Topic 10.

Integral. Multiple integral

$[a,b]$ in cross section determined $y = f(x)$ function for this don't cut all at the points

$$F'(x) = f(x)$$

equality if done , then $F(x)$ function this in cross section $f(x)$ of the function **initial function** is called

Primary of the function common $F(x)+C$ appearance

e given of the function **not clear integral** is called Here C – optional constant number and

$$\int f(x)\,dx$$

such as is determined .

In this $\int$ – integral symbol , $f(x)$-integral sub function , $f(x)\,dx$ under the - integral expression is called

$y = f(x)$ of the function $[a,b]$ in cross section sure integral

$$\int_a^b f(x)\,dx$$

such as is determined .

Two multiple the integral count two sure the integral consecutively to count is brought . D field $y = y_1(x), y = y_2(x)$ functions graphics and $x = a$ and $x = b$ right lines with limited be , that is

$$\begin{cases} a \le x \le b \\ y_1(x) \le y \le y_2(x) \end{cases}$$

if so , two the multiple integral is as follows is :

$$\iint_D f(x,y)\,ds = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy \right] dx = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy$$

➢ Main integration schedule

$\int dx = x + c$	$\int \frac{dx}{\cos x} = \ln\left\lvert tg\left(\frac{x}{2}+\frac{\pi}{2}\right)\right\rvert + c$
$\int x^a dx = \frac{x^{a+1}}{a+1} + c, a \neq -1$	$\int \frac{dx}{\sin x} = \ln\left\lvert tg\frac{x}{2}\right\rvert + c$
$\int a^x dx = \frac{a^x}{\ln a} + c$	$\int \frac{dx}{a^2+x^2} = \begin{cases} \frac{1}{a}arctg\frac{x}{a}+c \\ -\frac{1}{a}arcctg\frac{x}{a}+c \end{cases}$
$\int \frac{dx}{x} = \ln\lvert x\rvert + c$	$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert + c$
$\int \cos x dx = \sin x + c$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \begin{cases} \arcsin\frac{x}{a}+c \\ -\arccos\frac{x}{a}+c \end{cases}$
$\int \sin x dx = -\cos x + c$	$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left\lvert x+\sqrt{x^2 \pm a^2}\right\rvert + c$
$\int \frac{dx}{\cos^2 x} = tgx + c$	$\int shxdx = chx + c$
$\int \frac{dx}{\sin^2 x} = -ctgx + c$	$\int chxdx = shx + c$
$\int \frac{dx}{\sin x} = \ln\left\lvert tg\frac{x}{2}\right\rvert + c$	$\int \frac{dx}{ch^2 x} = thx + c$

Solved from examples samples

Example 10.1. Find integral : $\int x\sqrt{x^2-16}dx$

Solution :

This not clear the integral count for the following from replacement used :

$$t = x^2 - 16;$$

That is , as above t designation is entered and from this the following known will be :

$dt = 2xdx$, from this $xdx = \left(\frac{1}{2}\right)dt$ come comes out

Now in the integral x variable t to the variable replaced by :

$$\int x\sqrt{x^2-16}dx = \int \sqrt{x^2-16}\cdot xdx\Big|_{t=x^2-16} = \int\left(\frac{1}{2}\right)\sqrt{t}dt =$$

$$= \frac{1}{2}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}} = \frac{1}{2}\cdot\frac{2t^{\frac{3}{2}}}{3} = \frac{1}{3}t\sqrt{t}\Big|_{t=x^2-16} = \frac{1}{3}\left(x^2-16\right)^{3/2}+C\ .$$

Example 10.2. Given of the function not clear the integral find : $\int\frac{x^3}{9+16x^4}dx$

<u>Solution :</u>

This given of the function not clear the integral to find for fraction photo of the fraction from the denominator received from the derivation to bring from possible used :

$$\int\frac{x^3}{9+16x^4}dx = \frac{1}{4}\int\frac{d\left(x^4\right)}{9+16x^4} = \frac{1}{64}\int\frac{d\left(9+16x^4\right)}{9+16x^4} = \frac{1}{64}\ln\left(9+16x^4\right)+C\ .$$

Example 10.3. In pieces integration from the formula used without , not clear the integral find : $\int e^{2x}\cos xdx$

<u>Solution :</u>

Here both function very simply separately combined . In this in pieces integration from the formula use for as follows designations enter :

$$\int e^{2x}\cos xdx = \begin{Bmatrix} u = e^{2x} & du = 2e^{2x}dx \\ dv = \cos xdx & v = \sin x \end{Bmatrix} \Rightarrow u\cdot v - \int vdu\ ;$$

Above u and v s integration to the formula put in case the following result come output :

$$\int e^{2x}\cos xdx = e^{2x}\sin x - 2\int e^{2x}\sin xdx\ ;$$

In this $\int e^{2x}\sin xdx$ not clear the integral to find for again one times in pieces integration from the formula used :

$$\int e^{2x}\sin x dx = \begin{Bmatrix} u = e^{2x} & du = 2e^{2x}dx \\ dv = \sin x dx & v = -\cos x \end{Bmatrix} \Rightarrow u\cdot v - \int v du \Rightarrow$$

This in the case above u and v s integration to the formula put in case the following result come output :

$$\int e^{2x}\sin x dx = -e^{2x}\cos x + 2\int e^{2x}\cos x dx \text{ ;}$$

Now all received results summarizes :

$$\int e^{2x}\cos x dx = e^{2x}\sin x - 2\int e^{2x}\sin x dx =$$

$$= e^{2x}\sin x - 2\left(-e^{2x}\cos x + 2\int e^{2x}\cos x dx\right) = e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x dx \text{ ;}$$

In this

$$\int e^{2x}\cos x dx = e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x dx \text{ ;}$$

* If $\left\{I = \int e^{2x}\cos x dx\right\}$ designation if entered , then the following common the result get can :

$$I = e^{2x}\sin x + 2e^{2x}\cos x - 4I \Rightarrow 5I = e^{2x}\sin x + 2e^{2x}\cos x \text{ ;}$$

$$\Rightarrow I = \frac{1}{5}\left(e^{2x}\sin x + 2e^{2x}\cos x\right)$$

Result :

$$\int e^{2x}\cos x dx = \frac{1}{5}\left(e^{2x}\sin x + 2e^{2x}\cos x\right) + C \text{ .}$$

Example 10.4. Given sure the integral calculate : $\int_{-2}^{4}\left(8 + 2x - x^2\right)dx$.

<u>Solution :</u>

$$\int_{-2}^{4}\left(8 + 2x - x^2\right)dx = \left(8x + x^2 - \frac{x^3}{3}\right)\Bigg|_{-2}^{4} =$$

$$=\left(8\cdot4+4^2-\frac{4^3}{3}\right)-\left(8\cdot(-2)+(-2)^2-\frac{(-2)^3}{3}\right)=\left(32+16-\frac{64}{3}\right)-\left(-16+4+\frac{8}{3}\right)=\frac{80}{3}+\frac{28}{3}=36.$$

10. Example 5. Given two multiple the integral calculate : $\int_1^2\int_4^6\frac{x}{y^2}dxdy$.

Solution :

$\int_1^2\int_4^6\frac{x}{y^2}dxdy$ this two multiple the integral count for initially , y to relatively sure integral is , then while x to relatively is the definite integral :

$$\int_1^2\left(\int_4^6\frac{x}{y^2}dy\right)dx=\int_1^2\left(\left[-\frac{x}{y}\right]_{y=4}^{y=6}\right)dx=$$

$$=\int_1^2\left(\frac{x}{4}-\frac{x}{6}\right)dx=\int_1^2\frac{x}{12}dx=\left[\frac{x^2}{24}\right]_{x=1}^{x=2}=\frac{2^2}{24}-\frac{1^2}{24}=\frac{2^2-1^2}{24}=\frac{3}{24}=\frac{1}{8}.$$

Independent work for issues

Find the integral (10.1-10.24):

10.1 $\int(3x^2-6x+3)\,dx$;

10.2 $\int(8x^3-x^2+5x-1)\,dx$;

10.3 $\int\left(-\frac{x^4}{2}-\frac{x^3}{3}-\frac{x^2}{6}\right)dx$;

10.4 $\int\left(\frac{16}{x^5}-\frac{9}{x^4}+\frac{4}{x^3}\right)dx$;

10.5 $\int\left(\frac{1}{\sqrt[3]{3x^8}}-\frac{1}{\sqrt[7]{3x^5}}\right)dx$;

10.6 $\int(1-3x+x^3)\sqrt[3]{x}dx$;

10.7 $\int\left(\sqrt{x^5}-\sqrt[5]{x^2}\right)dx$;

10.8 $\int\sqrt{x\sqrt{x^3\sqrt{x^5\sqrt{x^7}}}}\,dx$;

10.9 $\int\left(1-x^{e}+e^{x}-e^{e}\right)dx$;

10.10 $\int\left(\pi^{2}+e-\sqrt{2}\right)dx$;

10.11 $\int\left(\sin x+\frac{1}{x}+\cos x\right)dx$;

10.12 $\int\sin 7x dx$;

10.13 $\int\left(\frac{1}{\cos^{2}x}-\frac{1}{\sin^{2}x}\right)dx$;

10.14 $\int\left(\frac{2}{1+x^{2}}-\frac{5}{\sqrt{1-x^{2}}}\right)dx$;

10.15 $\int\left(\frac{x^{2}-2x}{x^{3}-3x^{2}+1}\right)dx$;

10.16 $\int\frac{1}{x\ln x}dx$;

10.17 $\int\sqrt{5+2x}dx$;

10.18 $\int\frac{\ln^{2}x}{x}dx$;

10.19 $\int\frac{1}{x^{2}}\cos\frac{1}{x}dx$;

10.20 $\int\sin\left(\frac{3x-5}{2}\right)dx$;

10.21 $\int\frac{x}{x^{2}+1}dx$;

10.22 $\int\frac{3x^{3}}{\sqrt[3]{x^{4}+4}}dx$;

10.23 $\int\frac{2}{x^{2}+9}dx$;

10.24 $\int x^{2}e^{x^{3}}dx$;

In pieces integration from the formula using , of the function not clear find the integral (10.25-10.40):

10.25 $\int x\sin x dx$;

10.26 $\int x^2 \ln x dx$;

10.27 $\int x \cdot e^x dx$;

10.28 $\int (1-x) 2^x dx$;

10.29 $\int x^2 e^x dx$;

10.30 $\int x^2 \cos x dx$;

10.31 $\int x^3 \ln x dx$;

10.32 $\int x \ln (x-1) dx$;

10.33 $\int e^x \sin x dx$;

10.34 $\int 2e^x \cos x dx$;

10.35 $\int \frac{x}{x+5} dx$;

10.36 $\int \frac{3x-1}{x+2} dx$;

10.37 $\int \frac{9x^3}{3x+1} dx$;

10.38 $\int \frac{x+3}{x^2+4} dx$;

10.39 $\int \frac{2x-1}{(x-1)(x-2)} dx$;

10.40 $\int \frac{1}{x^2+4x+5} dx$;

The following sure the integral calculate (10.41-10.52):

10.41 $\int_{-1}^{2} 3x^2 dx$;

10.42 $\int_{1}^{8} \sqrt[3]{x} dx$;

10.43 $\int_{-2}^{1} x^3 dx$;

10.44 $\int_{0}^{2} (3x^3 - 2x + 5) dx$;

10.45 $\int_1^4 \left(-x+\frac{4}{x}\right)dx$;

10.46 $\int_0^1 \frac{x}{\left(x^2+1\right)^2}dx$;

10.47 $\int_0^1 \frac{e^x}{e^x+1}dx$;

10.48 $\int_0^{\pi} 2\sin x dx$;

10.49 $\int_0^{\pi} \frac{1}{\cos^2 x}dx$;

10.50 $\int_1^{e} \ln x dx$;

10.51 $\int_{-1}^{1} \frac{2x}{x^2-1}dx$;

10.52 $\int_{-\frac{\pi}{2}}^{0} \sin x\cos x dx$;

Below given two multiple the integral calculate (10.53-10.60):

10.53 $\int_0^2 dx \int_{x^2}^{2\sqrt{2x}} dy$;

10.54 $\int_0^6 dx \int_{\frac{x^2}{4}}^{\frac{3x}{2}} dy$;

10.55 $\int_1^5 dx \int_{\frac{(x-1)^2}{8}}^{\frac{(x-1)}{2}} dy$;

10.56 $\int_1^3 \int_2^4 9x^3y^2 dydx$;

10.57 $\int_{11}^{14} \int_7^{10} \left(x^2+4y\right)dydx$;

10.58 $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} x\cos\left(xy\right)dydx$;

10.59 $\int_0^1 \int_{x^2}^{x} (x+3)\,dydx$ **;**

10.60 $\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx$ **.**

Answers

10.1. $x^3 - 3x^2 + 3x + C$

10.2. $2x^4 - \frac{x^3}{3} + \frac{5}{2}x^2 - x + C$

10.3. $-\frac{x^5}{10} - \frac{x^4}{12} - \frac{x^3}{18} + C$

10.4. $-\frac{4}{x^4} + \frac{3}{x^3} - \frac{2}{x^2} + C$

10.5. $-\frac{3}{5\sqrt[3]{3x^5}} - \frac{7}{2\sqrt[7]{3x^2}} + C$

10.6. $\frac{3}{13}x^{\frac{13}{3}} - \frac{9}{7}x^{\frac{7}{3}} + \frac{3}{4}x^{\frac{4}{3}} + C$

10.7. $-\frac{5}{7}x^{\frac{7}{5}} + \frac{2}{7}x^{\frac{7}{2}} + C$

10.8. $\frac{16}{21}x^{\frac{21}{16}} + C$

10.9. $x - \frac{1}{e-1}x^{e-1} + e^x - e^e x + C$

10.10. $\left(\pi^2 + e - \sqrt{2}\right)x + C$

10.11. $\ln(x) + \sin x - \cos x + C$

10.12. $-\frac{1}{7}\cos 7x + C$

10.13. $tgx + ctgx + C$

10.14. $-5\arcsin x + 2arctgx + C$

10.15. $\frac{1}{3}\ln\left(x^3 - 3x^2 + 1\right) + C$

10.16. $\ln(\ln x) + C$

10.17. $\frac{1}{3}\sqrt{\left(2x+5\right)^3} + C$

10.18. $\frac{1}{3}\ln^3 x + C$

10.19. $-\sin\frac{1}{x} + C$

10.20. $-\frac{2}{3}\cos\left(\frac{3x-5}{2}\right)+C$

10.21. $\frac{1}{2}\ln\left(x^2+1\right)+C$

10.22. $\frac{9}{8}\sqrt[3]{\left(x^4+4\right)^2}+C$

10.23. $\frac{2}{3}arctg\left(\frac{x}{3}\right)+C$

10.24. $\frac{1}{3}e^{x^3}+C$

10.25. $x\cos x+\sin x+C$

10.26. $\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$

10.27. $\left(x-1\right)e^x+C$

10.28. $\frac{2^x\left(1-x\right)}{\ln 2}+\frac{2^x}{\ln^2 2}+C$

10.29. $\left(x^2-2x+2\right)e^x+C$

10.30. $x^2\sin x+2x\cos x-2\sin x+C$

10.31. $\frac{1}{4}x^4\ln x-\frac{1}{16}x^4+C$

10.32. $-\frac{x^2}{4}+\frac{1}{2}x\left(x+2\right)\ln x-x+C$

10.33. $\frac{1}{2}e^x\sin x-\frac{1}{2}e^x\cos x+C$

10.34. $e^x\sin x+e^x\cos x+C$

10.35. $x-5\ln\left(x+5\right)+C$

10.36. $3x-7\ln\left(x+2\right)+C$

10.37. $x^3-\frac{x^2}{2}+\frac{x}{3}-\frac{1}{9}\ln\left(9x+3\right)+C$

10.38. $\frac{1}{2}\ln\left(x^2+4\right)+\frac{3}{2}arctg\left(\frac{x}{2}\right)+C$

10.39. $3\ln\left(x-2\right)-\ln\left(x-1\right)+C$

10.40. $arctg\left(x+2\right)+C$

10.41. 9

10.42. $\frac{45}{4}$

10.43. $-\frac{15}{4}$

10.44. 18

10.45. $4\ln 4-\frac{15}{2}$

10.46. $\frac{1}{4}$

10.47. $\ln(e+1)-\ln 2$

10.48. 4

10.49. 0

10.50. 1

10.51. 0

10.52. $-\frac{1}{2}$

10.53. $4\sqrt{2}-\frac{8}{3}$

10.54. 9

10.55. $\frac{4}{3}$

10.56. 3360

10.57. 1719

10.58. $\frac{2}{\pi}\left(-\cos\left(\frac{\pi^2}{4}\right)+1\right)$

10.59. $\frac{7}{12}$

10.60. $\frac{\pi}{4}a^2$