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EFFECT OF A PLANE SHEAR WAVE ON CYLINDRICAL SHELL SURROUNDED BY ELASTIC MEDIUM

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Abstract. Unsteady-state strain of a cylindrical shell on the effect of a stepped shear wave propagating in an external elastic medium is studied in the paper. It is assumed that the shell is infinitely long and the front of the incident wave is parallel to its axis. In the paper under consideration, the expansion in a Fourier series in eigenmodes and finite differences in radial coordinate and time is applied. In calculations, a method of numerical dispersion minimization is used, which gives an almost exact description of frontal discontinuities. An asymptotic solution to the problem is obtained at $t \to \infty$ (t - time) by an analytical method. An integrated approach allows obtaining reliable and fairly complete information about the basic physical laws of unsteady-state strain of a cylindrical shell over the entire time interval of external load impact.

Keywords: shell, elastic medium, shear wave, unsteady-state, diffraction, asymptotics.

1.Introduction

The problems of elastic waves diffraction are the classical problems of the dynamics of deformable bodies, and their solution requires the involvement of a complex mathematical tools.

The possibilities for analyzing dynamic behavior of massive bodies and structures subject to seismic, explosive and shock loads, and the stress-strain state of the medium in the vicinity of these structures on the basis of full-scale experiments are significantly limited. In this situation, the calculation methods of wave mechanics of a deformable rigid body are becoming important. Theoretical estimates of the structure dynamics should be based on the solutions of non-stationary problems of wave diffraction on bodies of different geometrical contour.

The beginning of the analysis of unsteady-state problems of diffraction of elastic waves on cavities (reinforced and non-reinforced ones) was laid by the studies of Baron and Parnes [1]. To solve the problem, the sought for functions were expanded in a Fourier series in the circumferential coordinate; the Laplace transform in time was used. The inverse transform was carried out for the zero and second modes at $p \to 0$ (p - is the transform parameter). Thus, a solution that describes the stress state at $t \to \infty$ was obtained. In [3], to solve the problem, the numerical inversion of the Laplace transform was used by expanding the original in series in Jacobi polynomials. In [4-6], diffraction on rigid inclusions was considered, and while in [4-5] the problems were reduced to a numerical solution of the integrals of the Volterra second kind equations with respect to displacement potentials, in [6] an approximate solution was obtained for the initial points in time to calculate the total force acting on a rigid cylinder. To solve non-stationary problems with the medium, V.D. Kubenko has developed a method based on the use of Laplace integral transform in time and its inversion using the Volterra equations [2]. Some aspects of the unsteady-state diffraction of plane waves of expansion and shear were considered in [6–9, 12].

Along with this, dynamic behavior and wave phenomena in various systems were studied in [15–18], taking into account the design features.

Thus, the results were mostly obtained for rigid inclusions, and then only at initial or long periods of time, which does not allow obtaining complete information over the entire time interval. It should be noted that various statements of non-stationary problems and original methods for their solution were described in the monograph by Slepyan L.I. [10].

With the results obtained in [11], it became possible to apply the following approach to solving the unsteady-state problems of wave diffraction on rigid and deformable inclusions. In that paper, the equations of motion of the shell and elastic medium in displacements were expanded in the Fourier series in the angular coordinate. The resulting system of one-dimensional equations was solved numerically using an explicit finite-difference scheme.

Zero derivatives in the equations were replaced by a three-point approximation and the time step was chosen equal to the step along the radial coordinate, which minimizes the numerical dispersion [11]. These solutions determine the stresses in the shell and the medium over the entire interaction interval. In addition, the same problem was solved by analytical methods in order to obtain asymptotic solutions. Numerical and asymptotic solutions complement each other. A comparison of two solutions allows us to determine the applicability limits of asymptotic solutions.

2. Statement of the problem

Let us consider the effect of a stepped plane shear wave on an infinitely long elastic cylindrical shell surrounded by an elastic medium. The wave front is assumed to be parallel to the shell axes, thereby the task is reduced to a plane statement.

In the polar coordinate system (r , θ) associated with the cylinder, the stresses and displacements in the incident wave touching the frontal point with the coordinates $r = R$, $\theta = 0$ at $t = 0$ are specified in the form:

$$\sigma_r^0 = \sigma Sin2\theta H_0(z), \qquad \sigma_{r\theta}^0 = \sigma Cos2\theta H_0(z),$$

$$u_r^0 = \frac{\sigma}{\rho_1 c_2^2} Sin\theta z H_0(z), \qquad u_0^0 = -\frac{\sigma}{\rho_1 c_2^2} Cos\theta z H_0(z),$$

$$z = c_2 t - R + RCos\theta,$$

where: H_0 – is the Heaviside function, σ is the stress at the front of the wave propagating in direction z, R is the radius of the shell, c_1, c_2 are the velocities of the waves of expansion and shear waves in an elastic medium, ρ_1 is the density of the medium.

The motion of an elastic medium is described by wave equations for the scalar (φ) and vector (ψ) displacement potentials (the dot over the symbol denotes the time derivatives)

$$\ddot{\varphi} = c_1^2 \Delta\varphi \ , \ \ddot{\psi} = c_2^2 \Delta\psi \qquad (1)$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}$$

The shell motion is described by linear equations of the classical Kirchhoff – Love theory.

$$c^{-2}\ddot{\upsilon} = R^{-2}\left[\frac{\partial^2 \upsilon}{\partial\theta^2} + \frac{\partial w}{\partial\theta} + \delta\left(\frac{\partial^2 \upsilon}{\partial\theta^2} - \frac{\partial^3 w}{\partial\theta^2}\right)\right] + \beta^{-1}\sigma_{r\theta}^{\Sigma} /_{r=R}$$

$$c^{-2}\ddot{w} = R^{-2}\left[\frac{\partial \upsilon}{\partial\theta} + w + \delta\left(\frac{\partial^4 \upsilon}{\partial\theta^4} - \frac{\partial^2 w}{\partial\theta^2}\right)\right] + \beta^{-1}\sigma_{r}^{\Sigma} /_{r=R} \qquad (2)$$

$$\delta = h^2/(12R^2), \;\; \beta = \rho c^2 h$$

where υ, w are the tangential and normal displacements of the shell,

h and ρ are the shell thickness and density, c is the speed of sound in a thin plate,

$\sigma_{r\theta}^{\Sigma}$ and σ_{r}^{Σ} are the total stresses in the incident and reflected waves, radiated from the shell surface.

The interaction condition of structural elements with the medium is a separate topic of study [19] and here we restrict ourselves to considering one of the limiting cases of elastic interaction, i.e. a rigid contact when the boundary condition is written as follows

$$w = \frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta} + u_r^0, \qquad \upsilon = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} - \frac{\partial\psi}{\partial r} + u_\theta^0 \qquad (3)$$

The system of equations (1) and (2) are solved under zero initial conditions. The potentials φ and ψ must, in addition, be equal to zero outside the expanding region bounded by the excitation front.

To solve the problem, the expansion in a Fourier series in the angular coordinate is used. The equations of motion (1), (2) for the *m*-th mode of oscillations $(m = 0, 1, 2, ...)$ take the following form:

$$\begin{aligned} c^{-2}\ddot{\upsilon}_m &= -\alpha^2(1+\delta)\upsilon_m - \alpha R^{-1}(1+m^2\delta)w_m + \beta^{-1}\sigma_{r\theta m}^{\Sigma} /_{r=R} = 0 \\ c^{-2}\ddot{\upsilon}_m &= -\alpha R^{-1}(1+m^2\delta)\upsilon_m - R^{-2}(1+m^4\delta)w_m + \beta^{-1}\sigma_{r,m}^{\Sigma} /_{r=R} = 0 \end{aligned} \qquad (4)$$

$$\begin{aligned} c_1^{-2}\ddot{\varphi}_m &= \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_m}{\partial r} - \frac{m^2}{r^2}\varphi_m \\ c_2^{-2}\ddot{\varphi}_m &= \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_m}{\partial r} - \frac{m^2}{r^2}\varphi_m \\ \alpha &= m/R \end{aligned} \qquad (5)$$

Boundary conditions (3) are transformed into:

$$r = R: \qquad w_m = \frac{\partial\varphi_m}{\partial r} - \alpha\psi_m + u_r^0 \qquad \upsilon_m = \alpha\varphi_m - \frac{\partial\psi_m}{\partial r} + u_\theta^0 \qquad (6)$$

3. Problem solution

Apply to the system (4) - (5) the Laplace transform in time with a parameter p (the converted values are denoted by the superscript L). The solution to equations (5) in the images, with account for radiation and boundary conditions (6), takes the form:

$$\varphi_m = A_m K_m(\rho c_1^{-1} r), \qquad \psi_m = B_m K_m(\rho c_2^{-1} r),$$

$$\begin{aligned} A_m &= [K'_{m.2}(u_{r.m}^{0.L} - w_m^L) - mR^{-1}K_{m.2}(u_{\theta.m}^{0.L} - \upsilon_m^L)]\Delta_m^{-1} \\ B_m &= [mR^{-1}K_{m.1}(u_{r.m}^{0.L} - w_m^L) - K'_{m.1}(u_{\theta.m}^{0.L} - \upsilon_m^L)]\Delta_m^{-1} \end{aligned} \qquad (7)$$

$$\Delta_m = m^2 R^{-2} K_{m.1} K_{m.2} - K'_{m.1} K'_{m.2}$$

$$K_{m,q}^{(n)} = \frac{d^n}{dr^n} K_{m,q}\left(\frac{r\rho}{c_2}\right) /_{r=R} \qquad (q = 1, 2 \quad n = 0, 1)$$

where K_m is the McDonald's function of the *m*- th order.

To obtain harmonics images of the Fourier series of displacements and stresses in a direct wave, first we apply the Laplace transform, then expand in a Fourier series using the relation

$$H_o^L(p) = p^{-1}\exp\left(Rp\left(1 - rR^{-1}\cos\theta\right)/c_1\right)$$
$$I_m\left(R_p / c_1\right) = \frac{1}{2\pi}\int_0^{2\pi}\exp\left(Rp\cos\theta / c_1\right)\cos_m\theta d\theta$$

where I_m is the modified Bessel functions of the first kind of the *m*-th order; the sought for values are obtained at $r = R$

$$\sigma_{r.m}^{0.L} = \frac{2\sigma e_m e^{-\beta} c_2^2 m}{p^3 R}[I_{m.2}^{'} - R^{-1}I_{m.2}]$$
$$\sigma_{r\theta.m}^{0.L} = \frac{\sigma e_m e^{-\beta}}{p}[I_{m.2}^{'} - \frac{2c_2^2}{\rho^2 R}(I_{m.2}^{'} - m^2 R^{-1} I_{m.2})] \qquad (8)$$
$$u_{r.m}^{0.L} = \frac{\sigma e_m e^{-\beta} m}{\rho_1 p^3 R} I_{m.2}, \qquad u_{\theta.m}^{0.L} = \frac{\sigma e_m e^{-\beta}}{\rho_1 p^3} I_{m.2}^{'}$$

$$I_{m,2} = I_m(pc_2^{-1}R), \qquad \beta = pc_2^{-1}R, \qquad I_{m,2}^{'} = [\frac{d}{dr} I_m(pc_2^{-1}R)]/r = R$$

$$\varepsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m > 0 \end{cases}$$

Substituting (7) into the expression of stresses through potentials and taking into account relations (8), we determine the values of total stresses in the medium acting on the shell. To obtain this solution, the expression for Wronskian determinant of cylindrical functions is used, as well as the recurrence relations for I_m, K_m [21].

$$\begin{aligned}
\sigma_{r.m}^{\Sigma.L} &= \sigma_{r.m}^{0.L} + \sigma_{r.m}^{1.L} = \rho_1 p^2 m R^{-1} K_{m.1} K_{m.2} \Delta_m^{-1} \upsilon_m^L - \rho_1 p^2 K_{m.1}^{'} \times \\
&\times K_{m.2}^{'} \Delta_m^{-1} w_m^L - 2\rho_1 c_2^2 R^{-1}(w_m^L - m\upsilon_m^L) + \bar{\sigma}_{r.m}^L, \\
\sigma_{r\theta.m}^{\Sigma.L} &= \sigma_{r\theta.m}^{0.L} + \sigma_{r\theta.m}^{1.L} = \rho_1 p^2 m R^{-1} K_{m.1} K_{m.2} \Delta_m^{-1} w_m^L - \rho_1 p^2 K_{m.1}^{'} \times \\
&\times K_{m.2} \Delta_m^{-1} \upsilon_m^L - 2\rho_1 c_2^2 R^{-1}(\upsilon_m^L - m w_m^L) + \bar{\sigma}_{r\theta.m}^L,
\end{aligned} \qquad (9)$$

$$\bar{\sigma}_{r.m}^L = -\frac{\sigma e_m e^{-\beta 2} m}{pR^2 \Delta_m} K_{m.1}, \qquad \bar{\sigma}_{r\theta.m}^L = \frac{\sigma e_m e^{-\beta 2}}{pR\,\Delta_m} K_{m.1}^{'}$$

Here $\bar{\sigma}_r^L$, $\bar{\sigma}_{r\theta}^L$ -

are the stresses on the surface of a rigid stationary cylinder.

Substituting (9) in (4) we find the expression for the sought for functions

$$w_m^L = \frac{N_4 \bar{\sigma}_{r.m}^L - N_2 \bar{\sigma}_{r\theta.m}^L}{\rho h c^2 N}, \qquad \upsilon_m^L = \frac{N_1 \bar{\sigma}_{r\theta.m}^L - N_3 \bar{\sigma}_{r.m}^L}{\rho h c^2 N} \qquad (10)$$

$$N_1 = p^2 c^{-2} + R^{-2}(1 + m^4\delta) + \gamma R^{-2}(1 + p^2 c_2^{-2} R K_{m.1} K_{m.2}^{'} \Delta_m^{-1}/2),$$
$$N_2 = N_3 = mR^{-2}(1 + m^4\delta) + \gamma R^{-2} m(1 + p^2 c_2^{-2} K_{m.1} K_{m.2} \Delta_m^{-1}/2),$$
$$N_4 = p^2 c^{-2} + m^2 R^{-2}(1 + \delta) + \gamma R^{-2}(1 + p^2 c_2^{-2} R K_{m.1}^{'} K_{m.2} \Delta_m^{-1}/2)$$

Formula (10) determines the solution to the problem in images, which cannot be inversed exactly. Inverting it for small values of the transform parameter ($p \to 0$), we determine the asymptotic behavior of the coefficients of the Fourier series of each harmonic at $t \to \infty$

$$m = 0:$$

$$w_0 = 0, \qquad \frac{\upsilon_0}{R} = \frac{\sigma}{2\rho_1 c_2^2}, \qquad \sigma_{\theta,0}^{ц} = \sigma_{\theta,0}^{изг} = 0;$$

$$m = 1:$$

$$\frac{\partial w_1}{\partial t} = \frac{\partial \upsilon_1}{\partial t} = \frac{\sigma}{\rho_1 c_2}, \qquad \sigma_{\theta,1}^{ц} = \sigma_{\theta,1}^{изг} = 0;$$

$$m = 2:$$

$$\frac{w_2}{R} = \frac{2\sigma}{\rho_1 c_2^2 B}(2 + \gamma + h^2 R^{-2}/3), \qquad \frac{\upsilon_2}{R} = \frac{2\sigma}{\rho_1 c_2^2 B}(1 + \gamma + 2h^2 R^{-2}/3), \tag{11}$$

$$\sigma_{\theta,2}^{ц} = -\frac{4\sigma}{B}\frac{R}{h}(1 + h^2 R^{-2}/\gamma), \qquad \sigma_{\theta,2}^{изг} = \frac{4\sigma}{B}(1 + 3/\gamma);$$

$$m > 2:$$

$$w_m = \upsilon_m = 0,$$

$$B = (3-\varepsilon)(1 + h^2 R^{-2}/3) + \gamma(1-\varepsilon) + (\varepsilon + 3)h^2 R^{-2}/\gamma, \qquad \varepsilon = -\nu_1/(1-\nu_1),$$

$$\sigma_\theta^{ц} = \frac{E}{1-\nu^2}\left(w + \frac{\partial \upsilon}{\partial \theta}\right), \qquad \sigma_\theta^{изг} = \frac{Eh}{2(1-\nu^2)}\left(\frac{\partial \upsilon}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2}\right);$$

$\sigma_\theta^{ц}$, $\sigma_\theta^{изг}$ are the chain and bending stresses in the shell, ν is the Poisson's ratio of the shell material, ν_1 is the Poisson's ratio of the medium.

Asymptotics shows that the zero mode describes the shell rotation, the first mode describes the motion of a rigid body in a direction perpendicular to the wave propagation. The stress state is determined by the second mode of vibration, and the main contribution is made by chain stresses. The values of bending stresses are an order of magnitude less than the chain ones.

4. Algorithm for numerical solution and analysis of results

Next, it is necessary to find out the error of asymptotic estimates on a finite time interval, to determine the dynamic coefficients and to establish the limits of applicability of these estimates in specific cases with real parameters of the shell and medium. A numerical solution serves this purpose.

Numerical solution is carried out as follows. The motion of the medium is expressed by dynamic equations of the theory of elasticity in displacements, which after expansion into a Fourier series in the angular coordinate takes the following form

$$\frac{\partial^2 u_{r,m}}{\partial t^2} = c_1^2\left[\frac{\partial^2 u_{r,m}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{r,m}}{\partial r}\right] - \frac{c_1^2 + c_2^2 m^2}{r^2}u_{r,m} + \frac{(c_1^2 - c_2^2)m}{r}\frac{\partial u_{\theta,m}}{\partial r} - \frac{(c_1^2 + c_2^2)m}{r^2}u_{\theta,m}\,\frac{\partial u_{r,m}}{\partial t^2} =$$

$$= c_1^2\left[\frac{\partial^2 u_{r,m}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{\theta,m}}{\partial r}\right] - \frac{c_2^2 + c_1^2 m^2}{r^2}u_{\theta,m} - \frac{(c_1^2 - c_2^2)m}{r}\frac{\partial u_{r,m}}{\partial r} - \frac{(c_1^2 + c_2^2)m}{r^2}$$

An explicit-finite difference scheme is applied to the solution of these equations, and a method of numerical dispersion minimization (NDM) is used [11]. The essence of the method is as follows: in these equations, the first and second derivatives are replaced by central differences, the zero derivatives are approximated through three points.

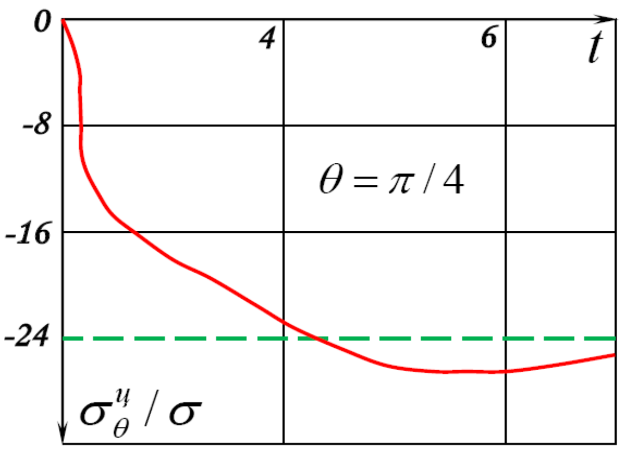
$$w_i = \frac{w_{i+1} + 2w_i + w_{i-1}}{4}, \quad w(u_r, u_\theta).$$

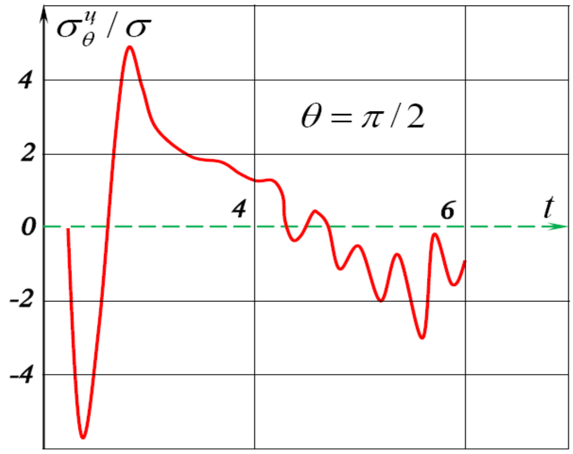
Such an approximation increases the stability limit of the scheme and allows choosing a time step equal to a step along the radial coordinate. The NDM method makes it possible to almost exactly describe the frontal discontinuities in solving numerous non-stationary problems [11, 19].

Calculations showed that already at $h_1 = 0{,}05R$ (h_1 is the step of the difference grid) a satisfactory accuracy is achieved; with further decrease in h_1 the change in results is observed in the third significant figure. When calculating the sums of the Fourier series, 11 terms were held ($m = 0, \ldots, 10$), an increase in the number of modes did not lead to a change in results by more than 3%. In calculations ρ_1, c_1 R were taken as the units of measurement.

In figures 1-4 the results of calculations are presented for the following parameters of the shell and the medium: $E = 23$, $\rho = 2.9$, $h = 0.04$, $\nu = 0{,}25$, $\nu_1 = 1/3$.

In the accepted initial parameters $c_2 = 0.5$ and the incident wave over time $t = 4$ travels the distance equal to the shell diameter.





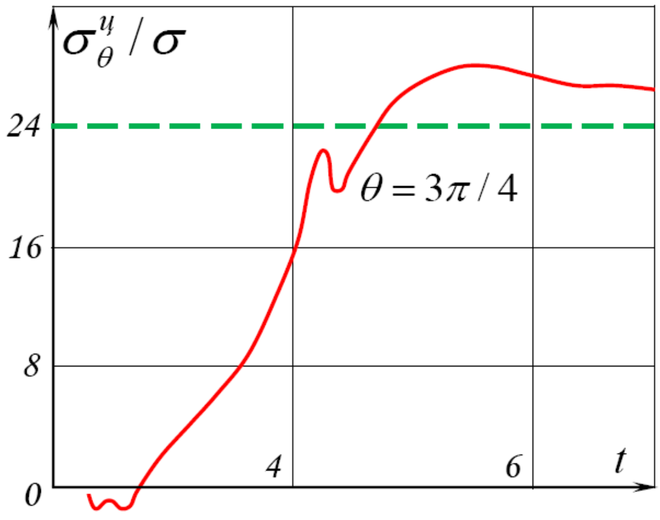


Fig. 1. Oscillogram of chain stresses

numerical solution

asymptotics (11)

5. Conclusions

Numerical and analytical study of the unsteady-state interaction of a plane wave with cylindrical shell allows us to draw the following conclusions:

1. An asymptotic solution to the problem is obtained. The main contribution is made by the second mode of vibration. Chain stresses are an order of magnitude greater than the bending stresses.
2. The proposed difference scheme, built on the basis of numerical dispersion minimization, allows exact description of frontal discontinuities. A comparison of two solutions shows that at $t \geq 6R/c_1$ the asymptotic solution completely describes the stress-strain state of a shell.
3. The maximum chain stresses are 10% greater than the stresses at asymptotic solution.

6. References

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