\partial \pi_\nu} + \sum_\tau \frac{\partial \tilde{T}}{\partial e_\tau} \cdot W_\nu^\tau = \tilde{Q}_\nu + \sum_s \mu_s A_{\mu s} \tag{8}$$

$$\frac{d}{dt} \cdot \frac{\partial \tilde{T}}{\partial e_{k+\sigma}} - \frac{\partial \tilde{T}}{\partial \pi_{k+\sigma}} - \sum_\tau \frac{\partial \tilde{T}}{\partial e_\tau} \cdot W_{k\cdot\sigma}^\tau = \tilde{Q}_{k+\sigma} + \lambda_{k+\sigma} + \sum_s \mu_s A_{k+\sigma,s}$$

where

$$W_\nu^\tau = -\sum_i \frac{\partial e_\nu}{\partial \dot{q}_i} \left(\frac{d}{dt} \frac{\partial \dot{q}_i}{\partial e_\nu} - \frac{\partial \dot{q}_i}{\partial \pi_\nu} \right) = \sum_i \frac{\partial \dot{q}_i}{\partial e_\nu} \left(\frac{d}{dt} \frac{\partial e_\tau}{\partial \dot{q}_i} - \frac{\partial e_\tau}{\partial \dot{q}_i} \right),$$

$$W_{k+\sigma}^\tau = -\sum_i \frac{\partial e_\nu}{\partial \dot{q}_i} \left(\frac{d}{dt} \frac{\partial \dot{q}_i}{\partial e_{k+\sigma}} - \frac{\partial \dot{q}_i}{\partial \pi_{k+\sigma}} \right) \quad .$$

With the exact implementation of servoconstraints (1), the first *k* equations (8) together with (4) make it possible to determine the motion, and the last d equations (8) serve to determine the reactions of nonholonomic servoconstraints (1), when the values of μ_s are given. Then the task is solved completely. If μ_s is not set in advance, then the problem of stabilization or optimal stabilization of the motion of mechanical systems with servoconstraints can be set.

Now let the system be superimposed with non-ideal stationary nonholonomic constraints, expressed by the equations *a*

$$\sum_i a_{\alpha i} \dot{q}_i = 0; \quad (\alpha = 1,2,...,a \,,\; i = 1,2,...,n\,). \tag{9}$$

imposing the following conditions on virtual displacements

$$\sum_i a_{\alpha i} \delta q_i = 0. \tag{10}$$

Introduce the release parameters ξ_α, taking

$$\xi_\alpha = \sum_i a_{\alpha i} \dot{q}_i \;; \quad \alpha = 1,2,...,a \;; i \;\; = 1,2,...,\mathrm{n} \;\;)$$

Often in practice there are problems in which the introduced relations (4) are nonlinear, although the equations of constraints (9) are linear. Choosing relations of (4) type for a mechanical system with servoconstraints (10), we can write the equations of motion in the form of (7) and (8).

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As an example, consider the Chaplygin's problem [10-13]. The problem is as follows. A solid body rests on horizontal plate with three points, two of which are freely sliding legs, and the third is the contact point of a sharp (pointed) wheel. The position of the body is determined by horizontal coordinates ξ and μ of the contact points of the wheel and the angle φ made by the axis, invariably connected with the body and lying in the wheel plane with a fixed axis $O\xi$. The horizontal projection of the center of gravity of the body is determined by the coordinates α and β along the moving axes. *A* servomotor is installed on a solid body to realize a constraint at which the speed of the point *A* of the moving plane is always directed along the moving axis Ax, i.e., its projection ϑ_{Ay} on the axis *Ay* is equal to zero at any time (*Ax* is directed along a sharp wheel). Then the servoconstraint equation can be written as

$$\vartheta_{Ay} = \dot{\xi}\cdot\sin\varphi - \dot{\eta}\cdot\cos\varphi = 0 \ .$$

We believe that no active forces act on the system other than gravity and servo motor reaction forces. The mass of the servomotor is neglected. The kinetic energy of the system is written in the form

$$T = \frac{M}{2}\left\{\left[\dot{\xi} - \dot{\varphi}(\alpha\sin\varphi + \beta\cos\beta)\right]^2 + \left[\dot{\eta} + \varphi(\alpha\cos\varphi - \beta\sin\varphi\right]^2 + k^2\dot{\varphi}^2\right\} ,$$

where *M* - is the mass of the body, *k* is the radius of inertia relative to the vertical axis passing through the center of the mass of the body.

Introduce the kinematic characteristics by independent relationships

$$e_1 = \dot{\varphi} \ , \quad e_2 = (\dot{\xi}^2 + \dot{\eta}^2)^{1/2}, \quad e_3 = \dot{\xi}\sin\varphi - \dot{\eta}\cos\varphi \ . \tag{11}$$

Then expressions (4) can be written as

$$\dot{\varphi} = e_1 \ ; \quad \dot{\xi} = (e_2^2 - e_3^2)^{1/2}\cdot\cos\varphi + e_3\cos\varphi ; \ \dot{\eta} = (e_2^2 - e_3^2)^{1/2}\cdot\sin\varphi - e_3\sin\varphi . \tag{12}$$

Note that relationships (11) and (12) are nonlinear, although the constraint equations are linear. Parameter e_3 characterizing the system deviation from the servoconstraint is taken as a release parameter.

Derive the equations of motion in the form (8). Given (12), we find the expression for kinetic energy

$$\tilde{T} = \frac{m}{2}\left[k^2 e_1^2 + \left(\beta e_1 - (e_2^2 - e_3^2)^{1/2}\right)^2 + (\alpha e_1 - e_3)^2\right]$$

The Voronets-Hamel coefficients are:

$$W_1^1 = 0, \quad W_1^2 = 0 \ , \quad W_1^3 = -(e_2^2 - e_3^2)^{1/2} \ ,$$

$$W_2^1 = 0 \ , \quad W_2^2 = \frac{e_3(e_3\dot{e}_2 - e_2\dot{e}_3)}{e_2(e_2^2 - e_3^2)} \ , \quad W_2^3 = \frac{e_3 e_2}{(e_2^2 - e_3^2)^{1/2}} \ .$$

$$W_3^1 = 0 \ , \quad W_3^2 = \frac{e_3\dot{e}_2 - \dot{e}_3 e_2 + e_1 e_2 \cdot (e_2^2 - e_3^2)^{1/2}}{e_2^2 - e_3^2} \ , \quad W_3^3 = \frac{e_1 e_3}{(e_2^2 - e_3^2)^{1/2}} \ .$$

The Voronets-Hamel type equations for the problem under consideration have the form

$$M\left[\dot{e}_1\gamma^2 - \beta\frac{e_2\dot{e}_2 - e_3\dot{e}_3}{(e_2^2 - e_3^2)^{1/2}} - \alpha\dot{e}_3 + \beta e_1 e_3 - e_1\alpha(e_2^2 - e_3^2)^{1/2}\right] = \mu_1 \ , \tag{13}$$

$$M\left\{\frac{\beta\dot{e}_1 e_2(e_2^2 - e_3^2) - \beta e_1 e_3(e_3\dot{e}_2 - e_2\dot{e}_3)}{(e_2^2 - e_3^2)^{3/2}} - \dot{e}_2 - \left(\frac{\beta e_1 e_2}{(e_2^2 - e_3^2)^{1/2}} - e_2\right)\cdot\right.$$

$$\left.\cdot\left[\frac{e_3(e_3\dot{e}_2 - e_2\dot{e}_3)}{e_2(e_2^2 - e_3^2)} + \frac{e_1 e_3}{(e_2^2 - e_3^2)^{1/2}}\right] + \frac{\alpha e_1^2 e_2}{(e_2^2 - e_3^2)^{1/2}} - \frac{\beta e_1^2 e_2 e_3}{(e_2^2 - e_3^2)}\right\} = \mu_2\frac{e_2^2}{e_2^2 - e_3^2} \ ,$$

$$M\left\{\alpha\dot{e}_1 - \frac{\beta e_3\dot{e}_1(e_2^2 - e_3^2) + \beta e_1 e_2(e_2\dot{e}_3 - e_3\dot{e}_2)}{(e_2^2 - e_3^2)^{3/2}} - \frac{1}{e_2^2 - e_3^2}\cdot\left[\frac{\beta e_3 e_1 - e_2\sqrt{e_2^2 - e_3^2}}{(e_2^2 - e_3^2)^{1/2}}\cdot\big(e_3\dot{e}_2 - \right.\right.$$

$$\left.\left. - \dot{e}_3 e_2 + e_1 e_2(e_2^2 - e_3^2)^{1/2}\big) + \alpha\cdot e_1^2 e_3(e_2^2 - e_3^2)^{1/2} + \beta e_1^2 e_2^2\right]\right\} = \lambda - \mu_2\frac{e_1 e_3}{e_2(e_2^2 - e_3^2)^{1/2}} \ ,$$

where

$$\gamma^2 = k^2 + \beta^2 + \alpha^2$$

If the servoconstraints are ideal, that is, $\mu_1 = \mu_2 = 0$ and $e_3 = 0$, then the equations obtained in [11] will follow from the first two equations (13)

$$\gamma^2\dot{e}_1 - \beta\dot{e}_2 - \alpha e_1 e_2 = 0, \qquad \dot{e}_2 - \beta\dot{e}_1 - \alpha e_1^2 = 0 \ .$$

From the last equation (13) we find

$$\lambda = M(\alpha\dot{e}_1 - \beta e_1^2 + e_1 e_2) \ .$$

To further investigate the system motion for stability or controllability, servoconstraints multipliers are taken as control parameters.

Now, as a second example, consider a gyroscope in gimbal mount with a nonholonomic servoconstraints

$$\omega_z = \dot{\gamma} + \dot{\alpha}\sin\beta = 0 \tag{14}$$

This constraint means that the projection of angular velocity of the gyro rotor on the axis Oz_1 is equal to zero at every point in time. (Ox_1, y_1, z_1) is connected with the inner ring [14], that is, the angular velocity always remains in the plane (Ox_1, y_1).

The left side of equation (14) is taken as a release parameter

$$\xi = \dot{\gamma} + \dot{\alpha}\sin\beta$$

We introduce the kinematic characteristics $e_i \quad (i = 1,2,3)$

$$e_1 = \dot{\alpha}, \quad e_2 = \dot{\beta}, \quad e_3 = \dot{\gamma} + \dot{\alpha}\sin\beta \ , \quad (\xi = e_3)$$

Then expression (4) takes the form

$$\dot{\alpha} = e_1, \quad \dot{\beta} = e_2, \quad \dot{\gamma} = e_3 - e_1 \sin\beta \tag{15}$$

Taking into account formulas (15), we obtain

$$\tilde{T} = \frac{1}{2}\left\{ \left[A_2 + (A_1 + A)\cos^2\beta + C_1\sin^2\beta\right]\cdot e_1^2 + Ce_3^2 + (A + B_1)e_2^2 \right\}.$$

For the Voronets-Hamel coefficients, the values are:

$$W_1^1 = W_1^2 = 0 \ , \quad W_2^1 = W_2^2 = 0, \quad W_3^1 = W_3^2 = W_3^3 = 0 \ ,$$

$$W_1^3 = e_2\cos\beta, \qquad W_2^3 = e_1\cos\beta \ .$$

Equations of type (8) for the problem in question have the form

$$\frac{d}{dt}\frac{\partial\tilde{T}}{\partial e_1} - \frac{\partial\tilde{T}}{\partial\pi_1} + \sum_\tau \frac{\partial\tilde{T}}{\partial e_\tau}\cdot W_1^\tau = \sum_s \mu_s A_{s1} \ ,$$

$$\frac{d}{dt}\frac{\partial\tilde{T}}{\partial e_2} - \frac{\partial\tilde{T}}{\partial\pi_2} + \sum_\tau \frac{\partial\tilde{T}}{\partial e_\tau}\cdot W_2^\tau = \sum_s \mu_s A_{s2} \ ,$$

$$\frac{d}{dt}\frac{\partial\tilde{T}}{\partial e_3} - \frac{\partial\tilde{T}}{\partial\pi_3} + \sum_\tau \frac{\partial\tilde{T}}{\partial e_\tau}\cdot W_3^\tau = \sum_s \mu_s A_{s3}$$

or

$$\left[A_2 + (A_1 + A)\cos^2\beta + C_1\sin^2\beta\right]\cdot\dot{e}_1 + C\cdot e_2\cdot e_3\cdot\cos\beta +$$

$$+ (C - A - A_1)\cdot\sin 2\beta\cdot e_1\cdot e_2 = (1 + \sin^2\beta)\cdot\mu_1 \ , \tag{16}$$

$$(B_1 + A)\cdot\dot{e}_2 - (C_1 - A - A_1)\cos\beta\cdot\sin\beta\cdot e_1^2 - C\cdot e_1\cdot e_3\cdot\cos\beta = \mu_2 \, ,$$

$$C \cdot \dot{e}_3 = \lambda - \mu_1 \cdot \sin\beta \quad .$$

Kinematic equations (14) should also be added to them.

At $e_3 = 0$, i.e., when the system is not released, we get

$$\left[A_2 + (A + A_1)\cos^2\beta + C_1 \sin^2\beta\right] \cdot \dot{e}_1 + (C_1 - A - A_1)\sin 2\beta \cdot e_1 \cdot e_2 = (1 + \sin^2\beta) \cdot \mu_1$$

$$(A + B_1) \cdot \dot{e}_2 - (C_1 - A - A_1)\cos\beta \cdot \sin\beta \cdot e_1^2 = \mu_2$$

$$\lambda = \mu_1 \cdot \sin\beta.$$

The first two equations give an extended equation of gyroscope motion with non-ideal servoconstraints in which deviations from servoconstraints (14) are absent.

If assume that the servoconstraint (14) is ideal, then from the last equation we obtain

$$\lambda = 0$$

This proves that equation (14) is an integral of motion.

4. Conclusion

Extended equations of motion such as the Chaplygin and Voronets-Hamel equations of mechanical systems with non-ideal nonholonomic servoconstraints, considering the release from servoconstraints were derived. It was accepted that the work of servoconstraints reactions on virtual displacements is nonzero for both non-released and a parametrically free from servoconstraints systems. A system of equations is obtained for such systems; it contains only the tangential components of the reaction force of the servoconstraints in a number that is equal to the number of degrees of freedom. It also contains normal and tangential components of the reaction forces of servoconstraints, which are equal in number to the number of equations of servoconstraints.

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