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Mathematical modelling of torsional vibrations of the three-layer cylindrical viscoelastic shell

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Abstract. The paper considers a circular cylindrical three-layer shell of arbitrary thickness from a viscoelastic material. It is believed that it consists of two outermost bearing layers and a middle layer between them, the materials of which are generally different. The problem of unsteady torsional vibrations of such a shell with rigid contact between the layers is formulated. Proceeding from the assumption that there is a rigid contact between the layers, the dynamic and kinematic contact conditions of the problem are formulated. On the basis of exact solutions in transformations of the three-dimensional problem of the linear theory of viscoelasticity for a circular cylindrical three-layer shell, a mathematical model of its unsteady torsional vibrations has been developed. The proposed model includes the derivation of the general equations of torsional vibrations of the shell with respect to two auxiliary functions, which are the main parts of the torsional displacement of the points of some intermediate surface of the middle layer of the shell. Along with the equations, an algorithm for calculating was created that allows, based on the results of solving the equations of vibration, to unambiguously determine the stress-strain state of the shell and its layers in their arbitrary sections.

1. Introduction

Dynamic calculation of shells is one of the mathematically difficult areas of deformable body mechanics. Therefore, it is natural to switch to a simplified model for calculating shells, which implies reducing the three-dimensional in spatial coordinates of the problem of the theory of shells to two-dimensional [1]. To reduce a three-dimensional problem to a two-dimensional one, various methods and approaches are used, using various simplifying hypotheses and prerequisites of a mechanical and geometric nature [2]. In the studies of many authors, it is indicated that the hypotheses and preconditions used in the construction of the theory, together with simplifications, lead to significant disadvantages and errors [3]. It should be emphasized that this study contains scientific works devoted to the creation of refined theories of plates, shells and rods, in particular, circular cylindrical shells, as well as a detailed review of various directions of this problem. In the classical and refined theories of plates and shells, the displacements of the middle surface of the shell are taken as the main unknown functions. In addition, when constructing new theories of shell vibrations, they try to derive refined equations of vibrations that take into account certain factors of a physical, mechanical or geometric nature. Depending on the factors taken into account, the methods for deriving the equations of vibration, based on the dynamic theory of elasticity, are divided into several directions [4].

One of the main directions of methods for reducing the three-dimensional in spatial coordinates of the problem of the theory of shells to the two-dimensional is the method of using general solutions in transformations [5]. In [6], this method was applied to substantiate the method of power series in the study of oscillatory processes in degenerate systems. The method is based on the use of integral transformations in coordinate and time, and the use of general solutions with their subsequent expansion in power series for approximate satisfaction of the dynamic conditions specified on the boundary surfaces of the elastic system under consideration [7, 8]. The essence of the method is to study the constructed solutions under various types of external influences and to find out the conditions under which the displacements or their "main parts" satisfy simple oscillation equations, and to find an algorithm that allows calculating the approximate field values from the field of these "main parts" displacements and stresses in any section for an arbitrary moment of time [9].

An essential and successful application of this method to problems of the dynamics of circular cylindrical shells was obtained in [10]. General equations of vibrations of rods, plates and shells are obtained in them, taking into account various rheological, anisotropic, temperature, inhomogeneous and other properties of the material, as well as with variable geometry of the section, variability of stiffness and other factors. The works [11,12] are also devoted to the generalization of this method to three-layer plates of an asymmetric structure, quadrangular in plan, where the equations of symmetric [11] anti-symmetric [12] unsteady oscillations are obtained, suitable for engineering calculations. Recently, attention has been paid to research based on vibrational and frequency analysis of two-layer and three-layer cylindrical shells made of functionally graded materials [13] and interacting with a viscous fluid [14]. Carrying out similar studies, but taking into account more complex physical and mechanical properties, in particular viscoelastic, are also devoted to works [15-17].

The analysis of vibrations of elements of engineering structures, such as rods, plates and shells, based on both classical (Kirchhoff-Love) and refined (Timoshenko-type) theories is relevant and is carried out at the present time [18,19]. At the same time, in most of these studies, the tendency to take into account the inertia of rotation, transverse shear deformation, and the multilayer structure prevail. In addition, a fairly large number of studies are carried out, where the influence of interacting deformable media [20], anisotropic, temperature and other physical and mechanical properties of the material are taken into account.

In monograph [21], the general equations of longitudinal and transverse vibrations of viscoelastic plates and rods of quadrangular and circular cross sections, as well as taking into account the environment and friction forces, are derived. The anisotropic properties and temperature of the plates and rods were taken into account (related theory). In [22], this method was developed for a circular cylindrical shell interacting with the deformable medium surrounding the shell and contained in the shell cavity by an ideal liquid, taking into account the viscoelastic properties of the material and with ideal contact between the shell and the liquid. In it, for the first time, an intermediate surface was introduced as the main surface, carrying information about the vibrations of the layer and passing, in limiting cases, to the middle, inner, outer or any other surface of the shell, depending on the values of a certain parameter χ having a continuous bounded spectrum of values.

Within the framework of this article, a circular cylindrical three-layer viscoelastic shell is considered. The problem is posed of studying its unsteady torsional vibrations on the basis of the above method of exact solutions in transformations. The construction of a mathematical model is envisaged, including the derivation of general and refined equations of oscillation and the creation of an algorithm that allows determining the stress-strain state of an arbitrary section of the shell in coordinate and in time from the field of the sought functions. The results obtained, in particular and limiting cases, will be compared with the known results of the authors [22].

2. Methods

2.1. Problem Statement

In a cylindrical coordinate system (r,θ,z) a three-layer circular cylindrical shell made of a viscoelastic material is considered. It is believed that the structure of the shell, taking into account the action of dynamic loads on it, will be such when the bulk of the rigid material in the form of two layers, hereinafter referred to as load-bearing layers, is separated by a certain distance using a thin wall or third layer. The third middle layer keeps the bearing layers at a distance equal to its thickness, carries out their joint work and can be made of the same material as the bearing layers. We direct the *Oz* axis of the coordinate system along the shell symmetry axis perpendicular to the cross section and number the layers as shown in Fig. 1. By *a* and *b* we denote the inner and outer radii of the shell, and by r_1 and r_2 - the inner and outer radii of the middle layer. When deriving the equations of oscillations, we will assume that both the cylindrical shell as a whole and the layers strictly obey the mathematical theory of viscoelasticity and, in the exact formulation, are described by its three-dimensional equations in a linear formulation. The torsional vibrations of the shell are axisymmetric, and therefore the displacements and deformations of the points of the layers, and, consequently, the stresses, do not depend on the angular coordinate. Only displacements v_m, stresses $\tau_{r\theta}^{(m)}$, $\tau_{\theta z}^{(m)}$ and deformations $\gamma_{r\theta}^{(m)}$, $\gamma_{\theta z}^{(m)}$ will be nonzero. Here and below, the index *m* takes the values 0, 1, 2. Therefore, in the future, we will not emphasize every time, implying that this is always the case. The dependencies between the nonzero components of stresses and strains at the points of the layers of a circular cylindrical three-layer shell are assumed to be given in the form [24]

$$\tau_{ij}^{(m)}(r,\theta,z,t)=2R_{\mu m}\left(\gamma_{ij}^{(m)}\right); \quad (i\neq j);\ (i,j=r,\theta,z),$$

where $R_{\mu m}$ -viscoelastic operators equal to

$$R_{\mu m}(\varsigma)=\mu_m\left[\varsigma(t)-\int_0^t K_{\mu m}(t-\tau)\varsigma(\tau)d\tau\right],$$

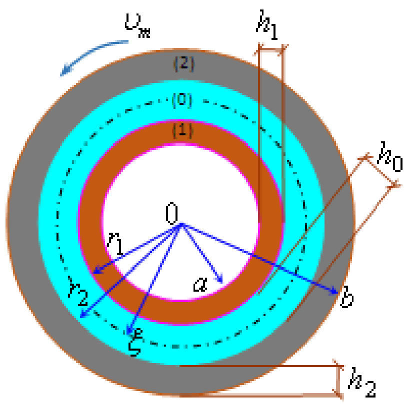

Figure 1. Cross section of the shell

μ_m- rotation coefficients (Lame) of layer materials, $K_{\mu m}(\tau)$-kernel of integral operators. It is assumed here that the viscoelastic operators $R_{\mu m}$ are invertible and their kernels $K_{\mu m}(\tau)$ are arbitrary. Taking into account the axisymmetry of the problem, the equations of motion of the points of the layers, as cylindrical three-dimensional bodies, in the absence of bulk forces have the form.

$$\frac{\partial\tau_{r\theta}^{(m)}}{\partial r}+\frac{\partial\tau_{z\theta}^{(m)}}{\partial z}+\frac{2}{r}\tau_{r\theta}^{(m)}=\rho_m\frac{\partial^2 v_m}{\partial t^2}, \quad m=0,1,2,$$

where v_m are the twisting displacements of the points of the layers. These equations in the potentials of transverse waves $\psi_m(r,z,t)$ are written as

$$R_{\mu m}(\Delta\psi_m)=\rho_m\frac{\partial^2\psi_m}{\partial t^2}, \quad m=0,1,2, \tag{1}$$

where $m=1$ at $a\le r\le r_1$, $m=0$ at $r_1\le r\le r_2$ and $m=2$ at $r_2\le r\le b$; ρ_m-density of layer materials; Δ-three-dimensional Laplace operator in a cylindrical coordinate system

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

In this case, displacements and deformations are expressed through potential functions, from which it follows that the torsional displacements of the points of the layers v_m depend only on the potentials ψ_m and are equal

$$v_m = -\frac{\partial \psi_m}{\partial r}, \quad m = 0,1,2. \tag{2}$$

It is assumed that at $t<0$ the shell is at rest, and at the moment $t=0$, stresses are applied to its boundary surfaces, causing its torsional vibrations, i.e. the boundary conditions are considered to be

$$\text{at } r=a \quad \tau_{r\theta}^{(1)}(a,z,t) = F_{r\theta}^{(1)}(z,t); \qquad \text{at } r=b \quad \tau_{r\theta}^{(2)}(b,z,t) = F_{r\theta}^{(2)}(z,t) \tag{3}$$

In addition, according to the conditions of rigid contact on the boundary surfaces between the layers, the conditions of equality of mixing and stresses must be satisfied, i.e.

$$\text{at } r=r_1 \qquad v_0(r_1,z,t) = v_1(r_1,z,t), \; \tau_{r\theta}^{(0)}(r_1,z,t) = \tau_{r\theta}^{(1)}(r_1,z,t) \tag{4}$$

$$\text{at } r=r_2 \qquad v_0(r_2,z,t) = v_2(r_2,z,t), \quad \tau_{r\theta}^{(0)}(r_2,z,t) = \tau_{r\theta}^{(2)}(r_2,z,t). \tag{5}$$

The initial conditions of the problem at *t = 0* are assumed to be zero, i.e.

$$\psi_m(r,z,t) = \frac{\partial \psi_m(r,z,t)}{\partial t} = 0. \tag{6}$$

2.2. Derivation of the Vibration Equations

To solve with the formulated problem of torsional vibrations of a three-layer cylindrical viscoelastic shell, the functions of external influences in the boundary conditions (3) are considered in the class of functions represented in the form [6]

$$F_{r\theta}^{(i)}(z,t) = \int_0^\infty \begin{Bmatrix} \sin kz \\ -\cos kz \end{Bmatrix} dk \int_{(l)} f_{r\theta}^{(i)}(k,p) e^{pt} dp, \; F_{rz}^{(i)}(z,t) = \int_0^\infty \begin{Bmatrix} \cos kz \\ \sin kz \end{Bmatrix} dk \int_{(l)} f_{rz}^{(i)}(k,p) e^{pt} dp, \; (i=1,2). \tag{7}$$

Here *(l)* is an open contour in the plane p adjacent to the right to the section $(-i\omega_0, i\omega_0)$ of the imaginary axis. In addition, the functions $F_r^{(i)}(z,t)$, $F_{rz}^{(i)}(z,t)$ and $F_{r\theta}^{(i)}(z,t)$ are assumed to be such that the functions $f_r^{(i)}(z,t)$, $f_{rz}^{(i)}(z,t)$ and $f_{r\theta}^{(i)}(z,t)$ (i=1,2) are negligible outside the region $\{0<k<k_0, \;\; \mathrm{Im}|p|<\omega_0\}$. Similarly to (7), the potential functions $\psi_m, (m=0,\,1,\,2)$ in formulas (6) can also be represented in the form

$$\psi_m(r,z,t) = \int_0^\infty \begin{Bmatrix} \sin kz \\ -\cos kz \end{Bmatrix} dk \int_{(l)} \tilde{\psi}_m(r,k,p) e^{pt} dp \; . \tag{8}$$

In this case, representations (8) allow, if the above conditions are satisfied with respect to the integrands, to differentiate the functions ψ_m as needed, both with respect to coordinate and with respect to time. Substituting the last representation into the wave equations of motion of the points of the layers, we obtain the ordinary differential Bessel equations for the transformed potential functions $\tilde{\psi}_m$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \beta_m^2\right)\tilde{\psi}_m = 0; \qquad (m=0,1,2), \tag{9}$$

where $\beta_m^2 = k^2 + \rho_m P^2 \tilde{R}_{\mu m}^{-1}, \quad \tilde{R}_{\mu m} = \mu_m\left[1-\tilde{K}_{\mu m}(p)\right], \qquad \tilde{K}_{\mu m}(p) = \int_0^\infty \tilde{K}_{\mu m}(t) e^{-pt} dt.$

The general solutions of equations (9) are

$$\tilde{\psi}_m(r) = B_m^{(1)} I_0(\beta_m r) + B_m^{(2)} K_0(\beta_m r); \qquad (m=0,1,2), \tag{10}$$

where $B_m^{(i)}(k,p)$ $(i=1,2)$, arbitrary, relative to the variable r, constants.

Our further task is to express all components of stresses and displacements at the points of all three layers through the obtained general solutions (10). For this purpose, we represent stresses $\tau_{r\theta}$, as well as (7)

$$\tau_{r\theta}^{(m)}(r,z,t)=\int_0^\infty \begin{Bmatrix} \sin kz \\ -\cos kz \end{Bmatrix} dk \int_{(l)} \tilde{\tau}_{r\theta}^{(m)}(r,k,p)\, e^{pt} dp \ . \tag{11}$$

and substitute representations (7), (8), and (11) into boundary conditions (3), we obtain

$$\left(\frac{1}{r}-\frac{d}{dr}\right)\frac{d\tilde{\psi}_1}{dr}\bigg|_{r=a}=\tilde{R}_{\mu 1}^{-1}[f_{r\theta}^{(1)}],\quad \left(\frac{1}{r}-\frac{d}{dr}\right)\frac{d\tilde{\psi}_2}{dr}\bigg|_{r=b}=\tilde{R}_{\mu 2}^{-1}[f_{r\theta}^{(2)}]. \tag{12}$$

Similarly transformed contact conditions (4) and (5) will have the following forms:

$$\text{at } r=r_1 \qquad \frac{d}{dr}\tilde{\psi}_1=\frac{d}{dr}\tilde{\psi}_0,\quad \left(\frac{1}{r}-\frac{d}{dr}\right)\frac{d\psi_0}{dr}=\tilde{R}_{\mu 0}^{-1}\tilde{R}_{\mu 1}\left(\frac{1}{r}-\frac{d}{dr}\right)\frac{d\psi_1}{dr}, \tag{13}$$

and

$$\text{at } r=r_2 \qquad \frac{d}{dr}\tilde{\psi}_0=\frac{d}{dr}\tilde{\psi}_2,\qquad \left(\frac{1}{r}-\frac{d}{dr}\right)\frac{d\tilde{\psi}_0}{dr}=\tilde{R}_{\mu 0}^{-1}\tilde{R}_{\mu 2}\left(\frac{1}{r}-\frac{d}{dr}\right)\frac{d\psi_2}{dr}\ . \tag{14}$$

General solutions (10) for all three layers have the same structure, taking into account the boundedness of solutions at $r\to 0$ and $r\to\infty$ simultaneously. In this case, the boundaries of the first layer are equal to a and r_1, $a\le r\le r_1$.. It is bounded from below (from the inside) by a surface $r=a$, which in the limit can tend to zero, i.e. $a\to 0$, but it cannot exceed the values r_1 in any way, i.e. cannot strive for infinity. Therefore, when writing a general solution to the potential function of the first layer - (s1), one can restrict oneself to taking into account its boundedness only at t = 0. Based on this, we take the general solution for the first layer in the form

$$\tilde{\psi}_1(r)=AI_0(\beta_1 r);\ (a\le r\le r_1), \tag{15}$$

where A is the constant of integration.

Similarly, the boundaries of the second, outer layer are cylindrical surfaces $r=r_2$ and $r=b$; $r_2\le r.\le b$. It is bounded from above (from the outside) by the surface $r=b$, the radius of which can tend to infinity, i.e. $b\to\infty$. On the other hand, the inner surface of this layer cannot be pulled to a straight line, b.c. this would lead to a uniform round rod with a radius $r=b$. Therefore, in the general solution for the potential function of the second layer $\tilde{\psi}_2(r)$, we can restrict ourselves to taking into account its boundedness only for the case $r\to\infty$. Based on this, we take the general solution for the second layer in the form

$$\tilde{\psi}_2(r)=CK_0(\beta_2 r),\qquad (r_2\le r\le b) \tag{16}$$

For the middle layer, we take a general solution, taking into account the fact that our solution, in the absence of two outer layers, should go over into the known solution for a homogeneous cylindrical layer, bounded at $r\to 0$ and $r\to\infty$, i.e.

$$\tilde{\psi}_0(r)=B_1 I_0(\beta_0 r)+B_2 K_0(\beta_0 r),\qquad r_1\le r\le r_2\ . \tag{17}$$

Thus, the number of integration constants to be determined from the contact conditions is reduced to two A and C. Therefore, there is no need for four contact conditions (4) and (5). Taking this circumstance into account, we restrict ourselves to only two contact conditions, leaving in (4) - (5) only the conditions of equality of the stress components. Substituting solutions (15), (16), and (17) into the transformed boundary (12) and contact (13) and (14) conditions, we obtain

$$\left[\frac{2\beta_1}{a}I_1(\beta_1 a)-\beta_1^2 I_0(\beta_1 a)\right]A=\tilde{R}_{\mu 1}^{-1}[f_{r\theta}^{(1)}(k,p)],\ \left[\frac{2\beta_2}{b}K_1(\beta_2 b)+\beta_2^2 K_0(\beta_2 b)\right]C=-\tilde{R}_{\mu 2}^{-1}[f_{r\theta}^{(2)}(k,p)],$$
$$\tilde{R}_{\mu 1}\beta_1^2 I_2(\beta_1 r_1)A=\tilde{R}_{\mu 0}\,\beta_0^2\left[I_2(\beta_0 r_1)B_1+K_2(\beta_0 r_1)B_2\right],$$
$$\tilde{R}_{\mu 2}\beta_2^2 K_2(\beta_2 r_2)C=\tilde{R}_{\mu 0}\,\beta_0^2\left[I_2(\beta_0 r_2)B_1+K_2(\beta_0 r_2)B_2\right].$$

From the last two equations, we find the constants *A*, *C* and substitute them into the first two equations. Next, we introduce the following notation

$$F_1(\beta_1,a,r_1)=\frac{I_0(\beta_1 a)-\dfrac{2}{\beta_1 a}I_1(\beta_1 a)}{I_2(\beta_1 r_1)},\qquad F_2(\beta_2,b,r_2)=\frac{K_0(\beta_2 b)+\dfrac{2}{\beta_1 b}K_1(\beta_2 b)}{K_2(\beta_2 r_2)}. \tag{18}$$

Then the boundary conditions can be written as

$$-F_1(\beta_1,a,r_1)\beta_0^2\left[I_2(\beta_0 r_1)B_1+K_2(\beta_0 r_1)B_2\right]=\tilde{R}_{\mu0}^{-1}[f_{r\theta}^{(1)}(k,p)], \tag{19}$$

$$-F_2(\beta_2,b,r_2)\beta_0^2\left[I_2(\beta_0 r_2)B_1+K_2(\beta_0 r_2)B_2\right]=\tilde{R}_{\mu0}^{-1}[f_{r\theta}^{(2)}(k,p)]. \tag{20}$$

Let us express the transformed displacements of layers $\tilde{v}_m$ $(m=0,1,2)$ in terms of solutions (15) - (17). To do this, it is sufficient to substitute in formulas (2) for $\tilde{v}_m(r,k,p)$, the solution (15) - (17). We get

$$\begin{cases} \tilde{v}_0(r,k,p)=-\beta_0 I_1(\beta_0 r)B_1+\beta_0 K_1(\beta_0 r)B_2\,, \\ \tilde{v}_1(r,k,p)=-\beta_1 I_1(\beta_1 r)A, \qquad \tilde{v}_2(r,k,p)=\beta_2 K_1(\beta_2 r)C. \end{cases} \tag{21}$$

Further, we use standard expansions in power series of Bessel functions $I_1(\beta_0 r)$ and $K_1(\beta_0 r)$ in powers $(\beta_0 r)$. We get

$$\tilde{v}_0(r,k,p)=\frac{1}{r}B_0^{(2)}+\sum_{n=0}^{\infty}\left\{-B_0^{(1)}+B_0^{(2)}[\ln\frac{\beta_0 r}{2}-\frac{1}{2}(\psi(n+1)+\psi(n+2))]\right\}\beta_0^{2n+2}\frac{(r/2)^{2n+1}}{n!(n+1)!}, \tag{22}$$

Here $\psi(n)-$ the logarithmic derivative of the Gamma function

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}.$$

In the classical study, the vibrations of a cylindrical shell are assumed to be the displacements of points on the middle surface of the shell. However, this choice is not the only one. For example, when considering a thick-walled shell, it is necessary to choose a surface that for a rod goes into the center line, and for thin shells into the middle surface. On the other hand, in experimental studies, information is obtained on the displacements of points on the outer or inner surface of the shell, by which it is necessary to determine the stress-strain state of the shell itself. In this regard, following work [25], we will take as the sought values of the displacement and stress at the points of a certain intermediate surface of the middle layer of a three-layer shell, the radius ξ of which (Fig.1.1) is defined in the gap $\xi\in[r_1,r_2]$ as

$$\xi=\frac{r_1}{2}\left(\chi-\frac{r_1}{r_2}\right)$$

where the constant χ satisfies the inequality

$$2+\frac{r_1}{r_2}\le\chi\le 2\frac{r_2}{r_1}+\frac{r_1}{r_2}.$$

Note that ξ can be the radius of the contacting surfaces between layers at $\xi=r_1$ or $\xi=r_2$. This “intermediate” surface passes into contact surfaces between layers, with values χ equal to

$$2+\frac{r_1}{r_2},\ 1+\frac{r_2}{r_1}+\frac{r_1}{r_2},\ 2\frac{r_2}{r_1}+\frac{r_1}{r_2},$$

and for $\xi=\frac{r_1+r_2}{2}$ it passes into the middle surface of the filler. At $r_1=r_2$, there is no middle layer and the radius of the intermediate surface ξ in this case passes into the radius of the contact surface between the bearing layers. For unknown values, we take the values of displacement and stresses calculated at the points of the introduced "intermediate" surface of the middle layer. We put $r=\xi$ in the expression of the transformed displacement (22) and select its main parts, assuming that they are determined as the first terms of a converging power series, we obtain

$$\tilde{v}_0(\xi)=\frac{1}{\xi}B_1+\left\{-B_1+B_2[\ln\frac{\beta_0\xi}{2}-\psi(1)-\frac{1}{2}]\right\}\beta_0^2(\frac{\xi}{2});$$

We introduce the following notation [25]

$$\tilde{v}_0^{(0)}=-\frac{1}{2}\beta_0^2B_0\,,\quad \tilde{v}_0^{(1)}=\frac{1}{\xi}B_2\,,\quad \eta_1(n,r)=\ln\frac{r}{\xi}+\frac{n}{2(n+1)}-\sum_{k=1}^{n}\frac{1}{k} \tag{23}$$

where $B_0=B_1-B_2[\ln\frac{\beta_0\xi}{2}-\psi(1)-\frac{1}{2}]$.

The transformed displacement $\tilde{v}_0(r,k,p)$ is introduced by new functions $\tilde{v}_0^{(0)}$ and $\tilde{v}_0^{(1)}$ is expressed as follows

$$\tilde{v}_0(r,k,p)=\frac{\xi}{r}\tilde{v}_0^{(1)}+2\sum_{n=0}^{\infty}\beta_0^{2n}\tilde{v}_0^{(0)}\frac{(r/2)^{2n+1}}{n!(n+1)!}+\xi\sum_{n=0}^{\infty}\beta_0^{2n+2}\cdot\tilde{v}_0^{(1)}\eta_1(n,r)\frac{(r/2)^{2n+1}}{n!(n+1)!} \tag{24}$$

Note that if the middle layer is thin (for example, a thin layer of glue, usually applied between layers), then we can assume that $r=\xi$. Then

$$\eta_1(n)=\frac{n}{2(n+1)n}-\sum_{k=1}^{n}\frac{1}{k}$$

which is number, for example $n=0$, $\eta_1(0)=0$ and at $n=1$, $\eta_1(1)=-3/4$.

Similarly, boundary conditions (19) - (20) are expressed through the main parts of the transformed torsional displacement $\tilde{v}_0^{(0)}$ and $\tilde{v}_0^{(1)}$. For combinations of Bessel functions in (18), restricting their expansion to zero and first approximations, we obtain

$$\begin{gathered}\frac{a^2}{r_1^2}\left(1+\frac{a^2-r_1^2}{12}\beta_1^2+\frac{r_1^2(a^2-r_1^2)}{144}\beta_1^4\right)\left\{\sum_{n=0}^{\infty}\beta_0^{2n+2}\left[2\tilde{v}_0^{(0)}+\xi\eta_{2,n}(r_1)\beta_0^2\,\tilde{v}_0^{(1)}\right]\frac{(r_1/2)^{2n+2}}{n!(n+2)!}+\right.\\ \left.+\xi\left(\frac{1}{2}\beta_0^2-\frac{2}{r_1^2}\right)\tilde{v}_0^{(1)}\right\}=\tilde{R}_{\mu1}^{-1}\left[f_{r\theta}^{(1)}(k,p)\right],\\ \frac{r_2^2}{b^2}\left[1+\frac{r_2^2-b^2}{4}\beta_2^2+\frac{r_2^2(r_2^2-b^2)}{16}\beta_2^4\right]\left\{\sum_{n=0}^{\infty}\beta_0^{2n+2}\left[2\tilde{v}_0^{(0)}+\xi\eta_{2,n}(r_2)\beta_0^2\,\tilde{v}_0^{(1)}\right]\frac{(r_2/2)^{2n+2}}{n!(n+2)!}+\right.\\ \left.+\xi\left(\frac{1}{2}\beta_0^2-\frac{2}{r_2^2}\right)\tilde{v}_0^{(1)}\right\}=\tilde{R}_{\mu2}^{-1}\left[f_{r\theta}^{(2)}(k,p)\right].\end{gathered} \tag{25}$$

We introduce new functions $v_0^{(0)}$, $v_0^{(1)}$ and operators λ_m^n by the formulae

$$\left[v_0^{(0)},v_0^{(1)}\right]=\int_0^{\infty}\left.\begin{matrix}\sin kz\\-\cos kz\end{matrix}\right\}dk\int_{(\ell)}\left(\tilde{v}_0^{0},\tilde{v}_0^{(1)}\right)e^{pt}dp\,,\quad \lambda_m^n(\zeta)=\int_0^{\infty}\left.\begin{matrix}\sin kz\\-\cos kz\end{matrix}\right\}dk\int_{(\ell)}\left(\beta_m^{2n}\zeta\right)e^{pt}dp\,. \tag{26}$$

Applying the operators in (26) to the left and right side of (25) we get the following equation

$$\begin{cases}\frac{a^2}{r_1^2}\left(1+\frac{a^2-r_1^2}{12}\lambda_1+\frac{r_1^2(a^2-r_1^2)}{144}\lambda_1^2\right)\left[C_{11}(r_1)v_0^{(0)}+\xi C_{21}(r_1)v_0^{(1)}\right]=R_{\mu1}^{-1}\left[F_{r\theta}^{(1)}(z,t)\right],\\ \frac{r_2^2}{b^2}\left[1+\frac{r_2^2-b^2}{4}\lambda_2+\frac{r_2^2(r_2^2-b^2)}{16}\lambda_2^2\right]\left[C_{12}(r_2)v_0^{(0)}+\xi C_{22}(r_2)v_0^{(1)}\right]=R_{\mu2}^{-1}\left[F_{2\theta}^{(2)}(z,t)\right].\end{cases} \tag{27}$$

where

$$C_{1i}(r_i)=2\sum_{n=0}^{\infty}\lambda_0^n\frac{(r_i/2)^{2n+1}}{n!(n+1)!},\quad C_{2i}(r_i)=\frac{1}{2}\lambda_0-\frac{2}{r^2}+\sum_{n=0}^{\infty}\eta_2(n,r)\lambda_0^{n+1}\frac{(r_i/2)^{2n+1}}{n!(n+1)!},$$

$$\eta_{2,n}(r)=\ln\frac{r}{\xi}+\frac{n^2+n-1}{2(n+1)(n+2)}-\sum_{k=1}^{n}\frac{1}{k}\ . \tag{28}$$

Based on the expressions (9) for $\beta_m\ (m=0,1,2)$, it is easy to obtain that the operators λ_m^n, introduced by formulas (26) under the reverse transition according to Fourier and Laplace, in the variables *z* and *t*, have the following forms

$$\lambda_m^n(\zeta)=\left[\rho_m R_{\mu m}^{-1}\left(\frac{\partial^2\zeta}{\partial t^2}\right)-\frac{\partial^2\zeta}{\partial z^2}\right]^n,\qquad m=0,1,2;\quad n=1,2,3,... \tag{29}$$

where $R_{\mu m}$ are the viscoelastic operators of the layer materials. Equations (27) in accordance with formulas (29) for the operators $\lambda_m^n\ (m=0,1,2;\ n=1,2,3,...)$, are integro-differential equations of unbounded high order. These equations contain the main parts of $v_0^{(0)}$ *and* $v_0^{(1)}$ the torsional displacement v_0 of points of a certain “intermediate” surface of the middle layer of a three-layer cylindrical shell. The specified "intermediate" surface has a radius, the values of which are enclosed in the interval $r_1\le\xi\le r_2$. In accordance with the numerical values of the radius, this “intermediate” surface can pass into the middle one at $\xi=\dfrac{r_1+r_2}{2}$ and contact between the layers of the shell surface at $\xi=r_1$, $\xi=r_2$.

Consequently, equations (27), depending on the values of the radius, can be the equations of vibration of a three-layer cylindrical shell relative to the main parts of the torsional displacement of the points of the middle or contact surfaces of the middle layer. These equations, in the absence of external layers, are general equations of torsional vibrations [25] of a circular cylindrical viscoelastic shell, relative to the main parts of the torsional displacement of the points of the intermediate surface of the shell. The resulting equations have, as indicated above, an unbounded order in derivatives, and therefore, in their structures, they contain derivatives of any order with respect to the longitudinal coordinate *z* and with respect to time *t*.

2.3. Determination of displacements and stresses

During torsional vibrations, as emphasized above, displacements v_m and stresses $\tau_{r\theta}^{(m)},\tau_{z\theta}^{(m)},\quad (m=0,1,2)$. are nonzero. To determine the stress-strain state of the shell points, it is necessary to determine the displacements v_m and stresses $\tau_{r\theta}^{(m)}$, $\tau_{z\theta}^{(m)}$ at these points. This procedure must be performed for all three layers of the shell. This will require expressing all displacements and stresses through the main parts $v_0^{(0)},v_0^{(1)}$ of the torsional displacement v_0 points of the intermediate surface of the middle layer of the shell.

First, we determine the displacement v_0 and the stresses $\tau_{r\theta}^{(0)}$ and $\tau_{z\theta}^{(0)}$ of the points of the middle layer. To find v_0, it is sufficient to invert expression (24) for $\tilde{v}_0$ with respect to *p* and *k*. Applying transformations (26) to (24), we obtain

$$v_0(r,z,t)=C_1v_0^{(0)}+\xi C_2v_0^{(1)}, \tag{30}$$

Where

$$C_1=C_{1i}\big|_{r_i=r}=2\sum_{n=0}^{\infty}\lambda_0^n\frac{(r/2)^{2n+1}}{n!(n+1)!},\quad C_2=C_{2i}\big|_{r_i=r}=\frac{1}{r}+\sum_{n=0}^{\infty}\eta_1(n,r)\lambda_0^{n+1}\frac{(r/2)^{2n+1}}{n!(n+1)!}.$$

According to the results of solving the system of equations (27) i.e. Using the found values of $v_0^{(0)}$, $v_0^{(1)}$ it is easy to calculate the displacement $v_0(r,z,t)$ of the points of an arbitrary section of the middle layer for any moment in time by formula (30) with the desired accuracy along the radial coordinate *r*, for an arbitrary moment in time.

The torsional displacements of the points of the bearing layers v_1 ,v_2 should be expressed in terms of the main parts $v_0^{(0)}$,$v_0^{(1)}$. To do this, use expressions (21) for $\tilde{v}_1$,$\tilde{v}_2$.. After some simple mathematical calculations, we get

$$
\begin{aligned}
v_1(r,z,t) &= \frac{r}{r_1}\left[1+\frac{r^2-r_1^2}{8}\lambda_1\right]\left[C_{11}v_0^{(0)}+\xi C_{21}v_0^{(1)}\right],\\
v_2(r,z,t) &= \frac{r_2}{r}\left\{1+\frac{1}{2}\left[r^2L(r)-r_2^2L(r_2)\right]\lambda_2\right\}\left[C_{12}v_0^{(0)}+\xi C_{22}v_0^{(1)}\right],
\end{aligned}
\tag{31}
$$

where

$$L(r)=\ln\frac{r}{r_2}-\frac{1}{2}[\psi(1)+\psi(2)].$$

Expressions (31) make it possible to calculate the torsional displacements of the points of the bearing layers using the found principal values of the torsional displacement $v_0^{(0)}$ and $v_0^{(1)}$ from the results of solving the equations of oscillations (27).

Similarly, formulas are derived for the stress components $\tau_{r\theta}^{(m)}$ and $\tau_{z\theta}^{(m)}$ of all layers, which are nonzero

$$\tau_{r\theta}^{(0)}=R_{\mu 0}\left\{\sum_{n=0}^{\infty}\lambda_0^{n+1}\left[2v_0^{(0)}+\xi\eta_{2,n}(r_1)\lambda_0 v_0^{(1)}\right]\frac{(r/2)^{2n+2}}{n!(n+2)!}+\xi\left(\frac{1}{2}\lambda_0-\frac{2}{r_1^2}\right)v_0^{(1)}\right\}, \tag{33}$$

$$\tau_{z\theta}^{(0)}=R_{\mu 0}\left\{\frac{\xi}{r}\frac{dv_0^{(1)}}{dz}+\frac{r}{2}\sum_{n=0}^{\infty}\lambda_0^{n}\left(2\frac{dv_0^{(0)}}{dz}+\xi\eta_1(n,r)\lambda_0\frac{dv_0^{(1)}}{dz}\right)\frac{(r/2)^{2n}}{n!(n+1)!}\right\}. \tag{34}$$

The obtained formulas for stresses(33), (34) make it possible to calculate, with a given accuracy, the stresses at the points of an arbitrary section of the middle layer of a three-layer shell. The stresses $\tau_{r\theta}^{(m)}$ and $\tau_{z\theta}^{(m)}$ are determined similarly in the case when *m = 1* and *m = 2*.

3. Results and Discussions

The obtained equations of torsional vibrations of a three-layer circular cylindrical viscoelastic shell - (27), are general. Several limiting cases and particular types of oscillation equations follow from them.

3.1. Vibrations of a two-layer viscoelastic shell.

At $a=r_1$, the three-layer cylindrical shell transforms into a two-layer shell. The intermediate surface of the shell of radius ξ passes into the intermediate surface of the inner layer of the shell. In this case, it should be assumed that the function of external action $F_{r\theta}^{(1)}(z,t)$ acts on the surface $r=r_1$, and the operator $\lambda_1=0.$ Then from the system of equations (27) we obtain the following system of equations for a two-layer cylindrical viscoelastic shell

$$\left[C_{11}(r_1)v_0^{(0)}+\xi C_{21}(r_1)v_0^{(1)}\right]=R_{\mu 0}^{-1}\left[F_{r\theta}^{(1)}(z,t)\right], \qquad r_1\le r\le r_2\,,$$

$$\frac{r_2^2}{b^2}\left[1+\frac{r_2^2-b^2}{4}\lambda_2+\frac{r_2^2\left(r_2^2-b^2\right)}{16}\lambda_2^2\right]\left[C_{12}(r_2)v_0^{(0)}+\xi C_{22}(r_2)v_0^{(1)}\right]=R_{\mu 2}^{-1}\left[F_{2\theta}^{(2)}(z,t)\right],\ \ r_2\le r\le r_2+b.$$

Similarly, one can obtain a system of equations for torsional vibrations of a two-layer cylindrical viscoelastic shell, where the main unknowns will be the main parts of the torsional displacement of the intermediate surface of the outer layer. To do this, it suffices to assume that there is no outer layer. In this case, $r_2=b$ and the system of equations (27) will be written as

$$\frac{a^2}{r_1^2}\left(1+\frac{a^2-r_1^2}{12}\lambda_1+\frac{r_1^2\left(a^2-r_1^2\right)}{144}\lambda_1^2\right)\left[C_{11}(r_1)v_0^{(0)}+\xi C_{21}(r_1)v_0^{(1)}\right]=R_{\mu 1}^{-1}\left[F_{r\theta}^{(1)}(z,t)\right],\quad a\le r\le r_2+b,$$

$$\left[C_{12}(r_2)v_0^{(0)}+\xi C_{22}(r_2)v_0^{(1)}\right]=R_{\mu 0}^{-1}\left[F_{2\theta}^{(2)}(z,t)\right],\quad r_1\le r\le r_2$$

If the inner and outer layers of the cylindrical shell are absent, i.e. the shell is homogeneous (single-layer), then $a=r_1$ and $b=r_2$ should be set in general equations (27). In this case, we will have a system of equations

$$\begin{cases}\left[C_{11}(r_1)v_0^{(0)}+\xi C_{21}(r_1)v_0^{(1)}\right]=R_{\mu 0}^{-1}\left[F_{r\theta}^{(1)}(z,t)\right],\\ \left[C_{12}(r_2)v_0^{(0)}+\xi C_{22}(r_2)v_0^{(1)}\right]=R_{\mu 0}^{-1}\left[F_{2\theta}^{(2)}(z,t)\right],\end{cases}\qquad r_1\le r\le r_2. \tag{35}$$

In the last equations, the integro-differential operators $C_{jk}(r_i)$ $(j,k=1,2;i=1,2)$ are found by formulas (28). Note that the resulting system of equations for torsional vibrations of a cylindrical homogeneous shell (35) exactly coincides with the system of equations derived in [21].

3.2. Cylindrical shell with a thin middle layer

If $r_2=r_1(1+\varepsilon)$, where $\varepsilon > 0$ is a small parameter, then the middle layer of the shell is thin (for example, a thin layer of glue, usually applied between layers). In this case, the values of $\ln(r_i/\xi)$ can be assumed to be zero. Then expression (28) for $\eta_2(n,r_i)$ is simplified and takes the form

$$\eta_{2,n}(r)=\frac{n^2+n-1}{2(n+1)(n+2)}-\sum_{k=1}^{n}\frac{1}{k},\qquad n=0,1,2,\ldots\ .$$

Consequently, the equations of torsional vibrations of a three-layer cylindrical shell with a thin middle layer are also equations (27), but with a different value for $\eta_2(n,r_i)$ determined by the last formula.

3.3. Three-layer elastic shell

If the materials of the layers are elastic, then in the expressions for viscoelastic operators the equalities will take place $K_{\mu m}(t)=0$, and therefore, we will have $R_{\mu m}=\mu_m$. Then, replacing the integral operators $R_{\mu 1}$ and $R_{\mu 2}$, respectively, by the Lamé coefficients μ_1 and μ_2, we obtain the system of equations

$$\begin{cases}\frac{a^2}{r_1^2}\left(1+\frac{a^2-r_1^2}{12}\lambda_1+\frac{r_1^2\left(a^2-r_1^2\right)}{144}\lambda_1^2\right)\left[C_{11}(r_1)v_0^{(0)}+\xi C_{21}(r_1)v_0^{(1)}\right]=\mu_1^{-1}\left[F_{r\theta}^{(1)}(z,t)\right],\\ \frac{r_2^2}{b^2}\left[1+\frac{r_2^2-b^2}{4}\lambda_2+\frac{r_2^2\left(r_2^2-b^2\right)}{16}\lambda_2^2\right]\left[C_{12}(r_2)v_0^{(0)}+\xi C_{22}(r_2)v_0^{(1)}\right]=\mu_2^{-1}\left[F_{2\theta}^{(2)}(z,t)\right].\end{cases}\quad a\le r\le r_2+b, \tag{36}$$

In this case, the integro-differential operators $\lambda_m^n(\zeta)$ defined by formulas (29) * transform into the following differential operators

$$\lambda_m^n(\zeta)=\left[\frac{1}{b_m^2}\left(\frac{\partial^2\zeta}{\partial t^2}\right)-\frac{\partial^2\zeta}{\partial z^2}\right]^n,\qquad m=0,1,2;\quad n=1,2,3,\ldots \tag{37}$$

Where b_m, $(m=0,1,2)$ are the velocities of propagation of transverse waves in the materials of the layers.

Thus, the system of equations (36), taking into account (37), are general equations of torsional vibrations of a three-layer cylindrical elastic shell. In particular cases, from equations (36) it is easy to obtain the equations of a two-layer elastic shell and a three-layer elastic shell with a thin middle layer similar to the limiting cases considered above.

3.4. Round viscoelastic rod

Consider one more limiting case that follows from the results obtained, torsional vibrations of a uniform round rod. Let's assume that the shell is homogeneous. Then the equations of its torsional vibrations are described by the system of equations (35). In the case when $r_1 = 0$, the homogeneous cylindrical shell transforms into a round rod of radius r_2. Equality to zero of the inner radius of a homogeneous shell implies equality to zero of the radius of the intermediate surface of the shell, i.e. $\xi = 0$. Consequently, in this case, the intermediate surface of the shell passes into the centerline of the rod. Therefore, the function of external influences $F_{2\theta}^{(1)}(z,t)$ should be considered equal to zero. Then the first equation of system (35) is satisfied identically. Only the second equation remains for the desired function $v_0^{(0)}$. Taking into account the above factors, the equation of torsional vibrations of a circular viscoelastic rod of radius r_2, following from (35), has the form

$$C_{12}(r_2)v_0^{(0)} = R_{\mu 0}^{-1}\left[F_{2\theta}^{(2)}(z,t)\right], \qquad 0 \le r \le r_2,$$

or in expanded form we obtain the equation

$$2\sum_{n=0}^{\infty}\frac{(r_2/2)^{2n+2}}{n!(n+2)!}\lambda_0^{n+1}v_0^{(0)} = R_{\mu 0}^{-1}\left[F_{2\theta}^{(2)}(z,t)\right], \tag{38}$$

where $v_0^{(0)}$ is the main part of the torsional displacement of the center line of the rod. In this case, integro-differential operator (29) takes the form

$$\lambda_0^n(\zeta) = \left[\rho_0 R_{\mu 0}^{-1}\left(\frac{\partial^2\zeta}{\partial t^2}\right) - \frac{\partial^2\zeta}{\partial z^2}\right]^n, \qquad n = 1,2,3,...$$

In the elastic case, assuming $R_{\mu 0} = \mu_0$ and limiting ourselves to zero, first, and other approximations in the infinite sum (38), one can obtain classical (Bernoulli-Euler type), refined (Timoshenko type) and other equations of a higher order.

4. Conclusions

- a new technique has been created for deriving general, refined and approximate equations of torsional vibrations of a circular cylindrical three-layer viscoelastic shell of an asymmetric structure and arbitrary thickness;

-proposed a new mathematical model of torsional vibrations of a circular cylindrical three-layer viscoelastic shell, including general and refined equations of vibration and an algorithm for calculating the stress-strain state of an arbitrary point of the shell. The oscillation equations in their structures automatically take into account the influence of rotational inertia and transverse shear deformation and are derived for external arbitrary dynamic loads acting on the shell surfaces. In the absence of external layers, the results obtained completely coincide with the results of [22] and are general equations of torsional vibrations of a circular cylindrical viscoelastic shell, relative to the main parts of the torsional displacement of the points of the intermediate surface of the shell;

- new general, refined and approximate equations of torsional vibrations of a circular cylindrical three-layer elastic shell are proposed. The resulting equations have, as indicated above, an unbounded order in derivatives, and therefore, in their structures, they contain derivatives of any order with respect to the longitudinal coordinate *z* and with respect to time *t*. Under constraints in infinite series by zero, first and other approximations, oscillation equations of the Kirchhoff Love type, Hermann-Mirsky [3] and other refined equations of higher orders follow from them;

-algorithms have been developed that allow to unambiguously determine the stress-strain state at any point of an arbitrary section of a circular cylindrical three-layer viscoelastic shell based on the results of solving the corresponding problems of its axisymmetric vibrations;

-created a new method for dynamic calculation of circular cylindrical three-layer elastic and viscoelastic shells on the action of various external dynamic loads. The method consists in deriving oscillation equations, both refined of the Timoshenko type, and the classical Kirchhoff-Love type, and in the development of an algorithm for calculating the Stress-Strain State of the system.

References

[1] Markus S 1988 *The mechanics of vibrations of cylindrical shells* (Amsterdam: Elsevier) p 195
[2] Yalgashev B F 2007 *Reports of the Academy of Sciences of the Republic of Uzbekistan* **2** 47-51
[3] Grigolyuk E I and Selezov I T 1973 *Results of science and technology Series Deformable Solid Mechanics* vol **5** (Moskov: VINITI) p 273
[4] Khudoynazarov K, Yalgashev B F 2016 *Scientific and technical journal Problems of Architecture and Construction* **4** 119-25
[5] Filippov I G, & Filippov S I 2007 *Oscillation and wave processes in continuous compressible media* (Moskov: Publishing house of MGSU) p 430
[6] Petrashen G I 1966 *Investigations of elasticity and plasticity* (Leningrad: *Publishing House* LGU) pp 33-53
[7] Filippov I G, & Kudainazarov K 1990 *Soviet Appl. Mech.* **26**(2) 161–68
[8] Filippov I G, & Kudainazarov K 1990 *Soviet Appl. Mech* **26**(4) 351–57
[9] Filippov I G, & Kudainazarov K 1998 *Int.J. Appl. Mech.* **34**(12) 1204–10
[10] Khudoynazarov K, Burkutboyev SM 2017 *Mathem. Modeling and Numer. Methods* **16** 31–47
[11] Khudoynazarov K, Khudoyberdiyev Z, Khudoyberdiyeva S 2018 *Int. J. of Advanced Research in Science, Engineering and Technology* **5(**10) 7117-21
[12] Khudoynazarov K, Skripnyak V A, Yakhshiboyev S R 2018 *Uzbek J. Problems of Mechanics* **2** 27-32
[13] Arshad S H, and others 2011 *Arch.of Appl. Mech.* **81** 319–43
[14] Abassi W, Rodoudi AE, Razafimahery F *2016 Int.J.of Appl.Mech.* **08**(03)
[15] Abdikarimov RA, Khudayarov BA 2019 *Int. Appl.Mech.* **50**(4) 389-98
[16] Khudayarov BA, Turaev FZ 2019 *Appl. Mathem. Modelling **66** 662-79*
[17] Netrebko AV, Pshenichnov SG 2015 *Problems of strength and ductility* **77**(1) 67-74
[18] J Kudlička 2006 *J. Sound and Vibr.* **294** (1–2) 368-73
[19] Xuebin Li 2008 *J. Sound and Vibr.* **311(**3–5) 667-82
[20] Kim JO, Chun HY 2003 *J. Vibr. and Acoustics 125(1) 39– 45*
[21] Filippov IG, Cheban VG 1988 *Mathematical theory of vibrations of elastic and viscoelastic plates and rods* (Kishinev: Publishing house Shtiinsa) p 190
[22] Khudoynazarov K, 2003 ., *Nonstatsionary interaction of circular cylindrical elastic and viscoelastic shells and rods with a deformable medium(*Tashkent: Publishing House named after Abu Ali ibn Sino) p 325